

Positive Definiteness of Paired Symmetric Tensors and Elasticity Tensors

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Abstract

In this paper, we consider higher order paired symmetric tensors and strongly paired symmetric tensors. Elasticity tensors and higher order elasticity tensors in solid mechanics are strongly paired symmetric tensors. A (strongly) paired symmetric tensor is said to be positive definite if the homogeneous polynomial defined by it is positive definite. Positive definiteness of elasticity and higher order elasticity tensors is strong ellipticity in solid mechanics, which plays an important role in nonlinear elasticity theory. We mainly investigate positive definiteness of fourth order three dimensional and sixth order three dimensional (strongly) paired symmetric tensors. We first show that the concerned (strongly) paired symmetric tensor is positive definite if and only if its smallest M -eigenvalue is positive. Second, we propose several necessary and sufficient conditions under which the concerned (strongly) paired symmetric tensor is positive definite. Third, we study the conditions under which the homogeneous polynomial defined by a fourth order three dimensional or sixth order three dimensional (strongly) paired symmetric tensor can be written as a sum of squares of polynomials, and further, propose several necessary and/or sufficient conditions to judge whether the concerned (strongly) paired symmetric tensors are positive definite or not. Fourth, by using semidefinite relaxation we propose a sequential semidefinite programming method to compute the smallest M -eigenvalue of a fourth order three dimensional (strongly) paired symmetric tensor, by which we can check positive definiteness of the concerned tensor. The preliminary numerical results confirm our theoretical findings.

Key words. Paired symmetric tensor, elasticity tensor, positive definiteness of tensor, M -eigenvalue, semidefinite relaxation.

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1 Introduction

For any positive integers m and n , an m -th order n dimensional real tensor can be denoted by

$$\mathcal{A} := (a_{i_1 i_2 \dots i_m}), \text{ where } a_{i_1 i_2 \dots i_m} \in \mathbb{R} \text{ for all } i_j \in [n] \text{ with } j \in [m],$$

here $[l] := \{1, 2, \dots, l\}$ for any positive integer l . We use $\mathbb{T}_{m,n}$ to denote the set of all m -th order n dimensional real tensors. For any $\mathcal{A} = (a_{i_1 i_2 \dots i_m}), \mathcal{B} = (b_{i_1 i_2 \dots i_m}) \in \mathbb{T}_{m,n}$, we will use the inner product defined by

$$\langle \mathcal{A}, \mathcal{B} \rangle = \sum_{i_1, \dots, i_m=1}^n a_{i_1 i_2 \dots i_m} b_{i_1 i_2 \dots i_m}$$

and the Hilbert-Schmidt norm defined by

$$\|\mathcal{A}\|_{HS} = \sqrt{\langle \mathcal{A}, \mathcal{A} \rangle} = \left(\sum_{i_1, \dots, i_m=1}^n a_{i_1 i_2 \dots i_m}^2 \right)^{1/2}.$$

A tensor $\mathcal{A} \in \mathbb{T}_{m,n}$ is said to be symmetric if its entries are invariant under any permutation of indices $\{i_1, i_2, \dots, i_m\}$. For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, a tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in \mathbb{T}_{m,n}$ defines a homogeneous polynomial by

$$p(x) = \sum_{i_1, \dots, i_m=1}^n a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m}. \quad (1)$$

We denote the degree of a polynomial p by $\deg(p)$. The polynomial p defined by (1) is said to be positive semidefinite if $p(x) \geq 0$ holds for all $x \in \mathbb{R}^n$; and p is said to be positive definite if $p(x) > 0$ holds for all $x \in \mathbb{R}^n \setminus \{0\}$. Obviously, for nonzero tensors, m being an even integer is a necessity for positive semidefiniteness. A tensor is said to be positive (semidefinite) definite if its corresponding homogeneous polynomial is positive (semidefinite) definite. Positive (semidefiniteness) definiteness of the polynomial is very important in many areas, which is related to the Hilbert seventeenth problem [24].

For an arbitrary $2m$ -th order n dimensional tensor denoted by $\mathcal{A} = (a_{i_1 j_1 i_2 j_2 \dots i_m j_m})$, its indices can be divided into m adjacent blocks $\{i_1 j_1\}, \dots, \{i_m j_m\}$. If entries of \mathcal{A} are invariant under any permutation of indices in every block $\{i_l j_l\}$ for $l \in [m]$, i.e.,

$$a_{i_1 j_1 i_2 j_2 \dots i_m j_m} = a_{j_1 i_1 i_2 j_2 \dots i_m j_m} = a_{i_1 j_1 j_2 i_2 \dots i_m j_m} = \dots = a_{i_1 j_1 i_2 j_2 \dots j_m i_m},$$

then \mathcal{A} is called a *2m-th order n dimensional paired symmetric tensor*. It is well known that the most important representatives for fourth order three dimensional paired symmetric tensors are: the piezooptical tensor, the second order electrooptical effect (Kerr effect), electrostriction and second order magnetostriction [5]. Furthermore, if a paired symmetric tensor additionally satisfies block symmetry, i.e., if the tensor further satisfies that the entries are invariant under any permutation of every block, then \mathcal{A} is called a *2m-th order n dimensional strongly paired symmetric tensor*. The most significant representative of fourth order three dimensional strongly paired symmetric tensors is the elasticity tensor, in which the pairwise permutability is based

on the reversibility of mechanical deformation work [5]. In the elasticity tensor and higher order elasticity tensor [6, 9, 29], every entry $a_{i_1 j_1 i_2 j_2 \dots i_m j_m}$ is called an m -th order elastic constant, which is an important quantity in studies of elasticity theory. Positive definiteness of the elasticity tensor is called *strong ellipticity*, which plays an important role in elasticity theory and has been studied extensively (see, for example, [2, 4, 10, 11, 15, 20, 25, 26, 27, 28, 31, 33]).

In this paper, we consider positive definiteness of higher order three dimensional (strongly) paired symmetric tensors. For simplicity of symbols, we only investigate some properties of fourth order three dimensional and sixth order three dimensional (strongly) paired symmetric tensors, mainly in positive definiteness of the concerned tensors. The results we obtained can be similarly extended to the case of more higher order (strongly) paired symmetric tensors. Some basic properties of fourth order three dimensional and sixth order three dimensional (strongly) paired symmetric tensors are given in the next section.

Eigenvalues of higher order tensors, introduced by Qi [18] and Lim [13], have been studied extensively in the recent years [1, 7, 19, 21]. The concept of M -eigenvalue for fourth order paired symmetric tensor was introduced in [4, 20] and further studied in [32]. In Section 3, we extend the concept of M -eigenvalue to sixth order three dimensional (strongly) paired symmetric tensors and bi-block symmetric tensors, and further discuss some related properties. In particular, we show that a sixth order three dimensional (strongly) paired symmetric tensor is positive definite if and only if its smallest M -eigenvalue is positive.

Positive definiteness of polynomials has been being an important issue in many areas, which has been discussed extensively. In [4, 17, 20], the authors studied positive definiteness conditions of fourth order paired symmetric tensors, which plays an important role in elasticity theory. In Section 4, following the ideas given in [4, 20], we further discuss positive definiteness of fourth order three dimensional (strongly) paired symmetric tensors and propose several necessary and sufficient conditions for which the concerned tensor is positive definite. Furthermore, we extend the related results to the case of sixth order three dimensional (strongly) paired symmetric tensor.

A polynomial with real coefficients is called a sum of squares (SOS) if it can be expressed as a sum of several squares of polynomials with real coefficients [12, 16]. It is obvious that a polynomial is positive semidefinite if it is an SOS polynomial. In Section 5, we investigate the SOS properties of polynomials defined by a fourth order three dimensional or a sixth order three dimensional (strongly) paired symmetric tensor. We give several necessary and/or sufficient conditions of a fourth order three dimensional (strongly) paired symmetric tensor being an SOS tensor, and propose several conditions under which a fourth order three dimensional (strongly) paired symmetric tensor is positive definite. In particular, we extend the related results to the case of sixth order three dimensional (strongly) paired symmetric tensor.

In [8], the authors introduced the tensor conic linear programming problem and proposed a sequential semidefinite programming method to solve it. As an application, they showed that their method can be applied to find the smallest Z -eigenvalue of a symmetric tensor. In Section 6, by using the special structure of the (strongly) paired symmetric tensor, we propose a sequential semidefinite programming method to compute the smallest M -eigenvalue of a fourth order three

dimensional (strongly) paired symmetric tensor, which is an extension of the method proposed in [8]. By this method, we can check whether a fourth order three dimensional (strongly) paired symmetric tensor is positive definite or not.

In Section 7, we give some numerical results of our methods for judging whether a fourth order three dimensional or sixth order three dimensional (strongly) paired symmetric tensor is positive definite or not. The preliminary numerical results are consistent with our theoretical results.

Some concluding remarks are made in Section 8.

In the remaining parts of our paper, we will simply call a three dimensional strongly paired symmetric tensor an elasticity tensor as the main motivation of our paper is the strong ellipticity of elasticity and higher order elasticity tensors.

2 Preliminaries

In this section, we consider fourth order three dimensional and sixth order three dimensional paired symmetric (elasticity) tensor and discuss related basic properties.

2.1 Fourth order paired symmetric tensor. For any $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$, if

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{jilk}, \quad \forall i, j, k, l \in \{1, 2, 3\},$$

then \mathcal{A} is a *paired symmetric tensor* $\mathcal{A} \in \mathbb{T}_{4,3}$; and if

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{jilk} \quad \text{and} \quad a_{ijkl} = a_{klij}, \quad \forall i, j, k, l \in \{1, 2, 3\},$$

then \mathcal{A} is a *fourth order three dimensional elasticity tensor* [5].

For any tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$, the corresponding biquadratic form is defined by

$$\mathcal{A}x^2y^2 := \sum_{i,j,k,l=1}^3 a_{ijkl}x_ix_jy_ky_l, \quad \forall x, y \in \mathbb{R}^3. \quad (2)$$

Define

$$\mathcal{C} := \{\mathcal{A} \in \mathbb{T}_{4,3} : \mathcal{A}x^2y^2 \geq 0, \forall x, y \in \mathbb{R}^3\}. \quad (3)$$

Then, by a similar way as those in [22, 23], we can obtain the following results.

Proposition 2.1. *Suppose that the polynomial $\mathcal{A}x^2y^2$ and the set \mathcal{C} are defined by (2) and (3), respectively. Then, the following statements hold.*

(i) *The interior of \mathcal{C} , denoted by $\text{int}\mathcal{C}$, is nonempty, and*

$$\text{int}\mathcal{C} = \{\mathcal{A} \in \mathbb{T}_{4,3} : \mathcal{A}x^2y^2 > 0, \forall x, y \in \mathbb{R}^3 \setminus \{0\}\}. \quad (4)$$

(ii) *The set \mathcal{C} is a pointed closed convex cone.*

Proof. (i) It is obvious that \mathcal{C} has nonempty interior. We show that (4) holds. On the one hand, if $\mathcal{A} \in \mathcal{C}$ is not positive definite, then there exists two nonzero vectors $x, y \in \mathbb{R}^3$ such that $\mathcal{A}x^2y^2 = 0$; and hence, for any $\epsilon > 0$,

$$(\mathcal{A} - \epsilon\mathcal{E})x^2y^2 = -\epsilon(x^\top x)(y^\top y) < 0$$

where $\mathcal{E} = (e_{ijkl}) \in \mathbb{T}_{4,3}$ is defined by

$$e_{ijkl} := \begin{cases} 1 & \text{if } i = j, k = l, \\ 0 & \text{otherwise,} \end{cases} \quad \forall i, j, k, l \in \{1, 2, 3\}. \quad (5)$$

This implies that $\mathcal{A} \notin \text{int}\mathcal{C}$; and hence, $\text{int}\mathcal{C} \subseteq \{\mathcal{A} \in \mathbb{T}_{4,3} : \mathcal{A}x^2y^2 > 0, \forall x, y \in \mathbb{R}^3 \setminus \{0\}\}$. On the other hand, if $\mathcal{A} \notin \text{int}\mathcal{C}$, then there exists a sequence $\{(\mathcal{B}^{(k)}, \epsilon_{(k)})\}$ satisfying $\|\mathcal{B}^{(k)}\|_{HS} = 1$ and $\epsilon_{(k)} > 0$ for all $k \in \{1, 2, \dots\}$ such that

$$\mathcal{A} + \epsilon_{(k)}\mathcal{B}^{(k)} \notin \mathcal{C} \text{ for all } k \in \{1, 2, \dots\} \quad \text{and} \quad \lim_{k \rightarrow \infty} \epsilon_{(k)} = 0,$$

which leads to that there exist $\{x^{(k)}\}, \{y^{(k)}\} \subseteq \mathbb{R}^3$ satisfying $\|x^{(k)}\| = 1$ and $\|y^{(k)}\| = 1$ for all $k \in \{1, 2, \dots\}$ such that

$$(\mathcal{A} + \epsilon_{(k)}\mathcal{B}^{(k)}) \left(x^{(k)}\right)^2 \left(y^{(k)}\right)^2 \leq 0.$$

Let x^*, y^* be the limiting points of $\{x^{(k)}\}$ and $\{y^{(k)}\}$, respectively. Then, $\|x^*\| = 1, \|y^*\| = 1$ and $\mathcal{A}(x^*)^2(y^*)^2 \leq 0$, which implies that \mathcal{A} is not positive definite; and hence, $\{\mathcal{A} \in \mathbb{T}_{4,3} : \mathcal{A}x^2y^2 > 0, \forall x, y \in \mathbb{R}^3 \setminus \{0\}\} \subseteq \text{int}\mathcal{C}$. So, (4) holds.

(ii) For any $\mathcal{A}, \mathcal{B} \in \mathcal{C}$ and $\alpha, \beta \geq 0$, let $\mathcal{C} := \alpha\mathcal{A} + \beta\mathcal{B}$. Then, for any $x, y \in \mathbb{R}^3$,

$$\mathcal{C}x^2y^2 = (\alpha\mathcal{A} + \beta\mathcal{B})x^2y^2 = \alpha\mathcal{A}x^2y^2 + \beta\mathcal{B}x^2y^2 \geq 0,$$

which implies that \mathcal{C} is a convex cone.

For any $\mathcal{A} = (a_{ijkl}) \in \mathcal{C}$, if $-\mathcal{A} \in \mathcal{C}$, then it follows from (2) that

$$\sum_{i,j,k,l=1}^3 a_{ijkl}x_ix_jy_ky_l \equiv 0, \forall x, y \in \mathbb{R}^3,$$

which yields that $\mathcal{A} = 0$ by arbitrariness of x and y . This implies that \mathcal{C} is a pointed cone.

For any $\{\mathcal{A}^{(k)}\} \subseteq \mathcal{C}$ and $\lim_{k \rightarrow \infty} \mathcal{A}^{(k)} = \mathcal{A}$, it is easy to see that for any $x, y \in \mathbb{R}^n$,

$$\mathcal{A}x^2y^2 = \lim_{k \rightarrow \infty} \mathcal{A}^{(k)}x^2y^2 \geq 0,$$

which implies that \mathcal{C} is a closed cone.

Therefore, by combining (i) with (ii), we conclude that the results of proposition hold. \square

In the following, we define a class of matrices which is related to tensor $\mathcal{A} \in \mathbb{T}_{4,3}$.

Definition 2.2. For any $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$, we define a matrix by

$$M = (m_{st}) \text{ with } m_{st} = a_{is_{it}js_{jt}} \text{ for all } s, t \in \{1, 2, \dots, 9\} \quad (6)$$

where $i_1 i_2 \cdots i_9$ and $j_1 j_2 \cdots j_9$ are two arbitrary permutations of 123123123 which satisfy that, for any $r \in \{1, 2, 3\}$, at least one of $i_{3(r-1)+1} i_{3(r-1)+2} i_{3(r-1)+3}$ and $j_{3(r-1)+1} j_{3(r-1)+2} j_{3(r-1)+3}$ is an arbitrary permutation of 123. We say that the matrix M defined by (6) is an **unfolded matrix** of tensor \mathcal{A} with respect to indices $i_1 i_2 \cdots i_9$ and $j_1 j_2 \cdots j_9$.

The following result is easy to be obtained.

Proposition 2.3. Suppose that $\mathcal{A} \in \mathbb{T}_{4,3}$ is a paired symmetric tensor, and M is its unfolded matrix defined by Definition 2.2. Then, the matrix M is symmetric.

For any $\mathcal{A} \in \mathbb{T}_{4,3}$, we define

- matrix $M^1 = (m_{st}) \in \mathbb{R}^{9 \times 9}$ by

$$M^1 := (m_{st}^1) \quad \text{with } m_{st}^1 = m_{3(i-1)+k, 3(j-1)+l}^1 = a_{ijkl} \text{ for any } i, j, k, l \in \{1, 2, 3\}; \quad (7)$$

- and matrix $M^2 = (m_{st}) \in \mathbb{R}^{9 \times 9}$ by

$$M^2 := (m_{st}^2) \quad \text{with } m_{st}^2 = m_{3(k-1)+i, 3(l-1)+j}^2 = a_{ijkl} \text{ for any } i, j, k, l \in \{1, 2, 3\}. \quad (8)$$

Then, M^1 and M^2 are two unfolded matrices of \mathcal{A} .

Moreover, for any $i, j, k, l \in \{1, 2, 3\}$, we define two block sub-matrices of tensor \mathcal{A} by

$$A_{ij} := (a_{ijkl})_{kl} \quad \text{and} \quad B_{kl} := (a_{ijkl})_{ij}. \quad (9)$$

Then, we can easily obtain the following results.

Proposition 2.4. For any $\mathcal{A} \in \mathbb{T}_{4,3}$, suppose that matrices M^1 , M^2 , A_{ij} and B_{kl} are given by (7), (8) and (9), respectively.

(i) If $\mathcal{A} \in \mathbb{T}_{4,3}$ is a paired symmetric tensor, then the following statements hold.

(a) The matrices M^1 and M^2 are symmetric, and

$$M^1 = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad \text{and} \quad M^2 = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix}; \quad (10)$$

(b) All sub-matrices A_{ij} and B_{kl} are symmetric.

(ii) If $\mathcal{A} \in \mathbb{T}_{4,3}$ is an elasticity tensor, then $M^1 = M^2$.

2.2 Sixth order paired symmetric tensor. $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$ is a paired symmetric tensor if its entries satisfy

$$a_{ijklpq} = a_{jiklpq} = a_{ijlkpq} = a_{ijklqp}, \quad \forall i, j, k, l, p, q \in \{1, 2, 3\}. \quad (11)$$

Furthermore, a paired symmetric tensor $\mathcal{A} \in \mathbb{T}_{6,3}$ is a *sixth order elasticity tensor* [5] if

$$a_{ijklpq} = a_{klijpq} = a_{ijpqkl}, \quad \forall i, j, k, l, p, q \in \{1, 2, 3\}. \quad (12)$$

For any tensor $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$, the corresponding homogeneous polynomial is defined by

$$\mathcal{A}x^2y^2z^2 := \sum_{i,j,k,l,p,q=1}^3 a_{ijklpq}x_ix_jy_ky_ly_lz_pz_q, \quad \forall x, y, z \in \mathbb{R}^3. \quad (13)$$

Then, similar to Proposition 2.1, we have the following results.

Proposition 2.5. *Suppose that $\mathcal{A}x^2y^2z^2$ is defined by (13). Then, the following statements hold.*

- (i) *The interior of set \mathcal{D} defined by $\mathcal{D} := \{\mathcal{A} \in \mathbb{T}_{6,3} : \mathcal{A}x^2y^2z^2 \geq 0, \forall x, y, z \in \mathbb{R}^3\}$ is nonempty, and $\text{int}\mathcal{D} = \{\mathcal{A} \in \mathbb{T}_{6,3} : \mathcal{A}x^2y^2z^2 > 0, \forall x, y, z \in \mathbb{R}^3 \setminus \{0\}\}$.*
- (ii) *The set \mathcal{D} is a pointed closed convex cone.*

Similar to Definition 2.2, we define the unfolded matrix of sixth order three dimensional tensor as follows.

Definition 2.6. *For any $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$, we define a matrix by*

$$N = (n_{st}) \text{ with } n_{st} = a_{i_s i_t j_s j_t k_s k_t} \text{ for all } s, t \in \{1, 2, \dots, 27\} \quad (14)$$

where $i_1 i_2 \dots i_{27}$, $j_1 j_2 \dots j_{27}$ and $k_1 k_2 \dots k_{27}$ are three arbitrary permutations of $\underbrace{123123 \cdot 123}_{27}$ which satisfy that, for any $r \in \{1, 2, \dots, 9\}$, at least one of $i_{3(r-1)+1} i_{3(r-1)+2} i_{3(r-1)+3}$, $j_{3(r-1)+1} j_{3(r-1)+2} j_{3(r-1)+3}$ and $k_{3(r-1)+1} k_{3(r-1)+2} k_{3(r-1)+3}$ is an arbitrary permutation of 123. We say that the matrix N defined by (14) is an **unfolded matrix** of tensor \mathcal{A} with respect to indices $i_1 i_2 \dots i_{27}$, $j_1 j_2 \dots j_{27}$ and $k_1 k_2 \dots k_{27}$.

Obviously, the following result holds.

Proposition 2.7. *Suppose that $\mathcal{A} \in \mathbb{T}_{6,3}$ is a paired symmetric tensor, and N is its unfolded matrix defined by Definition 2.6. Then, the matrix N is symmetric.*

In the following, we give several specific examples of the unfolded matrix. That is, for any $\mathcal{A} \in \mathbb{T}_{6,3}$, we define

$$\begin{aligned} N^1 &:= (n_{st}) \in \mathbb{R}^{27 \times 27} \text{ with} \\ &\quad n_{st} = n_{3[3(i-1)+(k-1)]+p, 3[3(j-1)+(l-1)]+q} = a_{ijklpq}, \quad \forall i, j, k, l, p, q \in \{1, 2, 3\}; \\ N^2 &:= (n_{st}) \in \mathbb{R}^{27 \times 27} \text{ with} \\ &\quad n_{st} = n_{3[3(k-1)+(i-1)]+p, 3[3(l-1)+(j-1)]+q} = a_{ijklpq}, \quad \forall i, j, k, l, p, q \in \{1, 2, 3\}; \\ N^3 &:= (n_{st}) \in \mathbb{R}^{27 \times 27} \text{ with} \\ &\quad n_{st} = n_{3[3(p-1)+(k-1)]+i, 3[3(q-1)+(l-1)]+j} = a_{ijklpq}, \quad \forall i, j, k, l, p, q \in \{1, 2, 3\}; \\ N^4 &:= (n_{st}) \in \mathbb{R}^{27 \times 27} \text{ with} \\ &\quad n_{st} = n_{3[3(i-1)+(p-1)]+k, 3[3(j-1)+(q-1)]+l} = a_{ijklpq}, \quad \forall i, j, k, l, p, q \in \{1, 2, 3\}; \\ N^5 &:= (n_{st}) \in \mathbb{R}^{27 \times 27} \text{ with} \\ &\quad n_{st} = n_{3[3(k-1)+(p-1)]+i, 3[3(l-1)+(q-1)]+j} = a_{ijklpq}, \quad \forall i, j, k, l, p, q \in \{1, 2, 3\}; \\ N^6 &:= (n_{st}) \in \mathbb{R}^{27 \times 27} \text{ with} \\ &\quad n_{st} = f_{3[3(p-1)+(i-1)]+k, 3[3(q-1)+(j-1)]+l} = a_{ijklpq}, \quad \forall i, j, k, l, p, q \in \{1, 2, 3\}. \end{aligned} \quad (15)$$

Then, we have the following results.

Proposition 2.8. *For any paired symmetric tensor $\mathcal{A} \in \mathbb{T}_{6,3}$, we have the following results.*

- All matrices N^1, N^2, \dots, N^6 defined by (15) are symmetric.
- If \mathcal{A} is an elasticity tensor, then $N^1 = N^2 = \dots = N^6$.

For any tensor $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$ and any $i, j, k, l, p, q \in \{1, 2, 3\}$, we define three block sub-tensors of tensor \mathcal{A} by

$$\mathcal{A}_{ij} := (a_{ijklpq})_{klpq}, \quad \mathcal{B}_{kl} := (a_{ijklpq})_{ijpq} \quad \text{and} \quad \mathcal{C}_{pq} := (a_{ijklpq})_{ijkl}. \quad (16)$$

Then, we have the following results.

Proposition 2.9. *For any $\mathcal{A} \in \mathbb{T}_{6,3}$, let sub-tensors \mathcal{A}_{ij} , \mathcal{B}_{kl} and \mathcal{C}_{pq} be defined by (16).*

- If \mathcal{A} is a paired symmetric tensor, then all sub-tensors \mathcal{A}_{ij} , \mathcal{B}_{kl} and \mathcal{C}_{pq} are paired symmetric tensors.
- If \mathcal{A} is an elasticity tensor (i.e., its entries satisfy (11) and (12)), then $\mathcal{A}_{st} = \mathcal{B}_{st} = \mathcal{C}_{st}$ for all $s, t \in \{1, 2, 3\}$.

3 M -Eigenvalue and Properties

In this section, we extend the concept of M -eigenvalues for fourth order paired symmetric tensor introduced in [4, 20] to sixth order three dimensional paired symmetric (elasticity) tensor and bi-block symmetric tensor, and discuss some related properties.

For any paired symmetric tensor $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$, the corresponding homogeneous polynomial is given in (13). For any $x, y, z \in \mathbb{R}^3$, we let $\mathcal{A}xy^2z^2, \mathcal{A}x^2yz^2, \mathcal{A}x^2y^2z \in \mathbb{R}^3$ be defined by

$$\begin{aligned} (\mathcal{A}xy^2z^2)_i &:= \sum_{j,k,l,p,q=1}^3 a_{ijklpq} x_j y_k y_l z_p z_q, \quad \forall i \in \{1, 2, 3\}, \\ (\mathcal{A}x^2yz^2)_k &:= \sum_{i,j,l,p,q=1}^3 a_{ijklpq} x_i x_j y_l z_p z_q, \quad \forall k \in \{1, 2, 3\}, \\ (\mathcal{A}x^2y^2z)_p &:= \sum_{i,j,k,l,q=1}^3 a_{ijklpq} x_i x_j y_k y_l z_q, \quad \forall p \in \{1, 2, 3\}. \end{aligned}$$

Then, it is easy to see that

$$\langle x, \mathcal{A}xy^2z^2 \rangle = \mathcal{A}x^2y^2z^2, \quad \langle y, \mathcal{A}x^2yz^2 \rangle = \mathcal{A}x^2y^2z^2, \quad \langle z, \mathcal{A}x^2y^2z \rangle = \mathcal{A}x^2y^2z^2 \quad (17)$$

hold for any $x, y, z \in \mathbb{R}^3$.

Definition 3.1. For any paired symmetric tensor $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$, if there exist $\lambda \in \mathbb{R}$ and $x, y, z \in \mathbb{R}^3$ such that

$$\begin{cases} \mathcal{A}xy^2z^2 = \lambda x, & \mathcal{A}x^2yz^2 = \lambda y, & \mathcal{A}x^2y^2z = \lambda z, \\ x^\top x = 1, & y^\top y = 1, & z^\top z = 1, \end{cases}$$

then λ is called an M -eigenvalue of \mathcal{A} and x, y, z are the eigenvectors of \mathcal{A} associated with the M -eigenvalue λ .

Theorem 3.2. For any paired symmetric (elasticity) tensor $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$, its M -eigenvalues always exist. Moreover, if x, y, z are the eigenvectors of \mathcal{A} associated with the M -eigenvalue λ , then $\lambda = \mathcal{A}x^2y^2z^2$.

Proof. Let $\mathcal{A}x^2y^2z^2$ be defined by (13). We consider the following optimization problem:

$$\begin{cases} \min & \mathcal{A}x^2y^2z^2 \\ \text{s.t.} & x^\top x = 1, \quad y^\top y = 1, \quad z^\top z = 1. \end{cases} \quad (18)$$

It is easy to see that the feasible set of (18) is compact and the objective function of (18) is continuous. Thus, the optimization problem (18) has at least a minimizer, say (x^*, y^*, z^*) , which satisfies the first order optimality condition of (18), i.e., there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{cases} \mathcal{A}x^*(y^*)^2(z^*)^2 = \alpha x^*, & \mathcal{A}(x^*)^2y^*(z^*)^2 = \beta y^*, & \mathcal{A}(x^*)^2(y^*)^2z^* = \gamma z^*, \\ (x^*)^\top x^* = 1, & (y^*)^\top y^* = 1, & (z^*)^\top z^* = 1. \end{cases}$$

This, together with (17), implies that

$$\alpha = \beta = \gamma = \mathcal{A}(x^*)^2(y^*)^2(z^*)^2.$$

Thus, α is an M -eigenvalue of \mathcal{A} and x^*, y^*, z^* are the eigenvectors of \mathcal{A} associated with the M -eigenvalue α . We complete the proof. \square

By Theorem 3.2, we have the following result.

Theorem 3.3. A paired symmetric (elasticity) tensor $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$ is positive definite if and only if the smallest M -eigenvalue of \mathcal{A} is positive.

This theorem demonstrates that positive definiteness detection of a paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{6,3}$ can be done by computing the smallest M -eigenvalue of \mathcal{A} . Moreover, from Definition 3.1 and Theorem 3.3, it is easy to obtain the following results.

Theorem 3.4. For any paired symmetric (elasticity) tensor $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$, it follows that λ is an M -eigenvalue of \mathcal{A} if and only if $-\lambda$ is an M -eigenvalue of $-\mathcal{A}$; and furthermore, \mathcal{A} is positive definite if and only if the largest M -eigenvalue of $-\mathcal{A}$ is negative.

From the point of view of numerical calculation, Theorem 3.4 is useful since positive definiteness detection of a paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{6,3}$ can be done by computing the largest M -eigenvalue of $-\mathcal{A}$.

For any $(\lambda, x, y, z) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ and tensor $\mathcal{A} \in \mathbb{T}_{6,3}$, $\lambda x^2 y^2 z^2$ is a rank-one sixth order paired symmetric tensor with its entries being $\lambda x_i x_j y_k y_l z_p z_q$ for all $i, j, k, l, p, q \in \{1, 2, 3\}$. We say that $\lambda_*(x^*)^2(y^*)^2(z^*)^2$ is the best rank-one approximation of \mathcal{A} if $(\lambda_*, x^*, y^*, z^*) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ solves the optimization problem:

$$\begin{aligned} \min \quad & \|\mathcal{A} - \lambda x^2 y^2 z^2\|_{HS}^2 \\ \text{s.t.} \quad & \lambda \in \mathbb{R} \quad \text{and} \quad x^\top x = 1, y^\top y = 1, z^\top z = 1, \forall x, y, z \in \mathbb{R}^3. \end{aligned} \quad (19)$$

The best rank-one approximation has wide applications in signal and image processing, wireless communication systems, independent component analysis, and so on.

Theorem 3.5. *For any paired symmetric (elasticity) tensor $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$, if λ_* is an M -eigenvalue of \mathcal{A} with the largest absolute value among all M -eigenvalues of \mathcal{A} , and $x^*, y^*, z^* \in \mathbb{R}^3$ are the eigenvectors of \mathcal{A} associated with the M -eigenvalue λ_* , then $\lambda_*(x^*)^2(y^*)^2(z^*)^2$ is the best rank-one approximation of \mathcal{A} .*

Proof. We denote the feasible set of (19) by Ω , i.e.,

$$\Omega := \{(\lambda, x, y, z) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 : x^\top x = 1, y^\top y = 1, z^\top z = 1\}.$$

On the one hand, since

$$\begin{aligned} \min \{ & \|\mathcal{A} - \lambda x^2 y^2 z^2\|_{HS}^2 : (\lambda, x, y, z) \in \Omega \} \\ &= \min \{ \|\mathcal{A}\|_{HS}^2 - 2\lambda \mathcal{A} x^2 y^2 z^2 + \lambda^2 (x^\top x)(y^\top y)(z^\top z) : (\lambda, x, y, z) \in \Omega \} \\ &= \min \{ \|\mathcal{A}\|_{HS}^2 - 2\lambda \mathcal{A} x^2 y^2 z^2 + \lambda^2 : (\lambda, x, y, z) \in \Omega \} \end{aligned}$$

and when $\lambda = \mathcal{A} x x y y z z$,

$$\begin{aligned} \min \{ \|\mathcal{A} - \lambda x^2 y^2 z^2\|_{HS}^2 : (\lambda, x, y, z) \in \Omega \} &= \min \{ \|\mathcal{A}\|_{HS}^2 - (\lambda \mathcal{A} x^2 y^2 z^2)^2 : (\lambda, x, y, z) \in \Omega \} \\ &= \|\mathcal{A}\|_{HS}^2 - \max \{ (\mathcal{A} x^2 y^2 z^2)^2 : (\lambda, x, y, z) \in \Omega \}, \end{aligned}$$

it is easy to see that $\|\mathcal{A} - \lambda x^2 y^2 z^2\|_{HS}^2$ is the smallest if and only if the absolute value of λ is the largest among all $\lambda \in \mathbb{R}$ satisfying $\lambda = \mathcal{A} x^2 y^2 z^2$ and $(\lambda, x, y, z) \in \Omega$. On the other hand, since λ_* is an M -eigenvalue of \mathcal{A} and $x^*, y^*, z^* \in \mathbb{R}^3$ are the eigenvectors of \mathcal{A} associated with the M -eigenvalue λ_* , from Definition 3.1 it follows that $(\lambda_*, x^*, y^*, z^*) \in \Omega$; and from Theorem 3.2 it follows that $\lambda_* = \mathcal{A}(x^*)^2(y^*)^2(z^*)^2$.

Therefore, by the assumption that λ_* is an M -eigenvalue of \mathcal{A} with the largest absolute value among all M -eigenvalues of \mathcal{A} , we obtain that $\lambda_*(x^*)^2(y^*)^2(z^*)^2$ is the best rank-one approximation of \mathcal{A} , which completes the proof. \square

In the following, we introduce the *bi-block symmetric tensor* and study some related properties.

Definition 3.6. $\mathcal{A} = (a_{i_1 i_2 \dots i_{2m}}) \in \mathbb{T}_{2m,3}$ is called a *bi-block symmetric tensor* if its indices $\{i_1, i_2, \dots, i_{2m}\}$ are divided into two adjacent blocks $\{i_1, i_2, \dots, i_t\}$ and $\{i_{t+1}, i_{t+2}, \dots, i_{2m}\}$ with $t \in [1, 2m]$ being an even number and entries of \mathcal{A} being invariant under any permutation of indices in every block of $\{i_1, i_2, \dots, i_t\}$ and $\{i_{t+1}, i_{t+2}, \dots, i_{2m}\}$, i.e.,

$$a_{i_1 i_2 \dots i_t i_{t+1} i_{t+2} \dots i_{2m}} = a_{\sigma(i_1 i_2 \dots i_t) \delta(i_{t+1} i_{t+2} \dots i_{2m})} \quad (20)$$

for all $i_1, i_2, \dots, i_{2m} \in \{1, 2, 3\}$, where $\sigma(i_1 i_2 \dots i_t)$ denotes an arbitrary permutation of $i_1 i_2 \dots i_t$ and $\delta(i_{t+1} i_{t+2} \dots i_{2m})$ denotes an arbitrary permutation of $i_{t+1} i_{t+2} \dots i_{2m}$.

For any even number $t \in [1, 2m]$ and $x, y \in \mathbb{R}^3$, we use the following notation:

$$\begin{aligned} x^{[t]} &:= (x_1^t, x_2^t, x_3^t)^\top; \\ \mathcal{A} x^t y^{2m-t} &:= \sum_{i_1, \dots, i_t, i_{t+1}, \dots, i_{2m}=1}^3 a_{i_1 \dots i_t i_{t+1} \dots i_{2m}} x_{i_1} \dots x_{i_t} y_{i_{t+1}} \dots y_{i_{2m}} \\ \mathcal{A} x^{t-1} y^{2m-t} &\in \mathbb{R}^3 \text{ with} \\ (\mathcal{A} x^{t-1} y^{2m-t})_i &:= \sum_{i_2, \dots, i_t, i_{t+1}, \dots, i_{2m}=1}^3 a_{i i_2 \dots i_t i_{t+1} \dots i_{2m}} x_{i_2} \dots x_{i_t} y_{i_{t+1}} \dots y_{i_{2m}}, \quad \forall i \in \{1, 2, 3\}, \\ \mathcal{A} x^t y^{2m-t-1} &\in \mathbb{R}^3 \text{ with} \\ (\mathcal{A} x^t y^{2m-t-1})_i &:= \sum_{i_1, \dots, i_t, i_{t+2}, \dots, i_{2m}=1}^3 a_{i_1 \dots i_t i_{t+2} \dots i_{2m}} x_{i_1} \dots x_{i_t} y_{i_{t+2}} \dots y_{i_{2m}}, \quad \forall i \in \{1, 2, 3\}. \end{aligned}$$

In the following, we consider the case of $t = m$.

Definition 3.7. For any bi-block symmetric tensor $\mathcal{A} = (a_{i_1 \dots i_t i_{t+1} \dots i_{2m}}) \in \mathbb{T}_{2m,3}$ whose entries satisfy bi-block symmetry given by (20) with $t = m$, if there exist $\lambda \in \mathbb{R}$ and $x, y \in \mathbb{R}^3$ such that

$$\begin{cases} \mathcal{A} x^{m-1} y^m = \lambda x, & \mathcal{A} x^m y^{m-1} = \lambda y, \\ x^\top x = 1, & y^\top y = 1, \end{cases}$$

then λ is called an M -eigenvalue of \mathcal{A} and x, y are the eigenvectors of \mathcal{A} associated with the M -eigenvalue λ .

Then, similar to Theorem 3.2, we can obtain the following results.

Theorem 3.8. For any bi-block symmetric tensor $\mathcal{A} = (a_{i_1 \dots i_t i_{t+1} \dots i_{2m}}) \in \mathbb{T}_{2m,3}$ whose entries satisfy bi-block symmetry given by (20) with $t = m$, M -eigenvalues always exist. Moreover, if x, y are the eigenvectors of \mathcal{A} associated with the M -eigenvalue λ , then $\lambda = \mathcal{A} x^m y^m$.

By using Theorem 3.8, the following result holds.

Theorem 3.9. A bi-block symmetric tensor $\mathcal{A} = (a_{i_1 \dots i_t i_{t+1} \dots i_{2m}}) \in \mathbb{T}_{2m,3}$ whose entries satisfy bi-block symmetry given by (20) with $t = m$ is positive definite if and only if the smallest M -eigenvalue of \mathcal{A} is positive.

From Definition 3.7 and Theorem 3.9, we have the following results.

Theorem 3.10. For any bi-block symmetric tensor $\mathcal{A} \in \mathbb{T}_{2m,3}$ whose entries satisfy bi-block symmetry given by (20) with $t = m$, it follows that λ is an M -eigenvalue of \mathcal{A} if and only if $-\lambda$ is an M -eigenvalue of $-\mathcal{A}$; and furthermore, \mathcal{A} is positive definite if and only if the largest M -eigenvalue of $-\mathcal{A}$ is negative.

Moreover, similar to Theorem 3.5, we can obtain the following result.

Theorem 3.11. *For any bi-block symmetric tensor $\mathcal{A} = (a_{i_1 \dots i_t i_{t+1} \dots i_{2m}}) \in \mathbb{T}_{2m,3}$ whose entries satisfy bi-block symmetry given by (20) with $t = m$, if λ_* is an M -eigenvalue of \mathcal{A} with the largest absolute value among all M -eigenvalues of \mathcal{A} , and $x^*, y^* \in \mathbb{R}^3$ are the eigenvectors of \mathcal{A} associated with the M -eigenvalue λ_* , then $\lambda_*(x^*)^m(y^*)^m$ is the best rank-one approximation of \mathcal{A} .*

For any bi-block symmetric tensor $\mathcal{A} = (a_{i_1 \dots i_t i_{t+1} \dots i_{2m}}) \in \mathbb{T}_{2m,3}$ whose entries satisfy bi-block symmetry given by (20), when $m = 2$ and $t = 2$, the tensor \mathcal{A} reduces to a fourth order three dimensional paired symmetric tensor; and hence, the definition of M -eigenvalue and results of Theorems 3.8, 3.9 and 3.11 reduce to those given in [20]. For example, by Theorem 3.9 we have

Corollary 3.12. *A paired symmetric (elasticity) tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$ is positive definite if and only if the smallest M -eigenvalue of \mathcal{A} is positive.*

4 Eigenvalue and Positive Definiteness

In this section, following the ideas given in [4], we consider positive definiteness of fourth order three dimensional and sixth order three dimensional paired symmetric (elasticity) tensors. Specially, we propose several necessary and sufficient conditions under which the concerned tensors are positive definite.

4.1 Fourth order paired symmetric tensors. For any $x, y \in \mathbb{R}^3$, we define two matrices by

$$A(y) := \begin{bmatrix} y^\top A_{11}y & y^\top A_{12}y & y^\top A_{13}y \\ y^\top A_{21}y & y^\top A_{22}y & y^\top A_{23}y \\ y^\top A_{31}y & y^\top A_{32}y & y^\top A_{33}y \end{bmatrix} \quad \text{and} \quad B(x) := \begin{bmatrix} x^\top B_{11}x & x^\top B_{12}x & x^\top B_{13}x \\ x^\top B_{21}x & x^\top B_{22}x & x^\top B_{23}x \\ x^\top B_{31}x & x^\top B_{32}x & x^\top B_{33}x \end{bmatrix}, \quad (21)$$

where A_{ij} and B_{kl} for $i, j, k, l \in \{1, 2, 3\}$ are defined by (9). Recall that for any tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$, the corresponding biquadratic form is given by (2), i.e.,

$$\mathcal{A}x^2y^2 = \sum_{i,j,k,l=1}^3 a_{ijkl}x_i x_j y_k y_l, \quad \forall x, y \in \mathbb{R}^3.$$

Thus, it follows that

$$\mathcal{A}x^2y^2 = x^\top A(y)x = y^\top B(x)y, \quad \forall x, y \in \mathbb{R}^3.$$

Proposition 4.1. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{4,3}$, suppose that the matrices $A(\cdot)$ and $B(\cdot)$ are defined by (21). Then, the following results are equivalent.*

- (i) *The polynomial $\mathcal{A}x^2y^2$ defined by (2) is positive definite.*
- (ii) *The matrix $A(y)$ is positive definite for all $y \in \mathbb{R}^3 \setminus \{0\}$.*
- (iii) *The matrix $B(x)$ is positive definite for all $x \in \mathbb{R}^3 \setminus \{0\}$.*

It is well known that a symmetric matrix M is positive definite if and only if all leading principal minors of M are positive, which is the Sylvester's criterion. Thus, we have the following results, where $\det(M)$ denotes the determinant of the matrix M .

Theorem 4.2. *Suppose that $\mathcal{A} \in \mathbb{T}_{4,3}$ is a paired symmetric tensor, the polynomial $\mathcal{A}x^2y^2$ is defined by (2), and matrices $A(\cdot)$ and $B(\cdot)$ are defined by (21). Then, the polynomial $\mathcal{A}x^2y^2$ is positive definite if and only if one of the following results holds.*

(i) *The matrix A_{11} is positive definite, and*

$$y^\top A_{11}yy^\top A_{22}y - y^\top A_{12}yy^\top A_{21}y > 0 \quad \text{and} \quad \det(A(y)) > 0, \quad \forall y \in \mathbb{R}^3 \setminus \{0\}.$$

(ii) *The matrix B_{11} is positive definite, and*

$$x^\top B_{11}xx^\top B_{22}x - x^\top B_{12}xx^\top B_{21}x > 0 \quad \text{and} \quad \det(B(x)) > 0, \quad \forall x \in \mathbb{R}^3 \setminus \{0\}.$$

Furthermore, if \mathcal{A} is an elasticity tensor, then the above (i) and (ii) are the same.

It is easy to see that

- (a) $y^\top A_{11}yy^\top A_{22}y - y^\top A_{12}yy^\top A_{21}y$ is a special homogeneous polynomial of degree 4, and there exists a unique symmetric tensor $\mathcal{T}_A^1 \in \mathbb{T}_{4,3}$ such that $\mathcal{T}_A^1 y^4 = y^\top A_{11}yy^\top A_{22}y - y^\top A_{12}yy^\top A_{21}y$; and
- (b) $\det(A(y))$ is a special homogeneous polynomial of degree 6, and there exists a unique symmetric tensor $\mathcal{T}_A^2 \in \mathbb{T}_{6,3}$ such that $\mathcal{T}_A^2 y^6 = \det(A(y))$.

Similarly, there exist symmetric tensors $\mathcal{T}_B^1 \in \mathbb{T}_{4,3}$ and $\mathcal{T}_B^2 \in \mathbb{T}_{6,3}$ such that

- (c) $\mathcal{T}_B^1 x^4 = x^\top B_{11}xx^\top B_{22}x - x^\top B_{12}xx^\top B_{21}x$ and $\mathcal{T}_B^2 y^6 = \det(B(x))$.

By combining the theory of Z -eigenvalues of symmetric tensors with Theorem 4.2, we have the following results.

Theorem 4.3. *Suppose that $\mathcal{A} \in \mathbb{T}_{4,3}$ is a paired symmetric tensor, the polynomial $\mathcal{A}x^2y^2$ is defined by (2), and symmetric tensors $\mathcal{T}_A^1, \mathcal{T}_A^2, \mathcal{T}_B^1, \mathcal{T}_B^2$ are defined by the above (a)-(c). Then, the polynomial $\mathcal{A}x^2y^2$ is positive definite if and only if one of the following results holds.*

- (i) *The symmetric matrix A_{11} is positive definite, and the smallest Z -eigenvalues of the symmetric tensors \mathcal{T}_A^1 and \mathcal{T}_A^2 are positive.*
- (ii) *The symmetric matrix B_{11} is positive definite, and the smallest Z -eigenvalues of the symmetric tensors \mathcal{T}_B^1 and \mathcal{T}_B^2 are positive.*

Furthermore, if \mathcal{A} is an elasticity tensor, then the above (i) and (ii) are the same.

4.2 Sixth order paired symmetric tensors. In this part, we consider positive definiteness of sixth order three dimensional paired symmetric tensors and elasticity tensors.

For any $x, y, z \in \mathbb{R}^3$, we define three matrices by

$$\begin{aligned} A(y, z) &:= \begin{pmatrix} \mathcal{A}_{11}y^2z^2 & \mathcal{A}_{12}y^2z^2 & \mathcal{A}_{13}y^2z^2 \\ \mathcal{A}_{21}y^2z^2 & \mathcal{A}_{22}y^2z^2 & \mathcal{A}_{23}y^2z^2 \\ \mathcal{A}_{31}y^2z^2 & \mathcal{A}_{32}y^2z^2 & \mathcal{A}_{33}y^2z^2 \end{pmatrix}, \\ B(x, z) &:= \begin{pmatrix} \mathcal{B}_{11}x^2z^2 & \mathcal{B}_{12}x^2z^2 & \mathcal{B}_{13}x^2z^2 \\ \mathcal{B}_{21}x^2z^2 & \mathcal{B}_{22}x^2z^2 & \mathcal{B}_{23}x^2z^2 \\ \mathcal{B}_{31}x^2z^2 & \mathcal{B}_{32}x^2z^2 & \mathcal{B}_{33}x^2z^2 \end{pmatrix}, \\ C(x, y) &:= \begin{pmatrix} \mathcal{C}_{11}x^2y^2 & \mathcal{C}_{12}x^2y^2 & \mathcal{C}_{13}x^2y^2 \\ \mathcal{C}_{21}x^2y^2 & \mathcal{C}_{22}x^2y^2 & \mathcal{C}_{23}x^2y^2 \\ \mathcal{C}_{31}x^2y^2 & \mathcal{C}_{32}x^2y^2 & \mathcal{C}_{33}x^2y^2 \end{pmatrix}. \end{aligned} \quad (22)$$

Then, it is easy to see that

$$\mathcal{A}x^2y^2z^2 = x^\top A(y, z)x = y^\top B(x, z)y = z^\top C(x, y)z, \quad \forall x, y, z \in \mathbb{R}^3. \quad (23)$$

Proposition 4.4. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{6,3}$, suppose that matrices $A(\cdot, \cdot)$, $B(\cdot, \cdot)$ and $C(\cdot, \cdot)$ are defined by (22). Then, the following results are equivalent.*

- (i) *The polynomial $\mathcal{A}x^2y^2z^2$ defined by (13) is positive definite.*
- (ii) *The matrix $A(y, z)$ is positive definite for all $y, z \in \mathbb{R}^3 \setminus \{0\}$.*
- (iii) *The matrix $B(x, z)$ is positive definite for all $x, z \in \mathbb{R}^3 \setminus \{0\}$.*
- (iv) *The matrix $C(x, y)$ is positive definite for all $x, y \in \mathbb{R}^3 \setminus \{0\}$.*

Furthermore, if \mathcal{A} is an elasticity tensor, then the above (ii), (iii) and (iv) are the same.

Furthermore, by Sylvester's criterion we have the following results.

Theorem 4.5. *For any paired symmetric tensor $\mathcal{A} \in \mathbb{T}_{6,3}$, we assume that matrices $A(\cdot, \cdot)$, $B(\cdot, \cdot)$ and $C(\cdot, \cdot)$ are defined by (22). Then, the polynomial $\mathcal{A}x^2y^2z^2$ is positive definite if and only if one of the following results holds.*

- (i) $\mathcal{A}_{11}y^2z^2 > 0$, $(\mathcal{A}_{11}y^2z^2)(\mathcal{A}_{22}y^2z^2) - (\mathcal{A}_{12}y^2z^2)(\mathcal{A}_{21}y^2z^2) > 0$ and $\det(A(y, z)) > 0$ for all $y, z \in \mathbb{R}^3 \setminus \{0\}$.
- (ii) $\mathcal{B}_{11}x^2z^2 > 0$, $(\mathcal{B}_{11}x^2z^2)(\mathcal{B}_{22}x^2z^2) - (\mathcal{B}_{12}x^2z^2)(\mathcal{B}_{21}x^2z^2) > 0$ and $\det(B(x, z)) > 0$ for all $x, z \in \mathbb{R}^3 \setminus \{0\}$.
- (iii) $\mathcal{C}_{11}x^2y^2 > 0$, $(\mathcal{C}_{11}x^2y^2)(\mathcal{C}_{22}x^2y^2) - (\mathcal{C}_{12}x^2y^2)(\mathcal{C}_{21}x^2y^2) > 0$ and $\det(C(x, y)) > 0$ for all $x, y \in \mathbb{R}^3 \setminus \{0\}$.

Furthermore, if \mathcal{A} is an elasticity tensor, then the above (i), (ii) and (iii) are the same.

We consider the above result (i).

- It is easy to see that \mathcal{A}_{11} is a paired symmetric tensor; and hence, the positive definiteness of $\mathcal{A}_{11}y^2z^2$ can be checked by the minimum M -eigenvalue of the tensor \mathcal{A}_{11} given in the above subsection.
- It is easy to see that $(\mathcal{A}_{11}y^2z^2)(\mathcal{A}_{22}y^2z^2) - (\mathcal{A}_{12}y^2z^2)(\mathcal{A}_{21}y^2z^2)$ is a homogeneous polynomial of degree 8 with special structure. We can define the unique paired symmetric tensor $\mathcal{T}_A^1 \in \mathbb{T}_{8,3}$ such that

$$\mathcal{T}_A^1 y^4 z^4 = (\mathcal{A}_{11}y^2z^2)(\mathcal{A}_{22}y^2z^2) - (\mathcal{A}_{12}y^2z^2)(\mathcal{A}_{21}y^2z^2),$$

where $(\mathcal{T}_A^1)_{i_1 i_2 \dots i_8} = (\mathcal{T}_A^1)_{\sigma(i_1 i_2 i_3 i_4) \sigma(i_5 i_6 i_7 i_8)}$ with $\sigma(i_1 i_2 i_3 i_4)$ being an arbitrary permutation of $i_1 i_2 i_3 i_4$. Thus, $(\mathcal{A}_{11}y^2z^2)(\mathcal{A}_{22}y^2z^2) - (\mathcal{A}_{12}y^2z^2)(\mathcal{A}_{21}y^2z^2) > 0$ can be checked by the minimum M -eigenvalue of the tensor \mathcal{T}_A^1 .

- It is easy to see that $\det(A(y, z))$ is a homogeneous polynomial of degree 12 with special structure. We can define the unique paired symmetric tensor $\mathcal{T}_A^2 \in \mathbb{T}_{12,3}$ such that

$$\mathcal{T}_A^2 y^6 z^6 = \det(A(y, z)),$$

where $(\mathcal{T}_A^2)_{i_1 i_2 \dots i_{12}} = (\mathcal{T}_A^2)_{\sigma(i_1 \dots i_6) \sigma(i_7 \dots i_{12})}$ with $\sigma(i_1 \dots i_6)$ being an arbitrary permutation of $i_1 i_2 i_3 i_4 i_5 i_6$. Thus, $\det(A(y, z)) > 0$ can be checked by the minimum M -eigenvalue of the tensor \mathcal{T}_A^2 .

Similarly, we can define paired symmetric tensors $\mathcal{T}_B^1, \mathcal{T}_C^1 \in \mathbb{T}_{8,3}$ and $\mathcal{T}_B^2, \mathcal{T}_C^2 \in \mathbb{T}_{12,3}$ by

$$\begin{aligned} \mathcal{T}_B^1 x^4 z^4 &= (\mathcal{B}_{11}x^2z^2)(\mathcal{B}_{22}x^2z^2) - (\mathcal{B}_{12}x^2z^2)(\mathcal{B}_{21}x^2z^2), & \mathcal{T}_B^2 x^6 z^6 &= \det(B(x, z)); \\ \mathcal{T}_C^1 x^4 y^4 &= (\mathcal{C}_{11}x^2y^2)(\mathcal{C}_{22}x^2y^2) - (\mathcal{C}_{12}x^2y^2)(\mathcal{C}_{21}x^2y^2), & \mathcal{T}_C^2 x^6 y^6 &= \det(C(x, y)). \end{aligned}$$

By combining Theorem 3.9 with Theorem 4.5, we have the following results.

Theorem 4.6. *For any paired symmetric tensor $\mathcal{A} \in \mathbb{T}_{6,3}$, the polynomial $\mathcal{A}x^2y^2z^2$ defined by (13) is positive definite if and only if one of the following results holds.*

- (i) *The smallest M -eigenvalues of tensors \mathcal{A}_{11} , \mathcal{T}_A^1 and \mathcal{T}_A^2 are positive.*
- (ii) *The smallest M -eigenvalues of tensors \mathcal{B}_{11} , \mathcal{T}_B^1 and \mathcal{T}_B^2 are positive.*
- (iii) *The smallest M -eigenvalue of tensors \mathcal{C}_{11} , \mathcal{T}_C^1 and \mathcal{T}_C^2 are positive.*

Furthermore, if \mathcal{A} is an elasticity tensor, then the above (i), (ii) and (iii) are the same.

5 Sum of Squares and Positive Definiteness

In this section, we investigate the SOS properties of polynomials defined by fourth order three dimensional paired symmetric (elasticity) tensors and sixth order three dimensional paired symmetric (elasticity) tensors, respectively. In particular, we give necessary and/or sufficient conditions of the concerned tensor being positive definite.

5.1 Fourth order paired symmetric tensors. For any paired symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$, we assume that the biquadratic form is defined by (2), i.e.,

$$\mathcal{A}x^2y^2 = \sum_{i,j,k,l=1}^3 a_{ijkl}x_ix_jy_ky_l, \quad \forall x, y \in \mathbb{R}^3.$$

We first investigate sufficient conditions under which the biquadratic form defined by (2) is an SOS of bilinear forms, or is positive definite.

Theorem 5.1. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{4,3}$, let M defined by Definition 2.2 be an unfolded matrix of \mathcal{A} with respect to indices $i_1i_2 \cdots i_9$ and $j_1j_2 \cdots j_9$. Then, the following results hold.*

(i) *If M is positive semidefinite, then the biquadratic form defined by (2) is an SOS of bilinear forms.*

(ii) *If M is positive definite, then the biquadratic form defined by (2) is positive definite.*

Proof. (i) For any $x, y \in \mathbb{R}^3$, by given indices $i_1i_2 \cdots i_9$ and $j_1j_2 \cdots j_9$, we define $w \in \mathbb{R}^9$ by

$$w := (x_{i_1}y_{j_1}, x_{i_2}y_{j_2}, x_{i_3}y_{j_3}, x_{i_4}y_{j_4}, x_{i_5}y_{j_5}, x_{i_6}y_{j_6}, x_{i_7}y_{j_7}, x_{i_8}y_{j_8}, x_{i_9}y_{j_9})^\top.$$

Then, from Definition 2.2, it is not difficult to show that

$$\mathcal{A}x^2y^2 = \sum_{i,j,k,l=1}^3 a_{ijkl}x_ix_jy_ky_l = w^\top Mw, \quad \forall x, y \in \mathbb{R}^3, \quad (24)$$

and from Proposition 2.3 it follows that the matrix M is symmetric.

Since the matrix M is positive semidefinite, there are real numbers $\lambda_1, \lambda_2, \dots, \lambda_9$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq \lambda_{r+1} = \dots = \lambda_9 = 0$ and an orthogonal matrix $Q = (q_1 \ q_2 \ \cdots \ q_9)$ such that

$$Q^\top MQ = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_9), \quad \text{i.e.,} \quad M = \sum_{i=1}^r \lambda_i q_i q_i^\top. \quad (25)$$

This, together with (24), implies that

$$\mathcal{A}x^2y^2 = w^\top Mw = \sum_{i=1}^r \lambda_i w^\top q_i q_i^\top w = \sum_{i=1}^r \left(\sqrt{\lambda_i} w^\top q_i \right)^2, \quad (26)$$

which means that $\mathcal{A}x^2y^2$ is an SOS of bilinear forms.

(ii) Since the matrix M is positive definite, there are real numbers $\lambda_1, \lambda_2, \dots, \lambda_9$ with $\lambda_i > 0$ for all $i \in \{1, 2, \dots, 9\}$ and an orthogonal matrix $Q = (q_1 \ q_2 \ \dots \ q_9)$ such that (25) holds; and hence, (26) holds. If $x \neq 0$ and $y \neq 0$, then it is easy to see that $w \neq 0$, which further implies that there exists at least an index $i \in \{1, 2, \dots, 9\}$ such that $w^\top q_i \neq 0$ since the matrix Q is invertible. Thus, when the matrix M is positive definite, it follows from (26) that $\mathcal{A}x^2y^2 > 0$ for any $x \neq 0$ and $y \neq 0$, i.e., $\mathcal{A}x^2y^2$ is positive definite. \square

Especially, we have following results.

Theorem 5.2. *For any paired symmetric tensor $\mathcal{A} \in \mathbb{T}_{4,3}$, let matrices M^1 and M^2 be defined by (7) and (8), respectively. Then, the following results hold.*

- *The biquadratic form defined by (2) is an SOS of bilinear forms if the matrix M^1 is positive semidefinite or the matrix M^2 is positive semidefinite. Furthermore, if $\mathcal{A} \in \mathbb{T}_{4,3}$ is an elasticity tensor, then the results mentioned above are the same.*
- *The biquadratic form defined by (2) is positive definite if the matrix M^1 is positive definite or the matrix M^2 is positive definite. Furthermore, if $\mathcal{A} \in \mathbb{T}_{4,3}$ is an elasticity tensor, then the results mentioned above are the same.*

Proof. For any $x, y \in \mathbb{R}^3$, we define two vectors $u, v \in \mathbb{R}^9$ by

$$\begin{aligned} u &:= (x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3, x_3y_1, x_3y_2, x_3y_3)^\top, \\ v &:= (x_1y_1, x_2y_1, x_3y_1, x_1y_2, x_2y_2, x_3y_2, x_1y_3, x_2y_3, x_3y_3)^\top. \end{aligned} \quad (27)$$

Then, it is easy to show that

$$\mathcal{A}x^2y^2 = \sum_{i,j,k,l=1}^3 a_{ijkl}x_ix_jy_ky_l = u^\top M^1 u = v^\top M^2 v, \quad \forall x, y \in \mathbb{R}^3.$$

Thus, by a similar way as Theorem 5.1 and using Proposition 2.4, we can complete the proof. \square

Next, we define a class of tensors with the help of $\mathcal{A} \in \mathbb{T}_{4,3}$.

Definition 5.3. *For any paired symmetric tensor $\mathcal{A} \in \mathbb{T}_{4,3}$, we define a tensor $\mathcal{B} \in \mathbb{T}_{4,3}$ whose entries satisfy*

$$b_{ijkl} = b_{jilk}, \quad b_{ijlk} = b_{jikl}, \quad \text{and} \quad b_{ijkl} + b_{jilk} + b_{ijlk} + b_{jikl} = 4a_{ijkl}.$$

*We say that \mathcal{B} is a **semi-paired symmetric tensor** of \mathcal{A} .*

Especially, we give the following remark.

Remark 5.4. *We define two sets of indices by*

$$\begin{aligned} \mathcal{J} &:= \{(i, j, k, l) \in \{1, 2, 3\}^4 : i \neq j, k \neq l\}; \\ \mathcal{D} &:= \{(i, j, k, l) \in \{1, 2, 3\}^4 : i < j, l < k \text{ or } i > j, l > k\}. \end{aligned}$$

For any paired symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$, we define a tensor $\mathcal{B} = (b_{ijkl}) \in \mathbb{T}_{4,3}$ whose entries satisfy

$$b_{ijkl} = \begin{cases} 2a_{ijlk} & \text{if } (i, j, l, k) \in \mathcal{J}, \\ 0 & \text{if } (i, j, l, k) \in \mathcal{I} \setminus \mathcal{J}, \\ a_{ijlk} & \text{otherwise,} \end{cases} \quad \forall i, j, k, l \in \{1, 2, 3\},$$

then \mathcal{B} is a **semi-paired symmetric tensor** of \mathcal{A} .

It is easy to see that the biquadratic forms defined by a paired symmetric tensor $\mathcal{A} \in \mathbb{T}_{4,3}$ and by its semi-paired symmetric tensor \mathcal{B} are the same. Thus, the following results hold.

Theorem 5.5. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{4,3}$, let \mathcal{B} defined by Definition 5.3 be a semi-paired symmetric tensor of \mathcal{A} . Then, the quadratic form $\mathcal{A}x^2y^2$ is an SOS of bilinear forms if and only if the quadratic form $\mathcal{B}x^2y^2$ is an SOS of bilinear forms; and tensor \mathcal{A} is positive (semidefinite) definite if and only if tensor \mathcal{B} is positive (semidefinite) definite.*

Thus, SOS property and positive (semidefiniteness) definiteness of the paired symmetric tensor $\mathcal{A} \in \mathbb{T}_{4,3}$ can be studied by investigating SOS property and positive (semidefiniteness) definiteness of its semi-paired symmetric tensor.

By Theorems 5.1 and 5.5, we have the following results.

Theorem 5.6. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{4,3}$, let \mathcal{B} defined by Definition 5.3 be a semi-paired symmetric tensor of \mathcal{A} . Suppose that S is an unfolded matrix of tensor \mathcal{B} with respect to indices $i_1i_2 \cdots i_9$ and $j_1j_2 \cdots j_9$. Then, the matrix S is symmetric. Furthermore, if the matrix S is positive semidefinite, then the biquadratic form defined by (2) is an SOS of bilinear forms; and if the matrix S is positive definite, then the biquadratic form defined by (2) is positive definite.*

Corollary 5.7. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{4,3}$, suppose that $\mathcal{B} = (b_{ijkl})$ defined by Remark 5.4 is a semi-paired symmetric tensor of \mathcal{A} , and the unfolded matrix of \mathcal{B} is defined by*

$$M := (m_{st}) \quad \text{with } m_{st} = m_{3(i-1)+k, 3(j-1)+l} = b_{ijkl} \text{ for any } i, j, k, l \in \{1, 2, 3\}; \quad (28)$$

If the matrix M is positive semidefinite, then the biquadratic form defined by (2) is an SOS of bilinear forms; and if the matrix M is positive definite, then the biquadratic form defined by (2) is positive definite.

In the following, we propose a necessary and sufficient condition under which a fourth order three dimensional paired symmetric (elasticity) tensor is positive semidefinite. For this purpose, we need the following lemma.

Lemma 5.8. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{4,3}$, suppose that the biquadratic form defined by (2) can be written as*

$$\mathcal{A}x^2y^2 = \sum_{i,j,k,l=1}^3 a_{ijkl}x_ix_jy_ky_l = u^\top M u, \quad \forall x, y \in \mathbb{R}^3, \quad (29)$$

where the vector u is given by (27) and $M = (m_{st}) \in \mathbb{R}^{9 \times 9}$ is a symmetric matrix. Then, the matrix M is an unfolded matrix of some semi-paired symmetric tensor of \mathcal{A} .

Proof. We define a tensor $\mathcal{B} = (b_{ijkl}) \in \mathbb{T}_{4,3}$ by

$$\begin{pmatrix} b_{1111} & b_{1112} & b_{1113} & b_{1211} & b_{1212} & b_{1213} & b_{1311} & b_{1312} & b_{1313} \\ b_{1121} & b_{1122} & b_{1123} & b_{1221} & b_{1222} & b_{1223} & b_{1321} & b_{1322} & b_{1323} \\ b_{1131} & b_{1132} & b_{1133} & b_{1231} & b_{1232} & b_{1233} & b_{1331} & b_{1332} & b_{1333} \\ b_{2111} & b_{2112} & b_{2113} & b_{2211} & b_{2212} & b_{2213} & b_{2311} & b_{2312} & b_{2313} \\ b_{2121} & b_{2122} & b_{2123} & b_{2221} & b_{2222} & b_{2223} & b_{2321} & b_{2322} & b_{2323} \\ b_{2131} & b_{2132} & b_{2133} & b_{2231} & b_{2232} & b_{2233} & b_{2331} & b_{2332} & b_{2333} \\ b_{3111} & b_{3112} & b_{3113} & b_{3211} & b_{3212} & b_{3213} & b_{3311} & b_{3312} & b_{3313} \\ b_{3121} & b_{3122} & b_{3123} & b_{3221} & b_{3222} & b_{3223} & b_{3321} & b_{3322} & b_{3323} \\ b_{3131} & b_{3132} & b_{3133} & b_{3231} & b_{3232} & b_{3233} & b_{3331} & b_{3332} & b_{3333} \end{pmatrix} := (m_{st}). \quad (30)$$

By using the symmetry of matrix M , it is easy to show that

$$b_{ijkl} = b_{jikl} \quad \text{and} \quad b_{ijlk} = b_{ikjl}, \quad \forall i, j, k, l \in \{1, 2, 3\}. \quad (31)$$

Moreover, by combining (29) with (30) we have

$$\mathcal{A}x^2y^2 = \sum_{i,j,k,l=1}^3 a_{ijkl}x_ix_jy_ky_l = u^\top M u = \sum_{i,j,k,l=1}^3 b_{ijkl}x_ix_jy_ky_l = \mathcal{B}x^2y^2,$$

which, together with (31) and the paired symmetry of \mathcal{A} , implies that

$$b_{ijkl} + b_{jikl} + b_{ijlk} + b_{ikjl} = 4a_{ijkl}, \quad \forall i, j, k, l \in \{1, 2, 3\}. \quad (32)$$

Thus, by Definition 5.3, (31) and (32), it follows that \mathcal{B} is a semi-paired symmetric tensor of \mathcal{A} .

In addition, from (30) it is easy to see that

$$\begin{aligned} m_{st} &= b_{i_s i_t j_s j_t}, \quad \forall s, t \in \{1, 2, \dots, 9\} \text{ with} \\ (i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8, i_9) &= (1, 1, 1, 2, 2, 2, 3, 3, 3), \\ (j_1, j_2, j_3, j_4, j_5, j_6, j_7, j_8, j_9) &= (1, 2, 3, 1, 2, 3, 1, 2, 3). \end{aligned}$$

Thus, by Definition 2.2 we obtain that the matrix M is an unfolded matrix of tensor \mathcal{B} with respect to indices 111222333 and 123123123. \square

Theorem 5.9. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{4,3}$, the biquadratic form defined by (2) is an SOS of bilinear forms if and only if an unfolded matrix of some semi-paired symmetric tensor of \mathcal{A} is positive semidefinite.*

Proof. The sufficiency can be obtained by Theorem 5.6. We now show the necessity. Since $\mathcal{A}x^2y^2$ is an SOS of bilinear forms, there exists some positive integer $r \geq 1$ such that

$$\mathcal{A}x^2y^2 = \sum_{i,j,k,l=1}^3 a_{ijkl}x_ix_jy_ky_l = \sum_{s=1}^r \left(\sum_{i,j=1}^3 \alpha_{ij}^s x_i y_j \right)^2, \quad \forall x, y \in \mathbb{R}^3$$

where $\alpha_{ij}^s \in \mathbb{R}$ for any $s \in \{1, 2, \dots, r\}$ and $i, j \in \{1, 2, 3\}$. For any $s \in \{1, 2, \dots, r\}$, we denote

$$q_s := (\alpha_{11}^s, \alpha_{12}^s, \alpha_{13}^s, \alpha_{21}^s, \alpha_{22}^s, \alpha_{23}^s, \alpha_{31}^s, \alpha_{32}^s, \alpha_{33}^s)^\top.$$

Let the vector $u \in \mathbb{R}^n$ be defined by (27). Then,

$$\mathcal{A}x^2y^2 = \sum_{s=1}^r (q_s^\top u)^2 = u^\top \left(\sum_{s=1}^r q_s q_s^\top \right) u, \quad \forall x, y \in \mathbb{R}^3.$$

By Lemma 5.8, we can obtain that the matrix $\sum_{s=1}^r q_s q_s^\top$ is an unfolded matrix of some semi-paired symmetric tensor of \mathcal{A} , which is positive semidefinite. \square

In the following, we give a necessary condition of a fourth order three dimensional paired symmetric tensor being an SOS of bilinear forms, which is convenient for judging some fourth order three dimensional paired symmetric tensors not be an SOS.

Theorem 5.10. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{4,3}$, if the biquadratic form defined by (2) is an SOS of bilinear forms, then it follows that*

$$a_{iikk} \geq 0, \quad \forall i, k \in \{1, 2, 3\}.$$

Proof. By the assumption, we can let

$$\mathcal{A}x^2y^2 = \sum_{i,j,k,l=1}^3 a_{ijkl} x_i x_j y_k y_l = \sum_{s=1}^r \left(\sum_{i,k=1}^3 \alpha_{ik}^s x_i y_k \right)^2, \quad \forall x, y \in \mathbb{R}^3$$

where $r \geq 1$ is a positive integer and $\alpha_{ik}^s \in \mathbb{R}$ for any $s \in \{1, 2, \dots, r\}$ and $i, k \in \{1, 2, 3\}$. Then, it is easy to show that

$$a_{iikk} = \sum_{s=1}^r (\alpha_{ik}^s)^2, \quad \forall i, k \in \{1, 2, 3\},$$

which completes the proof. \square

5.2 Sixth order paired symmetric tensors. In this part, we consider the homogeneous polynomial defined by a tensor $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$, which is given by (13).

Similar to Theorem 5.1, we have the following results.

Theorem 5.11. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{6,3}$, suppose that N defined by Definition 2.6 is an unfolded matrix of \mathcal{A} with respect to indices $i_1 i_2 \dots i_{27}$ and $j_1 j_2 \dots j_{27}$. Then, the following results hold.*

- (i) *If N is positive semidefinite, then the polynomial defined by (13) is an SOS of trilinear forms.*
- (ii) *If N is positive definite, then the polynomial defined by (13) is positive definite.*

Especially, we have the following results.

Theorem 5.12. For any paired symmetric tensor $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$, let matrices N^1, N^2, \dots, N^6 be defined by (15). Then, the following results hold.

- (i) The polynomial defined by (13) is an SOS of trilinear polynomials if one of matrices N^1, N^2, \dots, N^6 is positive semidefinite. Furthermore, if $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$ is an elasticity tensor, then the results mentioned above are the same.
- (ii) The polynomial defined by (13) is positive definite if one of matrices N^1, N^2, \dots, N^6 is positive definite. Furthermore, if $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$ is an elasticity tensor, then the results mentioned above are the same.

Proof. For any $x, y, z \in \mathbb{R}^3$, we define six vectors $u^i \in \mathbb{R}^{27}$ ($i \in \{1, 2, \dots, 6\}$) by

$$\begin{aligned}
(u^1)^\top &:= (x_1y_1z_1, x_1y_1z_2, x_1y_1z_3, x_1y_2z_1, x_1y_2z_2, x_1y_2z_3, x_1y_3z_1, x_1y_3z_2, x_1y_3z_3, \\
&\quad x_2y_1z_1, x_2y_1z_2, x_2y_1z_3, x_2y_2z_1, x_2y_2z_2, x_2y_2z_3, x_2y_3z_1, x_2y_3z_2, x_2y_3z_3, \\
&\quad x_3y_1z_1, x_3y_1z_2, x_3y_1z_3, x_3y_2z_1, x_3y_2z_2, x_3y_2z_3, x_3y_3z_1, x_3y_3z_2, x_3y_3z_3); \\
(u^2)^\top &:= (x_1y_1z_1, x_1y_1z_2, x_1y_1z_3, x_2y_1z_1, x_2y_1z_2, x_2y_1z_3, x_3y_1z_1, x_3y_1z_2, x_3y_1z_3, \\
&\quad x_1y_2z_1, x_1y_2z_2, x_1y_2z_3, x_2y_2z_1, x_2y_2z_2, x_2y_2z_3, x_3y_2z_1, x_3y_2z_2, x_3y_2z_3, \\
&\quad x_3y_1z_1, x_3y_1z_2, x_3y_1z_3, x_3y_2z_1, x_3y_2z_2, x_3y_2z_3, x_3y_3z_1, x_3y_3z_2, x_3y_3z_3); \\
(u^3)^\top &:= (x_1y_1z_1, x_2y_1z_1, x_3y_1z_1, x_1y_2z_1, x_2y_2z_1, x_3y_2z_1, x_1y_3z_1, x_2y_3z_1, x_3y_3z_1, \\
&\quad x_1y_1z_2, x_2y_1z_2, x_3y_1z_2, x_1y_2z_2, x_2y_2z_2, x_3y_2z_2, x_1y_3z_2, x_2y_3z_2, x_3y_3z_2, \\
&\quad x_1y_1z_3, x_2y_1z_3, x_3y_1z_3, x_1y_2z_3, x_2y_2z_3, x_3y_2z_3, x_1y_3z_3, x_2y_3z_3, x_3y_3z_3); \\
(u^4)^\top &:= (x_1y_1z_1, x_1y_2z_1, x_1y_3z_1, x_1y_1z_2, x_1y_2z_2, x_1y_3z_2, x_1y_1z_3, x_1y_2z_3, x_1y_3z_3, \\
&\quad x_2y_1z_1, x_2y_2z_1, x_2y_3z_1, x_2y_1z_2, x_2y_2z_2, x_2y_3z_2, x_2y_1z_3, x_2y_2z_3, x_2y_3z_3, \\
&\quad x_3y_1z_1, x_3y_1z_2, x_3y_1z_3, x_3y_2z_1, x_3y_2z_2, x_3y_2z_3, x_3y_3z_1, x_3y_3z_2, x_3y_3z_3); \\
(u^5)^\top &:= (x_1y_1z_1, x_1y_2z_1, x_1y_3z_1, x_2y_1z_1, x_2y_2z_1, x_2y_3z_1, x_3y_1z_1, x_3y_2z_1, x_3y_3z_1, \\
&\quad x_1y_1z_2, x_1y_2z_2, x_1y_3z_2, x_2y_1z_2, x_2y_2z_2, x_2y_3z_2, x_3y_1z_2, x_3y_2z_2, x_3y_3z_2, \\
&\quad x_1y_1z_3, x_2y_1z_3, x_3y_1z_3, x_1y_2z_3, x_2y_2z_3, x_3y_2z_3, x_1y_3z_3, x_2y_3z_3, x_3y_3z_3); \\
(u^6)^\top &:= (x_1y_1z_1, x_2y_1z_1, x_3y_1z_1, x_1y_1z_2, x_2y_1z_2, x_3y_1z_2, x_1y_1z_3, x_2y_1z_3, x_3y_1z_3, \\
&\quad x_1y_2z_1, x_2y_2z_1, x_3y_2z_1, x_1y_2z_2, x_2y_2z_2, x_3y_2z_2, x_1y_2z_3, x_2y_2z_3, x_3y_2z_3, \\
&\quad x_1y_3z_1, x_1y_3z_2, x_1y_3z_3, x_2y_3z_1, x_2y_3z_2, x_2y_3z_3, x_3y_3z_1, x_3y_3z_2, x_3y_3z_3).
\end{aligned}$$

With the help of the definitions of matrices N^1, N^2, \dots, N^6 by (15), it is easy to verify that

$$\mathcal{A}x^2y^2z^2 = \sum_{i,j,k,l,p,q=1}^3 a_{ijklpq} x_i x_j y_k y_l z_p z_q = (u^1)^\top N^1 u^1 = (u^2)^\top N^2 u^2 = \dots = (u^6)^\top N^6 u^6.$$

Thus, similar to Theorem 5.2, we can obtain the desired results. \square

Similar to Definition 5.3, we give the concept of semi-paired symmetric tensor of $\mathcal{A} \in \mathbb{T}_{6,3}$.

Definition 5.13. For any paired symmetric tensor $\mathcal{A} \in \mathbb{T}_{6,3}$, we define a tensor $\mathcal{B} \in \mathbb{T}_{6,3}$ whose entries satisfy

$$\begin{aligned}
b_{ijklpq} &= b_{jilkqp}, \quad b_{jiklpq} = b_{ijlkqp}, \quad b_{ijlkpq} = b_{jiklpq}, \quad b_{ijklpq} = b_{jilkqp}, \\
&\text{and } b_{ijklpq} + b_{jiklpq} + b_{ijlkqp} + b_{jiklpq} = 4a_{ijklpq}.
\end{aligned}$$

We say that \mathcal{B} is a **semi-paired symmetric tensor** of \mathcal{A} .

Especially, we give the following remark.

Remark 5.14. We define several index sets by

$$\begin{aligned}
\mathcal{T}_1 &:= \{(i, j, k, l, p, q) \in \{1, 2, 3\}^6 : i \neq j, k \neq l \text{ and } p \neq q\}, \\
\mathcal{T}_2 &:= \left\{ (i, j, k, l, p, q) \in \{1, 2, 3\}^6 : \begin{array}{l} i \neq j, k \neq l \text{ and } p = q; \\ \text{or } i \neq j, k = l \text{ and } p \neq q; \\ \text{or } i = j, k \neq l \text{ and } p \neq q \end{array} \right\}, \\
\mathcal{T}_3 &:= \left\{ (i, j, k, l, p, q) \in \{1, 2, 3\}^6 : \begin{array}{l} i \neq j, k = l \text{ and } p = q; \\ \text{or } i = j, k \neq l \text{ and } p = q; \\ \text{or } i = j, k = l \text{ and } p \neq q \end{array} \right\}, \\
\mathcal{T}_4 &:= \{(i, j, k, l, p, q) \in \{1, 2, 3\}^6 : i = j, k = l \text{ and } p = q\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{S}_1 &:= \left\{ (i, j, k, l, p, q) \in \mathcal{T}_1 : \begin{array}{l} i < j, k < l \text{ and } p < q; \\ \text{or } i > j, k > l \text{ and } p > q \end{array} \right\}, \\
\mathcal{S}_2 &:= \left\{ (i, j, k, l, p, q) \in \mathcal{T}_2 : \begin{array}{l} i < j, k < l \text{ and } p = q; \\ \text{or } i > j, k > l \text{ and } p = q \end{array} \right\}, \\
\mathcal{S}_3 &:= \left\{ (i, j, k, l, p, q) \in \mathcal{T}_2 : \begin{array}{l} i < j, k = l \text{ and } p < q; \\ \text{or } i > j, k = l \text{ and } p > q \end{array} \right\}, \\
\mathcal{S}_4 &:= \left\{ (i, j, k, l, p, q) \in \mathcal{T}_2 : \begin{array}{l} i = j, k < l \text{ and } p < q; \\ \text{or } i = j, k > l \text{ and } p > q \end{array} \right\}.
\end{aligned}$$

For any paired symmetric tensor $\mathcal{A} \in \mathbb{T}_{6,3}$, we define a tensor $\mathcal{B} = (b_{ijklpq}) \in \mathbb{T}_{6,3}$ whose entries satisfy

$$b_{ijklpq} = \begin{cases} 4a_{ijklpq} & \text{if } (i, j, l, k, p, q) \in \mathcal{S}_1, \\ 0 & \text{if } (i, j, l, k, p, q) \in \mathcal{T}_1 \setminus \mathcal{S}_1, \\ 2a_{ijklpq} & \text{if } (i, j, l, k, p, q) \in \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4, \\ 0 & \text{if } (i, j, l, k, p, q) \in \mathcal{T}_2 \setminus \{\mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4\}, \\ a_{ijklpq} & \text{otherwise,} \end{cases} \quad \forall i, j, k, l, p, q \in \{1, 2, 3\},$$

then \mathcal{B} is a **semi-paired symmetric tensor** of \mathcal{A} .

Similar to Theorem 5.5, we have the following results.

Theorem 5.15. For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{6,3}$, suppose that \mathcal{B} defined by Definition 5.13 is an semi-paired symmetric tensor of \mathcal{A} . Then, the polynomial $\mathcal{A}x^2y^2z^2$ is an SOS of trilinear forms if and only if the polynomial $\mathcal{B}x^2y^2z^2$ is an SOS of trilinear forms; and tensor \mathcal{A} is positive (semidefinite) definite if and only if tensor \mathcal{B} is positive (semidefinite) definite.

Corollary 5.16. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{6,3}$, suppose that $\mathcal{B} = (b_{ijkl})$ defined by Remark 5.14 is a semi-paired symmetric tensor of \mathcal{A} , and the unfolded matrix of \mathcal{B} is defined by*

$$N := (n_{st}) \text{ with } n_{st} = n_{3[3(i-1)+(k-1)]+p, 3[3(j-1)+(l-1)]+q} = b_{ijklpq}, \forall i, j, k, l, p, q \in \{1, 2, 3\}; \quad (33)$$

If the matrix N is positive semidefinite, then the polynomial defined by (13) is an SOS of trilinear forms; and if the matrix N is positive definite, then the polynomial defined by (13) is positive definite.

Similar to Theorem 5.9, we have the following results.

Theorem 5.17. *For any paired symmetric (elasticity) tensor $\mathcal{A} \in \mathbb{T}_{6,3}$, the polynomial defined by (13) is an SOS of trilinear forms if and only if an unfolded matrix of some semi-paired symmetric tensor of \mathcal{A} is positive semidefinite.*

The following necessary condition can be obtained in a similar way as Theorem 5.10.

Theorem 5.18. *For any paired symmetric (elasticity) tensor $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$. If the homogeneous polynomial defined by (13) is an SOS of trilinear polynomials, then it follows that*

$$a_{iikkpp} \geq 0, \quad \forall i, k, p \in \{1, 2, 3\}.$$

6 Algorithm for the Smallest M -Eigenvalue

In order to judge positive definiteness of a fourth order three dimensional paired symmetric (elasticity) tensor, from Corollary 3.12 we may compute the smallest M -eigenvalue of the concerned tensor. Some methods can be applied to do it, such as the one given in [17]. In this section, by using the special structure of the paired symmetric (elasticity) tensor, we propose a sequential semidefinite programming method for computing the smallest M -eigenvalue of a fourth order three dimensional paired symmetric (elasticity) tensor, by which we can judge whether a fourth order three dimensional paired symmetric (elasticity) tensor is positive definite or not.

Let $\mathcal{A}x^2y^2$ be defined by (2) and r be some nonnegative integer, we define a polynomial $F_{r,s} : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$F_{r,s}(x, y) := \left(\sum_{i=1}^3 x_i^2 \right)^r \left(\sum_{i=1}^3 y_i^2 \right)^s \mathcal{A}x^2y^2, \quad (34)$$

which is a homogeneous polynomial with $\deg(F_r) = 2(r + s) + 4$. The following result can be obtained in a similar way as those in [24, Corollary], [8, Theorem 3.2] and [14, Lemma 3.5].

Theorem 6.1. *For any paired symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$, let $\text{int}\mathcal{C}$ and $F_{r,s}$ be defined by (4) and (34), respectively. If $\mathcal{A} \in \text{int}\mathcal{C}$, then it follows that $F_{r,s}(x, y)$ is an SOS for some sufficiently large integers $r, s \geq 0$.*

With Theorem 6.1, we define

$$\mathcal{K} := \{\mathcal{A} \in \mathbb{T}_{4,3} : F_{r,s}(x, y) \text{ is an SOS for some } r, s \geq 0\}. \quad (35)$$

Then, similar to Theorem 3.3 in [8], we can obtain the following result.

Theorem 6.2. *For any paired symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$, \mathcal{K} defined by (35) is a convex cone. In particular, it follows that $\text{int}\mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{C}$, where \mathcal{C} and $\text{int}\mathcal{C}$ are defined by (3) and (4), respectively.*

For any nonnegative integer r , by the multinomial theorem we have

$$\left(\sum_{i=1}^3 x_i^2\right)^r = \sum_{r_1+r_2+r_3=r} \frac{r!}{r_1! \times r_2! \times r_3!} x_1^{2r_1} x_2^{2r_2} x_3^{2r_3}.$$

Thus, the polynomial $F_{r,s}(x, y)$ defined by (34) is a linear combination of monomials

$$x_1^{2r_1} x_2^{2r_2} x_3^{2r_3} y_1^{2s_1} y_2^{2s_2} y_3^{2s_3} x_i x_j y_k y_l$$

for $i, j, k, l \in \{1, 2, 3\}$ and nonnegative integers $r_1, r_2, r_3, s_1, s_2, s_3$ satisfying $\sum_{i=1}^3 r_i = r$ and $\sum_{i=1}^3 s_i = s$. We assume that the corresponding vector of monomials is denoted by $v_{r+s+2}(x, y)$, which is first ordered in descending order of powers of x_1, x_2, x_3 and is then ordered in descending order of powers of y_1, y_2, y_3 given by

$$v_{r+s+2}(x, y)^\top : = (x_1^{r+1} y_1^{s+1}, x_1^{r+1} y_1^s y_2, \dots, x_1^{r+1} y_3^{s+1}, x_1^r x_2 y_2^{r+1}, \dots, x_3^{r+1} y_3^{s+1}). \quad (36)$$

Then, the following result is immediate.

Theorem 6.3. *For any paired symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$ and nonnegative integers r and s , suppose that the homogeneous polynomial $F_{r,s}$ and the vector of monomials $v_{r+s+2}(x, y)$ are defined by (34) and (36), respectively. Then, the polynomial $F_{r,s}(x, y)$ is an SOS if and only if $F_{r,s}(x, y) = v_{r+s+2}(x, y)^\top H v_{r+s+2}(x, y)$, where H is a positive semidefinite matrix.*

For any given nonnegative integers r and s , we let $u_{r+s+2}(x, y)$ denote the vector made up of different monomials in $v_{r+s+2}(x, y)^\top v_{r+s+2}(x, y)$, which is first ordered in descending order of powers of x_1, x_2, x_3 and then ordered in descending order of powers of y_1, y_2, y_3 ; and denote the dimensions of vectors $v_{r+s+2}(x, y)$ and $u_{r+s+2}(x, y)$ by $d_{vrs} := \dim(v_{r+s+2}(x, y))$ and $d_{urs} := \dim(u_{r+s+2}(x, y))$, respectively.

- For $r = 0$ and $s = 0$, the vector $v_{r+s+2}(x, y)$ contains monomials

$$x_i y_j, \quad \forall i, j \in \{1, 2, 3\},$$

and hence, $d_{v00} = C_3^1 C_3^1 = 9$; and the vector $u_{r+s+2}(x, y)$ contains monomials

$$x_i^2 y_k y_l \ (k < l), \ x_i^2 y_k^2, \ x_i x_j y_k y_l \ (i < j, k < l), \ x_i x_j y_k^2 \ (i < j), \quad \forall i, j, k, l \in \{1, 2, 3\},$$

and hence, $d_{u00} = C_3^1 C_3^2 + C_3^1 C_3^1 + C_3^2 C_3^2 + C_3^1 C_3^2 = 36$.

- For $r = 1$ and $s = 0$, the vector $v_{r+s+2}(x, y)$ contains monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3},$$

where $\alpha_i, \beta_i \geq 0$ for all $i \in \{1, 2, 3\}$, $\alpha_1 + \alpha_2 + \alpha_3 = 2$ and $\beta_1 + \beta_2 + \beta_3 = 1$, and hence, $d_{v10} = C_{3+2-1}^2 C_{3+1-1}^1 = 18$; and the vector $u_{r+s+2}(x, y)$ contains monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3},$$

where $\alpha_i, \beta_i \geq 0$ for all $i \in \{1, 2, 3\}$, $\alpha_1 + \alpha_2 + \alpha_3 = 4$ and $\beta_1 + \beta_2 + \beta_3 = 2$, and hence, $d_{u10} = C_{3+4-1}^4 C_{3+2-1}^2 = 90$.

- For $r = 0$ and $s = 1$, the vector $v_{r+s+2}(x, y)$ contains monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3},$$

where $\alpha_i, \beta_i \geq 0$ for all $i \in \{1, 2, 3\}$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\beta_1 + \beta_2 + \beta_3 = 2$, and hence, $d_{v01} = C_{3+1-1}^1 C_{3+2-1}^2 = 18$; and the vector $u_{r+s+2}(x, y)$ contains monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3},$$

where $\alpha_i, \beta_i \geq 0$ for all $i \in \{1, 2, 3\}$, $\alpha_1 + \alpha_2 + \alpha_3 = 2$ and $\beta_1 + \beta_2 + \beta_3 = 4$, and hence, $d_{u01} = C_{3+2-1}^2 C_{3+4-1}^4 = 90$.

- For $r = 1$ and $s = 1$, the vector $v_{r+s+2}(x, y)$ contains monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3},$$

where $\alpha_i, \beta_i \geq 0$ for all $i \in \{1, 2, 3\}$, $\alpha_1 + \alpha_2 + \alpha_3 = 2$ and $\beta_1 + \beta_2 + \beta_3 = 2$, and hence, $d_{v01} = C_{3+2-1}^2 C_{3+2-1}^2 = 36$; and the vector $u_{r+s+2}(x, y)$ contains monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3},$$

where $\alpha_i, \beta_i \geq 0$ for all $i \in \{1, 2, 3\}$, $\alpha_1 + \alpha_2 + \alpha_3 = 4$ and $\beta_1 + \beta_2 + \beta_3 = 4$, and hence, $d_{u01} = C_{3+4-1}^4 C_{3+4-1}^4 = 225$.

Furthermore, we introduce two operators \mathcal{V} and \mathcal{W} in the following:

\mathcal{V} : For any nonnegative integers r and s , we define an operator $\mathcal{V} : \mathbb{R}^{d_{vrs} \times d_{vrs}} \rightarrow \mathbb{R}^{d_{urs}}$ such that $[\mathcal{V}(H)]_i$ is the coefficient of the i th monomial in the vector $u_{r+s+2}(x, y)$ of the polynomial $v_{r+s+2}(x, y)^\top H v_{r+s+2}(x, y)$ for any $H \in \mathbb{R}^{d_{vrs} \times d_{vrs}}$.

\mathcal{W} : For any nonnegative integers r and s , we define an operator \mathcal{W} which maps a homogeneous polynomial $F_{r,s}(x, y)$ defined by (34) to a vector in $\mathbb{R}^{d_{urs}}$ satisfying $F_{r,s}(x, y) = u_{r+s+2}(x, y)^\top \mathcal{W}(F_{r,s}(x, y))$.

Thus, by Theorem 6.2 it follows that the set \mathcal{K} defined by (35) can be written as

$$\mathcal{K} := \left\{ \mathcal{A} \in \mathbb{T}_{4,3} : \mathcal{W}(F_{r,s}(x, y)) = \mathcal{V}(Q), r, s \in \{0, 1, 2, \dots\}, Q \in \mathcal{S}_+^{d_{vrs}} \right\}. \quad (37)$$

For any $r, s \in \{0, 1, 2, \dots\}$, we define

$$\mathcal{K}_{r+s} := \left\{ \mathcal{A} \in \mathbb{T}_{4,3} : \mathcal{W}(F_{r,s}(x, y)) = \mathcal{V}(Q), Q \in \mathcal{S}_+^{d_{vrs}} \right\}. \quad (38)$$

Then, the following result is immediate.

Theorem 6.4. *For any paired symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$ and nonnegative integers r and s , suppose that the set \mathcal{K}_{r+s} is defined by (38). Then,*

$$\mathcal{K}_{r+s} \subseteq \mathcal{K}_{r+s+1} \text{ for any } r, s \in \{0, 1, 2, \dots\} \quad \text{and} \quad \lim_{r+s \rightarrow \infty} \mathcal{K}_{r+s} = \bigcup_{r+s=0}^{\infty} \mathcal{K}_{r+s} = \mathcal{K}.$$

For any paired symmetric tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$, it follows from Definition 3.7 and Theorem 3.8 that the smallest M -eigenvalue of \mathcal{A} is the optimal value of the following biquadratic optimization over unit spheres [14]:

$$\begin{aligned} \min \quad & \mathcal{A} x^2 y^2 \\ \text{s.t.} \quad & x^\top x = 1, \quad y^\top y = 1, \end{aligned} \quad (39)$$

which is equivalent to

$$\begin{aligned} \max \quad & \gamma \\ \text{s.t.} \quad & \mathcal{A} x^2 y^2 \geq \gamma, \forall (x, y) \in \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 : x^\top x = 1, y^\top y = 1\}. \end{aligned} \quad (40)$$

Let tensor $\mathcal{E} = (e_{ijkl}) \in \mathbb{T}_{4,3}$ be defined by (5). Then, it is easy to see that problem (40) is equivalent to

$$\begin{aligned} \max \quad & \gamma \\ \text{s.t.} \quad & \mathcal{A} + \gamma \mathcal{E} \in \mathcal{K}. \end{aligned} \quad (41)$$

Furthermore, by replacing the constraint $\mathcal{A} + \gamma \mathcal{E} \in \mathcal{K}$ by $\mathcal{A} + \gamma \mathcal{E} \in \mathcal{K}_{r+s}$, we obtain the $r + s$ th order relaxation problem of (41), which can be written as

$$\begin{aligned} \max \quad & \gamma \\ \text{s.t.} \quad & \mathcal{V}(Q) = \mathcal{W}(F_{r,s}(x, y)) + \gamma \mathcal{W} \left((\sum_{i=1}^3 x_i^2)^{r+1} (\sum_{i=1}^3 y_i^2)^{s+1} \right), \\ & Q \in \mathcal{S}_+^{d_{vrs}}, \end{aligned} \quad (42)$$

where $r, s \in \{0, 1, 2, \dots\}$. For any $r, s \in \{0, 1, 2, \dots\}$, (42) is a semidefinite programming problem, which is denoted by $SDP(r+s)$. Thus, we can solve a sequence of $SDP(r+s)$ through increasing $r + s$ to obtain an approximation optimal solution of (39) up to a priori precision.

7 Numerical Results

In this section, we present preliminary numerical results to judge whether a fourth order three dimensional or a sixth order three dimensional paired symmetric tensor is positive definite or not. All numerical experiments were run in Matlab on a PC with 2.93 GHz CPU and 2.00 GB of RAM. We divide our experiments into the following two parts.

Part 1. In this part, by using the sequential semidefinite programming method proposed in Section 6, we present preliminary numerical results for computing the smallest M -eigenvalue of the fourth order three dimensional paired symmetric tensor. We use SDPT3 [30] to solve the resulted conic linear programming problem. In addition, it is known that Wang, Qi and Zhang [32] proposed a power method to find the largest M -eigenvalue of a fourth order paired symmetric tensor. By Theorem 3.4 it follows that their method can be applied to find the smallest M -eigenvalue of a fourth order three dimensional paired symmetric tensor \mathcal{A} . In fact, if we let M^1 be the unfolded matrix of tensor \mathcal{A} defined by (7), then the smallest M -eigenvalue of \mathcal{A} can be found by solving

$$\begin{aligned} \max \quad & (\tau \mathcal{E} - \mathcal{A})x^2y^2 \\ \text{s.t.} \quad & x^\top x = 1, y^\top y = 1, \end{aligned} \quad (43)$$

where τ is a given scalar satisfying $\tau > \rho := \max\{|\lambda| : \lambda \text{ is an eigenvalue of matrix } M^1\}$, and \mathcal{E} is defined by (5). In this part, we also report the numerical results of the Wang-Qi-Zhang power method for the fourth order three dimensional paired symmetric tensors we tested.

Example 7.1. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$ be given by

$$\begin{aligned} a_{1111} = 1, a_{2222} = 1, a_{3333} = 1, a_{1122} = 2, a_{2233} = 2, a_{3311} = 2, \\ a_{1212} = a_{1221} = a_{2112} = a_{2121} = -\frac{1}{2}, a_{2323} = a_{2332} = a_{3223} = a_{3232} = -\frac{1}{2}, \\ a_{3131} = a_{3113} = a_{1331} = a_{1313} = -\frac{1}{2}, \text{ other } a_{ijkl} = 0. \end{aligned} \quad (44)$$

The biquadratic form defined by this tensor is

$$\begin{aligned} f(x, y) = & x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2 - 2(x_1x_2y_1y_2 + x_2x_3y_2y_3 + x_3x_1y_3y_1) \\ & + 2(x_1^2y_2^2 + x_2^2y_3^2 + x_3^2y_1^2), \end{aligned} \quad (45)$$

which is a positive semidefinite biquadratic form that is not an SOS of bilinear forms [3].

Suppose $\lambda \in \mathbb{R}$ is an M -eigenvalue of \mathcal{A} and $x, y \in \mathbb{R}^3 \setminus \{0\}$ are the eigenvectors of \mathcal{A} associated with the M -eigenvalue λ , then it follows from the definition of the M -eigenvalue that

$$\begin{cases} x_1y_1^2 - (x_2y_1y_2 + x_3y_3y_1) + 2x_1y_2^2 = \lambda x_1, \\ x_2y_2^2 - (x_1y_1y_2 + x_3y_2y_3) + 2x_2y_3^2 = \lambda x_2, \\ x_3y_3^2 - (x_2y_2y_3 + x_1y_3y_1) + 2x_3y_1^2 = \lambda x_3, \\ x_1^2y_1 - (x_1x_2y_2 + x_3x_1y_3) + 2x_3^2y_1 = \lambda y_1, \\ x_2^2y_2 - (x_1x_2y_1 + x_2x_3y_3) + 2x_1^2y_2 = \lambda y_2, \\ x_3^2y_3 - (x_2x_3y_2 + x_3x_1y_1) + 2x_2^2y_3 = \lambda y_3, \\ x_1^2 + x_2^2 + x_3^2 = 1, y_1^2 + y_2^2 + y_3^2 = 1. \end{cases} \quad (46)$$

From (46) it is not difficult to show that $\lambda = 0$ is an M -eigenvalue of \mathcal{A} , and

$$((x_1, x_2, x_3), (y_1, y_2, y_3)) = ((1, 0, 0), (0, 0, 1)), ((0, 1, 0), (1, 0, 0)), ((0, 0, 1), (0, 1, 0))$$

are the eigenvectors of \mathcal{A} associated with the M -eigenvalue $\lambda = 0$. Furthermore, $\lambda = 0$ is the smallest M -eigenvalue of \mathcal{A} since the biquadratic form $f(x, y)$ in (45) is positive semidefinite.

We use the method proposed in Section 6 to the smallest M -eigenvalue of tensor \mathcal{A} , and the numerical result is listed in Table 1.

Table 1: The numerical results of the problem in Example 7.1

r	s	ITER	CPU(s)	OPT	VOL
0	0	9	1.313	-1.5243441995e-09	2.7246124908e-12
1	0	11	1.719	-2.6041279810e-10	5.7978715466e-12
0	1	11	1.813	-2.6041187999e-10	5.7989670655e-12
1	1	13	4.594	-2.4538473770e-10	1.5420658730e-11
2	1	14	8.797	-2.3617078649e-10	1.3527256926e-11
1	2	14	8.859	-2.3616477957e-10	1.3665548633e-11
2	2	16	29.344	-2.8204474346e-10	1.0935180661e-11

In Table 1, “ITER” means the iteration number of the SDP solver, “CPU(s)” means the total time in seconds spent for both setting up the problem and solving it, “OPT” means the approximation value computed, and “VOL” means the norm of the violation of the constraints of the approximation solution. From this table, we see that the method can find a good approximation solution even with the zero order relaxation.

We use the Wang-Qi-Zhang power method to find the smallest M -eigenvalue of tensor \mathcal{A} . We consider two cases: $\tau = 1.001\rho$ and $\tau = 2\rho$. The CPU time of the algorithm in 30 iterations is 0.0156 seconds. In both cases, the sequences of the smallest M -eigenvalues generated by the algorithm are monotonically decreasing, which are shown in Figure 1.

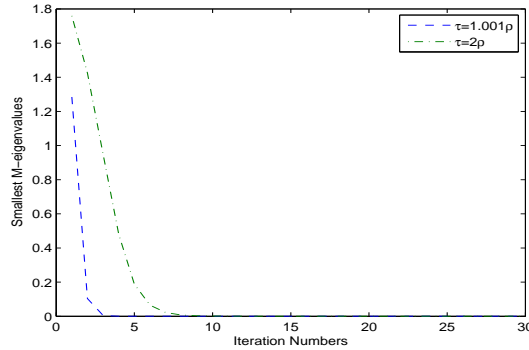


Figure 1: Graph of the power method for Example 7.1

In addition, when $\tau = 1.001\rho$, the obtained smallest M -eigenvalue is $7.8930677724e-05$ in the sixth step, and $4.0555680197e-05$ in the thirtieth step; and when $\tau = 2\rho$, the obtained smallest M -eigenvalue is $7.6951975782e-04$ in the eleventh step, and $2.8545893221e-04$ in the thirtieth step.

Example 7.2. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$ be given by

$$\begin{aligned} a_{1111} = a_{1122} = a_{1133} = a_{2211} = a_{2222} = a_{2233} = a_{3311} = 1, \\ a_{3322} = 3, a_{3333} = 3, a_{3323} = a_{3332} = -1, \text{ other } a_{ijkl} = 0. \end{aligned} \quad (47)$$

The biquadratic form defined by this tensor is

$$f(x, y) = x_1^2 y_1^2 + x_1^2 y_2^2 + x_1^2 y_3^2 + x_2^2 y_1^2 + x_2^2 y_2^2 + x_2^2 y_3^2 + x_3^2 y_1^2 + 3x_3^2 y_2^2 - 2x_3^2 y_2 y_3 + 3x_3^2 y_3^2. \quad (48)$$

Suppose $\lambda \in \mathbb{R}$ is an M -eigenvalue of \mathcal{A} and $x, y \in \mathbb{R}^3 \setminus \{0\}$ are the eigenvectors of \mathcal{A} associated with the M -eigenvalue λ , then it follows from the definition of the M -eigenvalue that

$$\begin{cases} x_1 y_1^2 + x_1 y_2^2 + x_1 y_3^2 &= \lambda x_1, \\ x_2 y_1^2 + x_2 y_2^2 + x_2 y_3^2 &= \lambda x_2, \\ x_3 y_1^2 + 3x_3 y_2^2 - 2x_3 y_2 y_3 + 3x_3 y_3^2 &= \lambda x_3, \\ x_1^2 y_1 + x_2^2 y_1 + x_3^2 y_1 &= \lambda y_1, \\ x_1^2 y_2 + x_2^2 y_2 + 3x_3^2 y_2 - x_3^2 y_3 &= \lambda y_2, \\ x_1^2 y_3 + x_2^2 y_3 - x_3^2 y_2 + 3x_3^2 y_3 &= \lambda y_3, \\ x_1^2 + x_2^2 + x_3^2 &= 1, \\ y_1^2 + y_2^2 + y_3^2 &= 1. \end{cases} \quad (49)$$

From the first three equalities and last two equalities in (49), we can obtain that

$$\lambda = 1 + x_3^2 y_2^2 + (x_3 y_2 - x_3 y_2)^2 + x_3^2 y_3^2 \geq 1,$$

which implies that all M -eigenvalues of \mathcal{A} given in (50) are greater than or equal to 1. Moreover, it is easy to see that $(\lambda_*, x^*, y^*) = (1, (1, 0, 0)^\top, (1, 0, 0)^\top)$ is a solution to (49). Therefore, $\lambda_* = 1$ is the smallest M -eigenvalues of \mathcal{A} given in (50).

The numerical result is listed in Table 2.

In Table 2, symbols “ITER”, “CPU(s)”, “OPT”, “VOL” are the same as in Example 7.1. From this table, we see that the method can find a good approximation solution even with the zero order relaxation.

We use the Wang-Qi-Zhang power method to find the smallest M -eigenvalue of tensor \mathcal{A} . We consider two cases: $\tau = 1.001\rho$ and $\tau = 2\rho$. The CPU time of the algorithm in 100 iterations is 0.0625 seconds. In both cases, the sequences of the smallest M -eigenvalues generated by the algorithm are monotonically decreasing, which are shown in Figure 2.

In addition, when $\tau = 1.001\rho$, the algorithm can find the smallest M -eigenvalue of tensor \mathcal{A} in the eighth step; and when $\tau = 2\rho$, the obtained smallest M -eigenvalue is 1.0088825298 in the eight-third step, and 1.0008862419 in the hundredth step.

Table 2: The numerical results of the problem in Example 7.2

r	s	ITER	CPU(s)	OPT	VOL
0	0	10	1.219	9.9999999970e-01	1.4472115766e-14
1	0	12	1.734	9.9999999963e-01	1.8617918560e-11
0	1	11	1.844	9.9999999504e-01	2.6733847183e-10
1	1	13	4.906	9.999999954e-01	1.7099952719e-11
2	1	14	9.000	9.999999949e-01	1.4822319679e-11
1	2	14	8.938	9.9999999812e-01	3.8117223117e-11
2	2	16	29.734	9.9999999749e-01	6.7937294113e-12

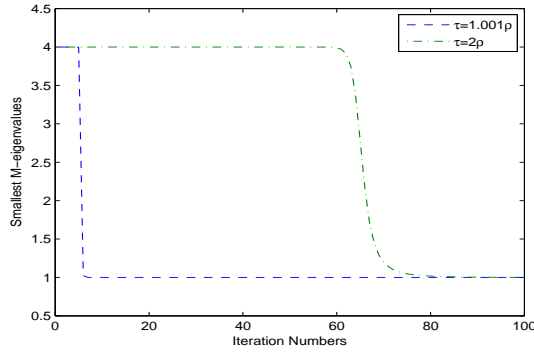


Figure 2: Graph of the power method for Example 7.2

From the numerical results given in this part, we have the following observations. (i) The Wang-Qi-Zhang power method is faster than the method proposed in this paper. (ii) The method proposed in this paper can always find a high precision approximation solution; while the Wang-Qi-Zhang power method strongly depends on selection of starting parameters, and it is generally difficult to find appropriate parameters in order to find a high precision approximation solution.

Part 2. In this part, by using some results obtained in Section 5, we compute the smallest eigenvalue of an unfolded matrix of some semi-paired symmetric tensor to judge whether the concerned paired symmetric tensor is positive definite or not.

Example 7.3. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{T}_{4,3}$ be given by

$$\begin{aligned} a_{1111} = 2, a_{2222} = 2, a_{3333} = 2, a_{1313} = a_{3113} = a_{1331} = a_{3131} = -\frac{1}{2}, \\ a_{2323} = a_{3223} = a_{2332} = a_{3232} = -\frac{1}{2}, \text{ other } a_{ijkl} = 0. \end{aligned} \quad (50)$$

The biquadratic form defined by this tensor is

$$f(x, y) = 2x_1^2y_1^2 + 2x_2^2y_2^2 + 2x_3^2y_3^2 - 2x_1x_3y_1y_3 - 2x_2x_3y_2y_3. \quad (51)$$

It is not difficult to obtain that $f(x, y) = x_1^2 y_1^2 + x_2^2 y_2^2 + (x_1 y_1 - x_3 y_3)^2 + (x_2 y_2 - x_3 y_3)^2$ and $f(x^*, y^*) = 0$ where $x^* = (1, 0, 0)^\top$ and $y^* = (0, 1, 0)$. These demonstrate that the biquadratic form in (51) is an SOS of bilinear forms but not positive definite.

Let \mathcal{A} be given by (50) and \mathcal{B} defined by Remark 5.4 be a semi-paired symmetric tensor, then

$$b_{1111} = 2, b_{2222} = 2, b_{3333} = 2, b_{1313} = d_{3131} = -1, b_{2323} = b_{3232} = -1, \text{ other } b_{ijkl} = 0.$$

Suppose $M \in \mathbb{R}^{9 \times 9}$ defined by (28) is the unfolded matrix of \mathcal{B} , then

$$m_{11} = 2, m_{55} = 2, m_{99} = 2, m_{19} = m_{91} = -1, m_{59} = m_{95} = -1, \text{ other } m_{ij} = 0.$$

It is easy to obtain that eigenvalues of M are $0, 0, 0, 0, 0, 0, 0.5858, 2, 3.4142$, respectively, which implies that the matrix M is positive semidefinite but not positive definite. Thus, by Corollary 5.7 we obtain that the quadratic form (51) is an SOS of bilinear forms but not positive definite.

Example 7.4. Let $\mathcal{A} = (a_{ijklpq}) \in \mathbb{T}_{6,3}$ be given by

$$\begin{aligned} a_{111111} &= a_{112233} = a_{221111} = a_{222222} = a_{331122} = a_{333333} = 1, \\ a_{111213} &= a_{111231} = a_{112113} = a_{112131} = -\frac{1}{2}, \quad a_{221212} = a_{221221} = a_{222112} = a_{222121} = -\frac{1}{2}, \\ a_{331323} &= a_{331332} = a_{333123} = a_{333132} = -\frac{1}{2}, \quad \text{other } a_{ijklpq} = 0. \end{aligned} \quad (52)$$

The homogeneous polynomial defined by this tensor is

$$\begin{aligned} f(x, y, z) &= x_1^2 y_1^2 z_1^2 + x_1^2 y_2^2 z_3^2 + x_2^2 y_1^2 z_1^2 + x_2^2 y_2^2 z_2^2 + x_3^2 y_1^2 z_2^2 + x_3^2 y_3^2 z_3^2 \\ &\quad - 2(x_1^2 y_1 y_2 z_1 z_3 + x_2^2 y_1 y_2 z_1 z_2 + x_3^2 y_1 y_3 z_2 z_3), \end{aligned} \quad (53)$$

It is not difficult to obtain that $f(x, y, z) = (x_1 y_1 z_1 - x_1 y_2 z_3)^2 + (x_2 y_1 z_1 - x_2 y_2 z_2)^2 + (x_3 y_1 z_2 - x_3 y_3 z_3)^2$ and $f(x^*, y^*, z^*) = 0$ where $x^* = (0, 0, 1)^\top$, $y^* = (0, 1, 0)$ and $z^* = (1, 0, 0)$. These demonstrate that the homogeneous polynomial in (53) is an SOS of bilinear forms but not positive definite.

Let \mathcal{A} be given by (52) and \mathcal{B} defined by Remark 5.14 be a semi-paired symmetric tensor, then

$$\begin{aligned} b_{111111} &= b_{112233} = b_{221111} = b_{222222} = b_{331122} = b_{333333} = 1, \\ b_{111213} &= b_{112131} = -1, \quad b_{221212} = b_{222121} = -1, \quad b_{331323} = b_{333132} = -1, \quad \text{other } b_{ijklpq} = 0. \end{aligned}$$

Suppose $N \in \mathbb{R}^{27 \times 27}$ defined by (33) is the unfolded matrix of \mathcal{B} , then

$$\begin{aligned} n_{11} &= n_{66} = n_{10,10} = n_{14,14} = n_{20,20} = n_{27,27} = 1, \\ n_{16} &= n_{61} = n_{10,14} = n_{14,10} = n_{20,27} = n_{27,20} = -1, \quad \text{other } n_{ij} = 0. \end{aligned}$$

It is easy to obtain that eigenvalues of N are $\underbrace{0, 0, \dots, 0}_{24}, 2, 2, 2$, respectively, which implies that the matrix N is positive semidefinite but not positive definite. Thus, by Corollary 5.16 we obtain that the homogeneous polynomial in (53) is an SOS of trilinear forms but not positive definite.

8 Concluding Remarks

In this paper, we investigated positive definiteness of the fourth order three dimensional and sixth order three dimensional paired symmetric (elasticity) tensor. With the help of the M -eigenvalue we gave several necessary and/or sufficient conditions under which a fourth order three dimensional or sixth order three dimensional paired symmetric (elasticity) tensor is positive definite. By introducing semi-paired symmetric tensor and the unfolded matrix of that tensor, we showed that positive definiteness of a fourth order three dimensional and sixth order three dimensional paired symmetric (elasticity) tensor can be obtained by investigating positive definiteness of some unfolded matrix of a semi-paired symmetric tensor of that tensor. We also proposed a necessary and sufficient condition under which a fourth order three dimensional or sixth order three dimensional paired symmetric (elasticity) tensor is positive semidefinite. These results can be extended to the case of higher order higher dimensional (strongly) paired symmetric tensors. We reported preliminary numerical results which confirm our theoretical findings.

We derived several necessary and sufficient conditions for positive definiteness of a sixth order three dimensional paired symmetric tensor in Section 4, where the most main condition needs to judge positive definiteness of a fourth order paired symmetric tensor, an eighth order bi-block symmetric tensor and a twelfth order bi-block symmetric tensor. It is well known that it is difficult to solve the polynomial optimization problem with the involved polynomials being defined by higher order higher dimensional tensors. Thus, one of further issues is how to design effective methods to judge positive definiteness of higher order (strongly) paired symmetric tensors or higher order elasticity tensors. Moreover, in this paper, we mainly gave some analysis for positive definiteness of the fourth order three dimensional and sixth order three dimensional paired symmetric (elasticity) tensor. It is worth investigating effective algorithms to check positive definiteness of paired symmetric (elasticity) tensors arisen from practical mechanical problems.

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