Riemann hypothesis for period polynomials of modular forms

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The period polynomial $r_f(z)$ for an even weight $k \geq 4$ newform $f \in S_k(\Gamma_0(N))$ is the generating function for the critical values of $L(f,s)$. It has a functional equation relating $r_f(z)$ to $r_f(-\frac{1}{Nz})$. We prove the Riemann hypothesis for these polynomials: that the zeros of $r_f(z)$ lie on the circle $|z| = 1/\sqrt{N}$. We prove that these zeros are equidistributed when either k or N is large.

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Let $f \in S_k(\Gamma_0(N))$ be a newform (1, 2) of even weight k and level N. Associated to f is its L-function $L(f,s)$, which has been normalized so that the completed L-function,

$$\Lambda(f,s) := \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma(s) L(f,s),$$

satisfies the functional equation $\Lambda(f,s) = \epsilon(f)\Lambda(f,k-s)$, with $\epsilon(f) = \pm 1$. Recall that the completed L-function arises as a period integral of the newform f:

$$\Lambda(f,s) = N^{s/2}\int_0^\infty f(iy)y^s \frac{dy}{y}. \qquad \textbf{[1.1]}$$

The focus of this paper is the period polynomial associated to f, the degree $k-2$ polynomial

$$r_f(z) := \int_0^{i\infty} f(\tau)(\tau - z)^{k-2} d\tau. \qquad \textbf{[1.2]}$$

Expanding $(\tau - z)^{k-2}$, and using Eq. **1.1**, we may also express the period polynomial by

$$r_f(z) = i^{k-1} N^{-\frac{k-1}{2}} \sum_{n=0}^{k-2} \binom{k-2}{n} \left(\sqrt{N} iz\right)^n \Lambda(f, k-1-n), \qquad \textbf{[1.3]}$$

or equivalently as

$$r_f(z) = -\frac{(k-2)!}{(2\pi i)^{k-1}} \sum_{n=0}^{k-2} \frac{(2\pi i z)^n}{n!} L(f, k-n-1). \qquad \textbf{[1.4]}$$

In other words, $r_f(z)$ is a generating function for the critical values $L(f,1), L(f,2), \ldots, L(f,k-1)$. For general facts on period polynomials, the reader is encouraged to see refs. 3–7; other papers broadly related to the themes of this paper are refs. 8 and 9.

Using the functional equation $\Lambda(f,s) = \epsilon(f)\Lambda(f,k-s)$ in Eq. **1.3**, we find that

$$r_f(z) = -i^k \epsilon(f)\left(\sqrt{N}z\right)^{\frac{k-2}{2}} r_f\left(-\frac{1}{Nz}\right),$$

so that if ρ is a zero of $r_f(z)$ then so is $-1/(N\rho)$. In analogy with the Riemann hypothesis, we may ask whether all of the zeros of $r_f(z)$ lie on the circle $|\rho| = 1/\sqrt{N}$. For Hecke eigenforms on $SL_2(\mathbb{Z})$, this was recently established by El-Guindy and Raj (10), who showed that the zeros of $r_f(z)$ (for $N = 1$) are all on the unit circle $|z| = 1$. Their work was inspired by the previous work by Conrey et al. (11), who proved an analogous result for odd period polynomials again for full level. We show that this "Riemann hypothesis" holds in general for all newforms of weight at least 4 and any level.

Theorem 1.1. *For any even integer k at least 4, and any level N, all of the zeros of the period polynomial $r_f(z)$ are on the circle $|z| = 1/\sqrt{N}$.*

Remark: Period polynomials for weight 2 newforms f are constant multiples of $L(f,1)$.

Example 1: The period polynomial for the normalized Hecke eigenform $\Delta(z) \in S_{12}(\Gamma_0(1))$ is

$$\begin{aligned} r_\Delta(z) &= \omega_\Delta^+ r_\Delta^+(z) + \omega_\Delta^- r_\Delta^-(z) \approx 0.114379i \\ &\times \left(\frac{36}{691}z^{10} - z^8 + 3z^6 - 3z^4 + z^2 - \frac{36}{691}\right) \\ &+ 0.00926927\left(4z^9 - 25z^7 + 42z^5 - 25z^3 + 4z\right). \end{aligned}$$

All 10 zeros of $r_\Delta(z)$ are on $|z| = 1$.

Example 2: For the unique weight 4 newform $f(z) = q - 4q^3 - 2q^5 + \cdots$ *on* $\Gamma_0(8)$, we have

$$\begin{aligned} L(f,1) &\approx 0.3545006\ldots, \\ L(f,2) &\approx 0.6900311\ldots, \\ L(f,3) &\approx 0.8746953\ldots, \end{aligned}$$

which in turn implies that $r_f(z) \approx 0.0564205361iz^2 + 0.0349573870z - 0.00705256701815496i$. The roots are $\approx \pm 0.17037672 + 0.30979311i$, and their norms are $\approx 1/(2\sqrt{2})$.

Remark: Manin (12) has used the work of Conrey et al. (11) to construct zeta functions that satisfy the Riemann hypothesis. He suggests that these polynomials arise from non-Tate motives and

Significance

Critical values of modular *L*-functions are objects of central importance in arithmetic geometry and number theory. These numbers are predicted to encode deep arithmetic information by the Birch and Swinnerton-Dyer conjecture and the Bloch–Kato conjecture. Here we consider the generating functions for these values, the so-called period polynomials. The Riemann hypothesis for these polynomials is the assertion that the zeros of these polynomials are located on the circle of symmetry that arises from the standard functional equations. The truth of this hypothesis places strong constraints on the size of the critical *L*-values. This assertion is proved here.

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geometric objects lying below Spec $\mathbf{Z}$ but not over $\mathbf{F}_1$. Using the $P_f(z)$ defined below, one obtains further such polynomials *mutatis mutandis*.

If the weight or level is large enough, then the zeros of r_f are regularly spaced on the circle $|z| = 1/\sqrt{N}$. To state this conveniently, and for our later work, we shall put $m := (k-2)/2$ throughout and define

$$P_f(z) = \frac{1}{2}\binom{2m}{m}\Lambda\left(f, \frac{k}{2}\right) + \sum_{j=1}^{m}\binom{2m}{m+j}\Lambda\left(f, \frac{k}{2}+j\right)z^j. \quad [1.5]$$

Then, using the functional equation, we see that

$$r_f\left(\frac{z}{i\sqrt{N}}\right) = i^{k-1}N^{-\frac{k-1}{2}}\epsilon(f)z^m\left(P_f(z) + \epsilon(f)P_f\left(\frac{1}{z}\right)\right). \quad [1.6]$$

Therefore, to understand the zeros of r_f, it is enough to understand the zeros of $P_f(z) + \epsilon(f)P_f(1/z)$, and *Theorem 1.1* states that this function has all its zeros on the unit circle $|z| = 1$. If we restrict to the unit circle $|z| = 1$, then $P_f(z) + \epsilon(f)P_f(1/z)$ is either a trigonometric cosine or a trigonometric sine polynomial [depending on whether $\epsilon(f)$ equals 1 or -1], and our proof of *Theorem 1.1* proceeds by finding the right number of sign changes as z varies over the unit circle. If k or N is large enough, the proof allows us to establish the following result on the location of the roots.

Theorem 1.2. *The following are true.*

i) *Suppose that $k = 4$. If $\epsilon(f) = -1$, then the zeros of $r_f(z)$ are $\pm i/\sqrt{N}$. If $\epsilon(f) = 1$ and N is sufficiently large, then the zeros of $r_f(z)$ are located at $\pm\left(1 + O\left(N^{-\frac{1}{4}+\epsilon}\right)\right)/\sqrt{N}$.*

ii) *If $k \geq 6$ and either N or k is large enough, then the roots of $r_f(z)$ may be written as*

$$\frac{1}{i\sqrt{N}}\exp\left(i\theta_\ell + O\left(\frac{1}{2^k\sqrt{N}}\right)\right),$$

where for $0 \leq \ell \leq 2m-1$ we denote by θ_ℓ the unique solution in $[0, 2\pi)$ to the equation

$$m\theta_\ell - \frac{2\pi}{\sqrt{N}}\sin\theta_\ell = \begin{cases} \frac{\pi}{2} + \ell\pi & \text{if } \epsilon(f) = 1 \\ \ell\pi & \text{if } \epsilon(f) = -1. \end{cases}$$

Our arguments readily allow us to quantify the results in *Theorem 1.2*. For example, the arguments in section 6 give that in part *ii* above, the implied O-constant may be taken as 10^9, although this is a gross overestimate. The arguments in section 5 locate sign changes even if the values of k or N are only moderately large.

Suppose that $\epsilon(f) = 1$. By counting sign changes, one consequence of *Theorem 1.1* is that $P_f(-1)$ has sign $(-1)^m$. In other words, if $\epsilon(f) = 1$, then we must have

$$\frac{1}{2}\binom{2m}{m}(-1)^m\Lambda\left(f, \frac{k}{2}\right) + \sum_{j=0}^{m-1}(-1)^j\binom{2m}{2m-j}\Lambda(f, k-1-j) > 0. \quad [1.7]$$

For any weight k, this inequality is clear for large enough N because the term $j = 0$ above dominates all other terms. However, it is interesting that such an inequality holds for all small weights and small level as well, and we wonder whether it has any other significance. In section 4 we give a proof of this inequality in the weight 6 case based on the Hadamard factorization formula. We also give there a more illuminating proof of this inequality based on the Riemann hypothesis for $\Lambda(f, s)$.

2. Preliminaries

Here we collect preliminary facts about L-functions that we shall find useful. The completed L-function $\Lambda(f, s)$ is an entire function of order 1. Its zeros all lie in the strip $\left|\mathrm{Re}(s) - \frac{k}{2}\right| < \frac{1}{2}$, with the Riemann hypothesis predicting that all zeros lie on the line $\mathrm{Re}(s) = \frac{k}{2}$. Recall also that the central value $\Lambda\left(f, \frac{k}{2}\right)$ is known to be nonnegative by the work of Waldspurger (13).

Hadamard's factorization formula applies to the entire function $\Lambda(f, s)$, and we may write

$$\Lambda(f, s) = e^{A+Bs}\prod_\rho\left(1 - \frac{s}{\rho}\right)e^{s/\rho}. \quad [2.1]$$

Here the product is over all of the zeros of $\Lambda(f, s)$ [that is, the nontrivial zeros of $L(f, s)$], and A and B are constants. Note that if ρ is a zero then so too are $\overline{\rho}$ and $k - \rho$. Because $\Lambda(f, s)$ is real-valued on the real line, and in view of the functional equation, we have that B is real-valued and

$$B = -\sum_\rho \mathrm{Re}\frac{1}{\rho} = -\sum_\rho \frac{\mathrm{Re}(\rho)}{|\rho|^2}.$$

These considerations also show that for real s

$$\Lambda(f, s) = e^A\prod_{\rho\in\mathbb{R}}\left(1 - \frac{s}{\rho}\right)\prod_{\mathrm{Im}(\rho)>0}\left|1 - \frac{s}{\rho}\right|^2, \quad [2.2]$$

where we have paired the complex conjugate roots together so that the product is convergent.

Lemma 2.1. *The function $\Lambda(f, s)$ is monotone increasing for $s \geq \frac{k}{2} + \frac{1}{2}$. Moreover, we have*

$$0 \leq \Lambda\left(f, \frac{k}{2}\right) \leq \Lambda\left(f, \frac{k}{2}+1\right) \leq \Lambda\left(f, \frac{k}{2}+2\right) \leq \ldots.$$

If $\epsilon(f)$ is -1, then $\Lambda\left(f, \frac{k}{2}\right) = 0$ and

$$0 \leq \Lambda\left(f, \frac{k}{2}+1\right) \leq \frac{1}{2}\Lambda\left(f, \frac{k}{2}+2\right) \leq \frac{1}{3}\Lambda\left(f, \frac{k}{2}+3\right) \leq \ldots.$$

Monotonicity results such as *Lemma 2.1* are familiar in the literature; for example, the Riemann hypothesis for L-functions is equivalent to the monotonicity of the absolute value of the completed L-function along horizontal lines starting from the critical line. In a different context Stark and Zagier (14) observed a similar result.

Proof: Because all of the zeros lie in $\left|\mathrm{Re}(s) - \frac{k}{2}\right| < \frac{1}{2}$, we see that $|1 - s/\rho|$ is increasing for $s \geq \frac{k}{2} + \frac{1}{2}$. So, by Eq. **2.2** it follows that $\Lambda(f, s)$ is increasing in $\mathrm{Re}(s) \geq \frac{k}{2} + \frac{1}{2}$. Further, we have

$$\left|1 - \frac{k/2}{\rho}\right| \leq \left|1 - \frac{k/2+1}{\rho}\right|,$$

and so $\Lambda(f, k/2) \leq \Lambda(f, k/2+1)$. When $\epsilon(f) = -1$, we apply the same reasoning and now take into account that there must be a zero of odd order at $k/2$.

We record a useful inequality for L-values in the range of absolute convergence.

Lemma 2.2. *If $0<a<b$ and k is the weight of f, then we have*

$$\frac{L\left(f,\frac{k+1}{2}+a\right)}{L\left(f,\frac{k+1}{2}+b\right)}\le\frac{\zeta(1+a)^2}{\zeta(1+b)^2}.$$

Proof: The Euler product for $L(f,s)$ gives rise to

$$-\frac{L'}{L}(f,s)=\sum_{n=1}^{\infty}\frac{\Lambda_f(n)}{n^s},$$

where $|\Lambda_f(n)|\le 2n^{\frac{k-1}{2}}\Lambda(n)$ for all n. Here $\Lambda(n)$ is the usual von Mangoldt function, and this estimate is an alternative way of encoding the Ramanujan bounds established by Deligne (15) [see also Li (2) for the Euler factors at primes dividing the level]. The point is that for prime powers $n=p^r$ we have $\Lambda_f(n)=(\alpha_p^r+\beta_p^r)\log p$, where the p th Fourier coefficient of f satisfies $a(p)=\alpha_p+\beta_p$. Therefore, we have

$$\frac{L\left(f,\frac{k+1}{2}+a\right)}{L\left(f,\frac{k+1}{2}+b\right)}=\exp\left(\int_a^b-\frac{L'}{L}\left(f,\tfrac{k+1}{2}+t\right)dt\right)$$

$$\le\exp\left(2\int_a^b-\frac{\zeta'}{\zeta}(1+t)dt\right)=\frac{\zeta(1+a)^2}{\zeta(1+b)^2}.$$

3. The Weight 4 Case

If f is a form of weight $k=4$ [so $m=(k-2)/2=1$], then $P_f(z)=\Lambda(f,2)+\Lambda(f,3)z$. If $\epsilon(f)=-1$, then the roots of $P_f(z)-P_f(1/z)=\Lambda(f,3)(z-1/z)$ are at $z=\pm1$ and so the period polynomial has roots at $\pm i/\sqrt{N}$.

If $\epsilon(f)=1$, then with $z=e^{i\theta}$ we have $P_f(z)+P_f(1/z)=2\Lambda(f,2)+2\Lambda(f,3)\cos\theta$. Because $\Lambda(f,2)<\Lambda(f,3)$ by *Lemma 2.1*, the above equation has two solutions for $\theta\in[0,2\pi)$, namely, θ satisfying $\cos\theta=-\Lambda(f,2)/\Lambda(f,3)$. This completes the proof of *Theorem 1.1* for weight 4.

Note that $\Lambda(f,3)\gg N^{\frac{3}{2}}$ for large N, whereas the Phrágmen–Lindelöf principle gives $\Lambda(f,2)\le\max_{t\in\mathbb{R}}\left|\Lambda\left(f,\frac{5}{2}+\epsilon+it\right)\right|\ll N^{\frac{5}{4}+\epsilon}$ (this is the "convexity bound" for L-functions). Therefore, the ratio $\Lambda(f,2)/\Lambda(f,3)$ is small (precisely $\ll N^{-\frac{1}{4}+\epsilon}$), and hence the corresponding values of θ tend to $\pi/2$ and $3\pi/2$. Thus, for large level, the zeros of the period polynomial [in the $\epsilon(f)=1$ case] are located at $\pm\left(1+O\left(N^{-\frac{1}{4}+\epsilon}\right)\right)/\sqrt{N}$.

4. The Weight 6 Case

If f is a form of weight $k=6$ (so that $m=2$) then

$$P_f(z)=3\Lambda(f,3)+4\Lambda(f,4)z+\Lambda(f,5)z^2.$$

If $\epsilon(f)=-1$, then we are interested in the roots of

$$P_f(z)-P_f(1/z)=\left(z-\frac{1}{z}\right)\left(4\Lambda(f,4)+\Lambda(f,5)\left(z+\frac{1}{z}\right)\right).$$

Clearly there are two solutions $z=\pm1$. Because $\epsilon(f)=-1$, we know that $2\Lambda(f,4)<\Lambda(f,5)$ by *Lemma 2.1*, and so there are two solutions in $[0,2\pi)$ to $\cos\theta=-2\Lambda(f,4)/\Lambda(f,5)$. Thus, we have shown *Theorem 1.1* in this case. Moreover, if N is large, then $\Lambda(f,4)/\Lambda(f,5)$ is small and θ tends to $\pi/2$ or $3\pi/2$. So, for large N (and odd sign) the period polynomial has two zeros exactly at $\pm i/\sqrt{N}$ and the other two zeros are very near $\pm1/\sqrt{N}$.

It remains now to consider when $\epsilon(f)=1$. With $z=e^{i\theta}$ we must show that

$$P_f(z)+P_f(1/z)=2\cos(2\theta)\Lambda(f,5)+8\cos\theta\,\Lambda(f,4)+6\Lambda(f,3) \quad \textbf{[4.1]}$$

has two zeros in $[0,\pi]$ (and therefore four zeros in $[0,2\pi)$). Differentiating the above with respect to θ gives $-8\sin\theta(\Lambda(f,4)+\cos\theta\,\Lambda(5))$ so that there are critical points at $\theta=0$, π, and at the solution $\theta_0\in(0,\pi)$ to $\cos\theta=-\Lambda(f,4)/\Lambda(f,5)$. We would like the quantity in Eq. **4.1** to be positive at $\theta=0$, negative at θ_0, and positive again at $\theta=\pi$, which would ensure two zeros in $(0,\pi)$ (and note that these conditions are also necessary for the period polynomial to have all zeros on a circle).

The value at $\theta=0$ is clearly positive. That the value should be positive at π is equivalent to

$$\Lambda(f,5)+3\Lambda(f,3)>4\Lambda(f,4). \quad \textbf{[4.2]}$$

The condition that the value should be negative at θ_0 is equivalent to

$$\Lambda(f,5)^2+2\Lambda(f,4)^2\ge3\Lambda(f,3)\Lambda(f,5). \quad \textbf{[4.3]}$$

Lemma 4.1. *Suppose a_1, a_2, b_1, b_2, and c_1, c_2 are all positive with $a_i\ge\max(b_i,c_i)$. Suppose that $a_i+\gamma c_i\ge(1+\gamma)b_i$, where γ is positive. Then $a_1a_2+\gamma c_1c_2\ge(1+\gamma)b_1b_2$.*

Proof: Multiply the relation $a_1+\gamma c_1\ge(1+\gamma)b_1$ by b_2. It suffices to show that

$$a_1a_2+\gamma c_1c_2\ge a_1b_2+\gamma c_1b_2;$$

or, rearranging that $a_1(a_2-b_2)\ge\gamma c_1(b_2-c_2)$. Because $(a_2-b_2)\ge0$, and $a_1\ge c_1$, the left-hand side above is at least $c_1(a_2-b_2)$ which is $\ge\gamma c_1(b_2-c_2)$.

Proof of 4.2: We use *Lemma 4.1* suitably, together with the Hadamard factorization formula (Eqs. **2.1** and **2.2**), proceeding zero by zero. We use the Hadamard formula for $\Lambda(f,3)$, $\Lambda(f,4)$, and $\Lambda(f,5)$; note that at all these values Λ is known to be nonnegative (this is clear for 4 and 5, and work of Waldspurger for 3), so we can also assume that the products are taken with absolute values.

Suppose first that $\rho=3+z$ is a real zero, and then $6-\rho=3-z$ is also a real zero. (Note that even if $\rho=3$, we get zeros of even multiplicity at the center, which may be paired.) Then note that this pair of zeros contributes to $\Lambda(f,5)$ the amount $a=(4-z^2)/(9-z^2)$, to $\Lambda(f,4)$ the amount $b=(1-z^2)/(9-z^2)$, and to $\Lambda(f,3)$ the amount $c=z^2/(9-z^2)$ (using here the absolute value remark). Note that with $\gamma=3$ we have the inequality $a+3c\ge4b$.

Now consider a zero $\rho=3+iy$ on the critical line, and pair it with its conjugate $3-iy$. These contribute to $\Lambda(f,5)$ the amount $a=(4+y^2)/(9+y^2)$, to $\Lambda(f,4)$ the amount $b=(1+y^2)/(9+y^2)$ and to $\Lambda(f,3)$ the amount $c=y^2/(9+y^2)$, and we check again that $a+3c\ge4b$ (and indeed equality holds).

Finally consider a zero $\rho=3+z$ not on the critical line with $z=x+iy$. This comes in a set of four zeros $3\pm x\pm iy$. Note that these four zeros contribute (multiply through by $|\rho|^2|6-\rho|^2$) to $\Lambda(f,5)$ an amount $a=|4-z^2|^2$, to $\Lambda(f,4)$ an amount $b=|1-z^2|^2$, and to $\Lambda(f,3)$ the amount $c=|z^2|^2$. We can check again that $a+3c\ge4b$.

Thus, when grouped as above, each group of zeros appearing in the Hadamard formula satisfies a version of **4.2**. By *Lemma 4.1*, taking products of these groups of zeros we again obtain a version of **4.2**. Letting these products run over all zeros and taking the limit, we obtain **4.2**.

Proof of 4.3: This proof is similar, appealing to *Lemma 4.1* with $\gamma=2$ and using Hadamard's formula and grouping zeros as before.

The inequality **4.2** is implied by the usual Riemann hypothesis for $\Lambda(f,s)$. Note that the Riemann hypothesis for $\Lambda(f,s)$ implies

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also that the derviatives $\Lambda^{(j)}(f,s)$ satisfy the Riemann hypothesis. Moreover, at the central point one sees that $\Lambda^{(j)}(f,3)=0$ for all odd j, and that $\Lambda^{(j)}(f,3)\geq 0$ for all even j. Therefore, taking Taylor expansions around 3, we see that

$$\Lambda(f,5)+3\Lambda(f,3)=4\Lambda(f,3)+\sum_{j=1}^{\infty}\frac{\Lambda^{(2j)}(f,3)}{(2j)!}2^{2j}\geq 4\Lambda(f,3)+4\sum_{j=1}^{\infty}\frac{\Lambda^{(2j)}(f,3)}{(2j)!}=4\Lambda(f,4).$$

This reasoning in general explains why the period polynomial has the right sign at π (see **1.7**).

5. Weights Between 8 and 14: Applications of Results of Pólya and Szegö

Classic work of Pólya (16) and Szegö (17) considers trigonometric polynomials

$$u(\theta)=a_0+a_1\cos\theta+a_2\cos(2\theta)+\ \dots\ +a_n\cos(n\theta),$$
$$v(\theta)=a_1\sin\theta+a_2\sin(2\theta)+\ \dots\ +a_n\sin(n\theta).$$

If $0\leq a_0\leq a_1\leq a_2\dots\leq a_{n-1}<a_n$, then Szegö (17) showed that u and v both have exactly n zeros in $[0,\pi)$ and that these zeros are simple. Each interval $\left(\frac{\ell-\frac{1}{2}}{n+\frac{1}{2}}\pi,\frac{\ell+\frac{1}{2}}{n+\frac{1}{2}}\pi\right)$ for $\ell=1,\dots,n$ has precisely one zero of u, and apart from $\theta=0$, each interval $\left(\frac{\ell}{n+\frac{1}{2}}\pi,\frac{\ell+1}{n+\frac{1}{2}}\pi\right)$ for $1\leq\ell\leq n-1$ has exactly one zero of v. His proof is a simple sign change argument using the positivity of the Fejér kernel.

When the level is suitably large, these results apply and provide a quick proof of *Theorem 1.1*. For weight k, for Szegö's theorem to apply we must verify the criteria

$$\binom{2m}{m}\Lambda\left(f,\frac{k}{2}\right)\leq 2\binom{2m}{m+1}\Lambda\left(f,\frac{k}{2}+1\right),\qquad\textbf{[5.1]}$$

and for all $1\leq j\leq m-1$ that

$$\binom{2m}{m+j}\Lambda\left(f,\frac{k}{2}+j\right)\leq\binom{2m}{m+j+1}\Lambda\left(f,\frac{k}{2}+j+1\right).\qquad\textbf{[5.2]}$$

Because $\Lambda\left(f,\frac{k}{2}\right)\leq\Lambda\left(f,\frac{k}{2}+1\right)$, the condition **5.1** is immediate for all $k\geq 4$. Now suppose $k\geq 6$. Using the definition of Λ, and simplifying a little, the condition **5.2** becomes (for $1\leq j\leq m-1$)

$$\sqrt{N}\geq\frac{2\pi}{(k/2-j-1)}\frac{L\left(f,\frac{k}{2}+j\right)}{L\left(f,\frac{k}{2}+j+1\right)},$$

and by *Lemma 2.2* we conclude that our criterion (**5.2**) is met if

$$N\geq\max_{1\leq j\leq k/2-2}\left(\frac{2\pi}{k/2-j-1}\right)^2\frac{\zeta(j+1/2)^4}{\zeta(j+3/2)^4}.\qquad\textbf{[5.3]}$$

For any given k, we can compute the bound (**5.3**). Thus, for $k=8$, it suffices to take $N\geq 142$; for $k=10$ it suffices to have $N\geq 64$; for $k=12$ it suffices to have $N\geq 45$; for $k=14$ it suffices to have $N\geq 42$. We have used sage to check **5.2** for those newforms not covered by **5.3** for weights $8\leq k\leq 14$. The zeros of those newforms that do not satisfy **5.2** still lie on $|z|=1/\sqrt{N}$.

Remark: Eventually, this cannot furnish a bound better than $4\pi^2$ for N, and so we must turn to another approach for large k and small N, which we carry out in the next section.

6. Larger Weights: A Second Approach

Here we consider larger weights by reformulating the previous approach of refs. 11 and 10. Recast the definition (Eq. **1.5**) of $P_f(z)$ as

$$P_f(z)=(2m)!\left(\frac{\sqrt{N}}{2\pi}\right)^{2m+1}L(f,2m+1)Q_f(z),$$

where

$$Q_f(z)=z^m\sum_{j=0}^{m-1}\frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j\frac{L(f,2m+1-j)}{L(f,2m+1)}+\frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1}\frac{\Lambda\left(f,\frac{k}{2}\right)}{L(f,2m+1)}.\qquad\textbf{[6.1]}$$

We wish to show that on the unit circle $|z|=1$, the real and imaginary parts of $Q_f(z)$ (which correspond to the even and odd signs of the functional equation) have exactly $2m$ zeros.

Now let us write

$$Q_f(z)=z^m\exp\left(\frac{2\pi}{z\sqrt{N}}\right)+S_1(z)+S_2(z)+S_3(z),$$

with

$$S_1(z)=z^m\sum_{j=1}^{m-1}\frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j\left(\frac{L(f,2m+1-j)}{L(f,2m+1)}-1\right),$$

$$S_2(z)=-z^m\sum_{j=m}^{\infty}\frac{1}{j!}\left(\frac{2\pi}{z\sqrt{N}}\right)^j,$$

and

$$S_3(z)=\frac{1}{2(m!)^2}\left(\frac{2\pi}{\sqrt{N}}\right)^{2m+1}\frac{\Lambda\left(f,\frac{k}{2}\right)}{L(f,2m+1)}.$$

For $z=e^{i\theta}$ on the unit circle, the argument of $z^m\exp(2\pi/(z\sqrt{N}))$ is $m\theta-2\pi(\sin\theta)/\sqrt{N}$, which is monotone increasing as θ varies from 0 to 2π, and changes by $2\pi m$ overall. Therefore, the real and imaginary parts of $z^m\exp(2\pi/(z\sqrt{N}))$ both have exactly $2m$ zeros. More precisely, consider first the real part of $z^m\exp(2\pi/(z\sqrt{N}))=\cos(m\,\theta-2\pi(\sin\theta)/\sqrt{N})\exp(2\pi(\cos\theta)/\sqrt{N})$, and clearly we can find m values of θ with $\cos(m\,\theta-2\pi(\sin\theta)/\sqrt{N})=1$ and m interlacing values where it is -1. Between two such interlacing values there must be a zero of the real part. Further, because $\exp(2\pi(\cos\theta)/\sqrt{N})\geq\exp(-2\pi/\sqrt{N})$ for all θ, if

$$|S_1(z)+S_2(z)+S_3(z)|<\exp\left(-\frac{2\pi}{\sqrt{N}}\right),\qquad\textbf{[6.2]}$$

then the real part of $Q_f(z)$ will also have sign changes and thus a zero in these intervals. A similar argument applies to the imaginary part of $Q_f(z)$, and so it suffices to check the criterion **6.2**.

Now by *Lemma 2.2* we see that $L(f,2m+1-j)/L(f,2m+1)-1\leq\zeta\left(\frac{1}{2}+m-j\right)^2-1$ so that

$$|S_1(z)+S_2(z)|\leq\sum_{j=1}^{m-1}\frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j\left(\zeta\left(\frac{1}{2}+m-j\right)^2-1\right)+\sum_{j=m}^{\infty}\frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j.$$

For the term $j=m-1$, note that $\zeta\left(\frac{3}{2}\right)^2-1\leq\frac{35}{6}$ by direct computation. Note that for $2^x\left(\zeta\left(\frac{1}{2}+x\right)^2-1\right)$ is decreasing in $x\geq 2$ and so may be bounded by $4(\zeta(5/2)^2-1)\leq\frac{16}{5}$. Using this observation for smaller values of j, we obtain

$$|S_1(z)+S_2(z)|\leq\frac{16}{5}\sum_{j=1}^{m-1}\frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j\frac{2j}{2^m}+\frac{17}{4}\frac{1}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}+\sum_{j=m}^{\infty}\frac{1}{j!}\left(\frac{2\pi}{\sqrt{N}}\right)^j\frac{2j}{2^m}.$$

Combining the first and third terms, we conclude that

$$|S_1(z)+S_2(z)|\leq\frac{16}{5}2^{-m}\left(\exp\left(\frac{4\pi}{\sqrt{N}}\right)-1\right)+\frac{17}{4}\frac{1}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}. \qquad [6.3]$$

To bound $S_3(z)$, note that $\Lambda(f,\frac{k}{2})\leq\Lambda(f,\frac{k}{2}+1)\leq\left(\frac{\sqrt{N}}{2\pi}\right)^{m+2}(m+1)!\zeta\left(\frac{3}{2}\right)^2$, so for $m\geq7$ we have

$$|S_3(z)|\leq\frac{m+1}{2(m!)}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}\zeta\left(\frac{3}{2}\right)^2\leq\frac{4}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}.$$

Combining this with **6.3**, we conclude that

$$|S_1(z)|+|S_2(z)|+|S_3(z)|\leq\frac{16}{5}\frac{1}{2^m}\left(\exp\left(\frac{4\pi}{\sqrt{N}}\right)-1\right)+\frac{33}{4}\frac{1}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}. \qquad [6.4]$$

Thus, to verify the condition **6.2**, we need only ensure that

$$\frac{16}{5}\frac{1}{2^m}\left(\exp\left(\frac{4\pi}{\sqrt{N}}\right)-1\right)+\frac{33}{4}\frac{1}{(m-1)!}\left(\frac{2\pi}{\sqrt{N}}\right)^{m-1}<\exp\left(-\frac{2\pi}{\sqrt{N}}\right). \qquad [6.5]$$

For values of m at least as large as the figure in the first row, the table below gives a bound $N(m)$ such that estimate **6.5** holds for all $N\geq N(m)$:

m	29	21	18	16	14	13	12	11	10	9	8	7
$N(m)$	1	2	3	4	5	6	7	9	11	14	20	28

We used sage to confirm *Theorem 1.1* for the finitely many newforms missed by **6.5**.

7. Proof of *Theorem 1.2*

The weight 4 case was already treated in section 3. For $m\geq2$ (that is, weights $k\geq6$), the argument in section 6 shows that for $z=e^{i\theta}$ on the unit circle we have

$$Q_f(z)=\exp\left(im\ \theta+\frac{2\pi}{\sqrt{N}}e^{-i\theta}\right)+O\left(\frac{1}{2^m\sqrt{N}}\right).$$

Thus, we have that

$$\mathrm{Re}\big(Q_f(z)\big)=\exp\left(\frac{2\pi}{\sqrt{N}}\cos\ \theta\right)\cos\left(m\ \theta-\frac{2\pi}{\sqrt{N}}\sin\ \theta\right)+O\left(\frac{1}{2^m\sqrt{N}}\right).$$

For $\theta\in[0,2\pi)$ the first term above vanishes when $m\theta-2\pi(\sin\theta)/\sqrt{N}=\frac{\pi}{2}+\ell\pi$ with $0\leq\ell\leq2m-1$. For such a point θ_ℓ, if we consider the values at $\theta_\ell-C/(2^m\sqrt{N})$ and $\theta_\ell+C/(2^m\sqrt{N})$ for a suitable constant $C>0$ (and if $2^m\sqrt{N}$ is large enough) then $\mathrm{Re}(Q_f(z))$ has differing signs at these points, and hence a zero in between. When $\epsilon(f)=1$, the zeros of the period polynomial $r_f(z)$ are located at $1/(i\sqrt{N})$ times the zeros of $\mathrm{Re}(Q_f(z))$, and this proves *Theorem 1.2* in this case. The case when $\epsilon(f)=-1$ corresponds to $\mathrm{Im}(Q_f(z))$, and a similar argument applies here.

8. Remarks on the Calculations

In the previous sections we proved *Theorem 1.1* for $k=4,6$ and $k\geq42$. For $8\leq k\leq40$ finitely many newforms remain to complete the proof (see the discussions after **5.3** and **6.5**). We used inequality (**5.3**) for $8\leq k\leq14$. The most levels remain for weight $k=8$; we are left to consider those newforms with $N\leq141$. For weights $16\leq k\leq40$ we used **6.5**. The table after **6.5** gives the remaining levels. The most levels remain for weight $k=16$; we are left with $N\leq27$.

Using sage we confirmed *Theorem 1.1* for these remaining newforms. Running the commands CuspForms and newforms on a laptop, we had no difficulty computing these newforms. We then used Dokchitser's sage L-functions calculator to compute the values $\Lambda(f,1),\ldots,\Lambda(f,k-1)$ to very high precision. We tested inequality **5.2** and found that it held for many of the remaining newforms. However, **5.2** fails for some newforms with low weight and level. For example, **5.2** fails for some weight $k=8$ newforms with $N\in\{2,3,5-17,19\}$.

For the forms that do not satisfy (5.2), we computed the trigonometric polynomials and checked that on the unit disk that they have the required number of sign changes for the truth of *Theorem 1.1*. As an example, consider the unique newform $f\in S_{10}(\Gamma_0(12))$. We have that

$$L(f,1)\approx343.041936898889, L(f,2)\approx140.422365373567,$$
$$L(f,3)\approx32.9164131544840, L(f,4)\approx6.41626479306637,$$
$$L(f,5)\approx1.71889934464323,\ldots,$$

which in turn implies for $z=e^{i\theta}$ that

$$\big(P_f(z)+\epsilon(f)P_f(1/z)\big)/2\approx189.128932153817\cos(4\theta)+341.466246468159\cos(3\theta)+308.910589184567\cos(2\theta)+199.188643773093\cos(\theta)+73.5501402820398.$$

This has four zeros for $\theta\in[0,\pi)$ as required, and they are in the intervals

$$\left(\frac{4\pi}{20},\frac{5\pi}{20}\right),\left(\frac{10\pi}{20},\frac{11\pi}{20}\right),\left(\frac{14\pi}{20},\frac{15\pi}{20}\right),\left(\frac{18\pi}{20},\frac{19\pi}{20}\right).$$

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