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COMON'S CONJECTURE, RANK DECOMPOSITION, AND SYMMETRIC RANK DECOMPOSITION OF SYMMETRIC TENSORS*

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Abstract. Comon's Conjecture claims that for a symmetric tensor, its rank and its symmetric rank coincide. We show that this conjecture is true under an additional assumption that the rank of that tensor is not larger than its order. Moreover, if its rank is less than its order, then all rank decompositions are necessarily symmetric rank decompositions.

Key words. tensor, rank, symmetric rank, rank decomposition, symmetric rank decomposition

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1. Introduction.

A tensor is a multidimensional array. The order of a tensor is the number of dimensions, also known as degree. A first order tensor is a vector, and a second order tensor is a matrix. Tensors of order three or higher are called higher order tensors. Unless otherwise specialized, tensors appearing in this paper are higher order tensors. A tensor is called square or cubical if all its dimensions are identical. A square tensor is called symmetric if its elements are invariant under any permutation of their indices. Symmetric tensors have wide applications such as in signal and image processing and blind source separation; we refer to [9, 21, 23, 24, 26] and references therein.

Decompositions of higher order tensors are the extensions of matrix singular value decomposition. For example, we have CANDECOMP/PARAFAC (CP) decomposition, Tucker decomposition, and PARAFAC2 decomposition; see [19] and references therein. In this paper, we will consider the CP decomposition of higher order tensors. The CP decomposition was introduced by Hitchcock in 1927 [17, 18] and has attracted much attention in the areas of machine learning, biomedical engineering, signal processing, independent component analysis, psychometrics, and chemometrics [1, 7, 8, 10, 13, 14, 19, 29]. For symmetric tensors, there are two types of CP decompositions: the outer product decomposition and the symmetric outer product decomposition, which are also called the CP decomposition and the symmetric CP decomposition, respectively. One may regard them as the generalizations of singular value decomposition and eigenvalue decomposition of symmetric matrices. The symmetric CP decomposition has wide applications in blind identification of underdetermined mixtures, speech, and so on. Furthermore, the bijection between symmetric tensors and homogeneous polynomials is another motivation to study the symmetric CP decompositions [2, 6, 25].

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For symmetric tensors, CP decomposition leads to the CP rank, and symmetric CP decomposition leads to the symmetric CP rank. For ease, we will call them the rank and the symmetric rank, respectively, throughout this paper. Furthermore, the decompositions corresponding to the rank and the symmetric rank are called the rank decomposition and the symmetric rank decomposition, respectively.

From [11], we know that the rank and the symmetric rank always exist. As the generalization of matrix rank, the rank and the symmetric rank of tensors are distinguishing themselves from the matrix rank. First, to determine the rank of a tensor is NP-hard [15], while the matrix rank can be determined by polynomial time algorithms. Another difference is about the relationship between the rank and the symmetric rank. For any symmetric matrix, the rank and the symmetric rank coincide, while for symmetric tensors, a similar relationship is not known to us. In 2008, the following conjecture, termed Comon's Conjecture in [20, 22], is raised in [11].

Comon's Conjecture. For a symmetric tensor, its rank and its symmetric rank always coincide.

Comon's Conjecture has attracted much attention since 2008 [3, 16, 20, 22, 27]. As far as we know, this conjecture has been proved at least in the following cases: (1) the border rank is 2 [3] or 3 [20]; (2) the rank is less than the dimension [11]; (3) the symmetric rank is 1 or 2 [11]; (4) the flattening rank condition and Kruskal's condition hold [22]. For other cases, Comon's Conjecture remains open.

This paper is concerned with Comon's Conjecture, the rank decomposition, and the symmetric rank decomposition for symmetric tensors. In section 2, we present preliminaries on the rank and the rank decomposition of tensors. In section 3, for a symmetric tensor whose rank is less than its order, any rank decomposition is shown to be a symmetric rank decomposition. As a corollary, Comon's Conjecture is true for such tensors. In section 4, we present an example to show that a rank decomposition need not be symmetric in general. Furthermore, we give a positive answer to Comon's Conjecture for the case that the rank of a symmetric tensor is equal to its order.

2. Preliminaries.

In this paper, m, $n_1, n_2, \ldots, n_m$ are positive integers, $\mathtt{F}$ is the real number field $\mathtt{R}$ or complex number field $\mathtt{C}$, and Dim is the abbreviation for dimension of a vector space. $\mathtt{Span}(\{\ldots\})$ denotes the linear span of a set of vectors $\{\ldots\}$. $(\cdot)^*$ denotes the dual space of space $(\cdot)$. Throughout this paper, we assume that $m \geq 2$ and $n_i \geq 2$ $(i = 1, \ldots, m)$.

An m-order $(n_1 \times \cdots \times n_m)$-dimensional tensor $\mathcal{A} = (\mathcal{A}_{i_1 i_2 \ldots i_m})$ is a multidimensional array of entries $\mathcal{A}_{i_1 i_2 \ldots i_m} \in \mathtt{F}$ with $i_j = 1, \ldots, n_j$ and $j = 1, \ldots, m$. All such tensors form a linear space of dimension $n_1 \times n_2 \times \cdots \times n_m$, which is denoted by $\mathtt{F}^{n_1 \times \cdots \times n_m}$. In particular, such tensors are said to be square if $n_1 = \cdots = n_m := n$, which are then called m-order n-dimensional tensors. Let $\mathtt{T}^m(\mathtt{F}^n)$ be the space of all m-order n-dimensional square tensors. A square tensor $\mathcal{A} = (\mathcal{A}_{i_1 \ldots i_m}) \in \mathtt{T}^m(\mathtt{F}^n)$ is called symmetric if $\mathcal{A}_{i_1 \ldots i_m}$ is invariant under all permutations of $(i_1, \ldots, i_m)$. We will denote by $\mathtt{S}^m(\mathtt{F}^n)$ the space of all m-order n-dimensional symmetric tensors.

For any given tensor, fibers are defined by fixing every index but one; slices are second order sections, defined by fixing all but two indices. It is clear that any slice of a symmetric tensor is a symmetric matrix. The mode-k unfolding of tensor $\mathcal{A} \in \mathtt{F}^{n_1 \times \cdots \times n_m}$ is a matrix, denoted by $\mathcal{A}_{(k)}$, with entries

$$(\mathcal{A}_{(k)})_{ij} = \mathcal{A}_{i_1 \ldots i_{k-1} i i_{k+1} \ldots i_m}, \quad j = 1 + \sum_{l=1, l \neq k}^{n_l} (i_l - 1) J_l, \quad J_l = \prod_{l=1, l \neq k} n_l.$$

An m-order $(n_1 \times n_2 \times \cdots \times n_m)$-dimensional tensor $\mathcal{A} \in \mathtt{F}^{n_1 \times n_2 \times \cdots \times n_m}$ is called rank-1 if it can be written as an outer product of m vectors $x^{(i)} \in \mathtt{F}^{n_i}$ $(i = 1, 2, \ldots, m)$. We denote it by $\mathcal{A} = x^{(1)} \otimes x^{(2)} \otimes \cdots \otimes x^{(m)}$ with entries

$$\mathcal{A}_{i_1 i_2 \ldots i_m} = x^{(1)}_{i_1} x^{(2)}_{i_2} \cdots x^{(m)}_{i_m}, \quad i_j = 1, \ldots, n_m, \quad j = 1, \ldots, m.$$

Here, the symbol "$\otimes$" denotes the vector outer product. Furthermore, an m-order n-dimensional symmetric tensor $\mathcal{A} \in \mathtt{S}^m(\mathtt{F}^n)$ is called symmetric rank-1 if $\mathcal{A}$ can be written as $\mathcal{A} = \alpha x^{\otimes m} := \alpha \underbrace{x \otimes x \otimes \cdots \otimes x}_{m}$ for a vector $x \in \mathtt{F}^n$ and a scalar $\alpha \in \mathtt{F}$. The rank of tensor $\mathcal{A} \in \mathtt{F}^{n_1 \times n_2 \times \cdots \times n_m}$ is the smallest r such that

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \cdots + \mathcal{A}_r, \tag{2.1}$$

where $\mathcal{A}_i \in \mathtt{F}^{n_1 \times n_2 \times \cdots \times n_m}$ $(i = 1, 2, \ldots, r)$ are rank-1 tensors. For convenience, the rank of $\mathcal{A}$ is denoted by $r(\mathcal{A})$, and (2.1) is called a rank decomposition of $\mathcal{A}$. The symmetric rank of tensor $\mathcal{A} \in \mathtt{S}^m(\mathtt{F}^n)$ is the minimal number s (denoted as $r_S(\mathcal{A})$) such that

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \cdots + \mathcal{A}_s \tag{2.2}$$

for some symmetric rank-1 tensors $\mathcal{A}_i \in \mathtt{S}^m(\mathtt{F}^n)$ $(i = 1, 2, \ldots, s)$. Equation (2.2) is referred to as a symmetric rank decomposition. It is not hard to see that $r(\mathcal{A}) \le r_S(\mathcal{A})$ for any $\mathcal{A} \in \mathtt{S}^m(\mathtt{F}^n)$. However, it is unknown whether the equality holds, which is conjectured to be true in Comon's Conjecture.

To study Comon's Conjecture and the rank decomposition, we introduce a relation "$\sim$" between two vectors in $\mathtt{F}^n$. Specifically, for two nonzero vectors $x, y \in \mathtt{F}^n$, $x \sim y$ if and only if $x = \tau y$ for some nonzero scalar $\tau \in \mathtt{F}$. Clearly, such a relation is an equivalence relation. A set of vectors that are mutually equivalent is called an equivalence class. For a tensor $\mathcal{A} \in \mathtt{S}^m(\mathtt{F}^n)$ with $r_S(\mathcal{A}) = s$,

$$\mathcal{A} = \sum_{i=1}^{s} x^{(i,1)} \otimes x^{(i,2)} \otimes \cdots \otimes x^{(i,m)}$$

is also a symmetric rank decomposition if $x^{(i,j)} \sim x^{(i,1)}$ for $i = 1, \ldots, s$ and $j = 2, \ldots, m$.

For a linearly independent vector set $\{v_1, \ldots, v_t\} \subset \mathtt{F}^q$, there exist covectors $\phi_1, \ldots, \phi_t \in (\mathtt{F}^q)^*$ (also known as dual basis of dual space $(\mathtt{Span}(v_1, \ldots, v_t))^*$) that are dual to a $v_1, \ldots, v_t$, such that

$$\phi_k(v_i) = \delta_{ik} := \begin{cases} 1, & i = k, \\ 0, & i \ne k, \end{cases}$$

for $i, k = 1, \ldots, t$. Suppose that

$$\begin{aligned}
&I = \{j_1, \ldots, j_s\}, \\
&v_i = x^{(i,j_1)} \otimes x^{(i,j_2)} \otimes \cdots \otimes x^{(i,j_s)}, \quad i = 1, \ldots, t, \\
&\mathcal{A} = \sum_{i=1}^{t} x^{(i,1)} \otimes \cdots \otimes x^{(i,j_1)} \otimes \cdots \otimes x^{(i,j_s)} \otimes \cdots \otimes x^{(i,m)}.
\end{aligned}$$

By contracting $\mathcal{A}$ with covector ϕ_j $(j = 1, \ldots, t)$ in I-modes, we mean a tensor

$$\mathcal{A} \cdot_I \phi_k = \sum_{i=1}^{t} \phi_k(v_i) x^{(i,\overline{j_1})} \otimes \cdots \otimes x^{(i,\overline{j_{m-s}})} = x^{(k,\overline{j_1})} \otimes \cdots \otimes x^{(k,\overline{j_{m-s}})},$$

where $\{\overline{j_1}, \ldots, \overline{j_{m-s}}\} = \{1, \ldots, m\} \backslash I$. Clearly, $\mathcal{A} \cdot_I \phi_k$ is a symmetric rank-1 tensor if $\mathcal{A}$ is symmetric.

The mode-k inner product $\mathcal{A} \cdot_k x \in \mathtt{F}^{n_1 \times n_{k-1} \times n_{k+1} \times \cdots \times n_m}$ is defined between a tensor $\mathcal{A} \in \mathtt{F}^{n_1 \times n_2 \times \cdots \times n_m}$ and a vector $x \in \mathtt{F}^{n_k}$ with entries

$$(\mathcal{A} \cdot_k x)_{j_1 j_2 \ldots j_{m-1}} = \sum_{i=1}^{n_k} \mathcal{A}_{j_1 \ldots j_{k-1} i j_{k+1} \ldots j_m} x_i, \quad j_l = 1, \ldots, n_j, l = 1, \ldots, k-1, k, \ldots, m.$$

If $\mathcal{A}$ is symmetric, $\mathcal{A} \cdot_k x$ is symmetric too. The multilinear transformation of tensor $\mathcal{A} \in \mathtt{F}^{n_1 \times n_2 \times \cdots \times n_m}$ by matrices $P^i \in \mathtt{F}^{\bar{n}_i \times n_i}$, $i = 1, 2, \ldots, m$, is a tensor $\mathcal{Y} = (P^1, \ldots, P^m) \cdot \mathcal{A} \in \mathtt{F}^{\bar{n}_1 \times \bar{n}_2 \times \cdots \times \bar{n}_m}$, whose entries are

$$\mathcal{Y}_{i_1 i_2 \ldots i_m} = \sum_{j_1=1}^{n_1} \cdots \sum_{j_m=1}^{n_m} P^1_{i_1 j_1} P^2_{i_2 j_2} \ldots P^m_{i_m j_m} \mathcal{X}_{j_1 j_2 \ldots j_m}, \quad i_l = 1, 2, \ldots, \bar{n}_l, \quad l = 1, \ldots, m.$$

3. Rank decomposition and symmetric rank decomposition. In this section, the relationship between the rank decomposition and the symmetric rank decomposition is investigated for any symmetric tensor with its rank being less than its order. To begin with, we present several properties of the rank decomposition.

LEMMA 3.1. *Let*

$$\mathcal{A} = \sum_{i=1}^{r} x^{(i,1)} \otimes x^{(i,2)} \otimes \cdots \otimes x^{(i,m)} \tag{3.1}$$

be a rank decomposition of $\mathcal{A} \in \mathtt{F}^{n_1 \times \cdots \times n_m}$. Then for any index set $J = \{j_1, j_2, \ldots, j_{m-1}\} \subset \{1, 2, \ldots, m\}$ with $|J| = m - 1$, the set

$$\{x^{(i,j_1)} \otimes x^{(i,j_2)} \otimes \cdots \otimes x^{(i,j_{m-1})} \,|\, i = 1, 2, \ldots, r\}$$

is linearly independent.

Proof. This result is a corollary of Proposition 2.4 of [5], and the proof is omitted here. ∎

LEMMA 3.2. *For $\mathcal{A} \in \mathtt{S}^m(\mathtt{F}^n)$, let* (3.1) *be a rank decomposition of $\mathcal{A}$ with $r \geq 2$ and $W := \mathtt{Span}(\{x^{(1,j)}, x^{(2,j)}, \ldots, x^{(r,j)}\})$ for some $j \in \{1, \ldots, m\}$. Then $x^{(i,k)} \in W$ for $i = 1, 2, \ldots, r$ and $k = 1, 2, \ldots, m$.*

Proof. This result can be regarded as a direct corollary of Proposition 3.1.3.1 of [20] and section 3.1 in [4]. Hence the proof is omitted here. ∎

COROLLARY 3.3. *Let $\mathcal{A} \in \mathtt{S}^m(\mathtt{F}^n)$ and* (3.1) *be a rank decomposition of $\mathcal{A}$ with $r \geq 2$. Then there is no index k such that $x^{(i,k)} \sim x^{(j,k)}$ for all $i, j \in \{1, \ldots, r\}$ and $i \neq j$.*

Proof. Otherwise, we assume without loss of generality that $x^{(i,1)} \sim x^{(j,1)}$ for $i, j \in \{1, \ldots, r\}$. Letting $W = \mathtt{Span}(\{x^{(1,1)}, x^{(2,1)}, \ldots, x^{(r,1)}\})$, we have $\mathrm{Dim}(W) = 1$. From Lemma 3.2, $x^{(i,j)} \in W$ for $i = 1, \ldots, r$ and $j = 1, \ldots, m$. We then have $r(\mathcal{A}) = 1$, which contradicts the assumption $r(\mathcal{A}) \geq 2$. The desired result is established now. ∎

Concerned with the relationship between the rank decomposition and the symmetric rank decomposition, let us begin with the case $r(\mathcal{A}) = 1$ or 2.

LEMMA 3.4. *Let $\mathcal{A} \in \mathcal{S}^m(\mathbb{F}^n)$ $(m \geq 3)$ and $r(\mathcal{A}) = 1$ or 2. Then any rank decomposition of $\mathcal{A}$ is a symmetric rank decomposition.*

Proof. Case 1. $r(\mathcal{A}) = 1$. Let (3.1) be a rank decomposition with $r = 1$. Since $\mathcal{A}$ is symmetric, any slice of $\mathcal{A}$ is a symmetric matrix. This indicates that $x^{(1,i)} \sim x^{(1,1)}$, $i = 2, \ldots, m$, and the result holds clearly.

Case 2. $r(\mathcal{A}) = 2$. Let (3.1) be a rank decomposition of $\mathcal{A}$ with $r = 2$. From Corollary 3.3, $x^{(1,j)} \not\sim x^{(2,j)}$ for $j \in \{1, \ldots, m\}$. For the case $j = 1$, vectors $x^{(1,1)}, x^{(2,1)}$ are linearly independent. Hence, there exist covectors ϕ_1, ϕ_2 such that $\phi_i(x^{(k,1)}) = \delta_{ik}$. Contracting $\mathcal{A}$ with ϕ_i in $\{1\}$-modes, we get

$$\mathcal{A} \cdot_1 \phi_i = \sum_{k=1}^{2} \phi_i(x^{(k,1)}) x^{(k,2)} \otimes \cdots \otimes x^{(k,m)} = x^{(i,2)} \otimes \cdots \otimes x^{(i,m)},$$

which is a symmetric rank-1 tensor. From Case 1, $x^{(i,2)} \sim x^{(i,l)}$ for $i = 1, 2$ and $l = 3, \ldots, m$. In the same way, for the case $j = 2$, we can show that $x^{(i,1)} \sim x^{(i,l)}$ for $l = 3, \ldots, m$ and $i = 1, 2$. So we can assert that $x^{(i,1)} \sim x^{(i,l)}$ for $l = 2, \ldots, m$ and $i = 1, 2$. This completes the proof. ∎

Proposition 5.5 of [11] gives a similar conclusion under the condition $r_S(\mathcal{A}) = 1$ or 2. As $r(\mathcal{A}) \leq r_S(\mathcal{A})$, Lemma 3.4 is then an extension of Proposition 5.5 of [11]. To generalize above result, we need the following lemmas.

LEMMA 3.5. *For $i = 1, \ldots, r$, let $x^{(i,j)} \in \mathbb{F}^{n_j} \backslash \{0\}$, $j = s+1, \ldots, m$, and $\mathcal{B}_i \in \mathbb{F}^{n_1 \times n_2 \times \cdots \times n_s}$ be s-order tensors. If the tensors*

$$\mathcal{B}_i \otimes x^{(i,s+1)} \otimes \cdots \otimes x^{(i,m)}, i = 1, 2, \ldots, r$$

are linearly independent and $\mathrm{Dim}\mathtt{Span}(\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_r\}) = p < r$, *then there exists an index j_0, $s + 1 \leq j_0 \leq m$, such that*

$$\mathrm{Dim}\mathtt{Span}(\{\mathcal{B}_1 \otimes x^{(1,j_0)}, \mathcal{B}_2 \otimes x^{(2,j_0)}, \ldots, \mathcal{B}_r \otimes x^{(r,j_0)}\}) > p.$$

Proof. Without loss of generality, the set $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p\}$ is assumed to be linearly independent and $\mathcal{B}_j$ $(p+1 \leq j \leq r)$ can be expressed linearly by $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$. From Lemma 1 of [28], we have

$$\mathrm{Dim}\mathtt{Span}(\{\mathcal{B}_1 \otimes x^{(1,j)}, \mathcal{B}_2 \otimes x^{(2,j)}, \ldots, \mathcal{B}_p \otimes x^{(p,j)}\}) = p. \tag{3.2}$$

Hence

$$\mathrm{Dim}\mathtt{Span}(\{\mathcal{B}_1 \otimes x^{(1,j)}, \mathcal{B}_2 \otimes x^{(2,j)}, \ldots, \mathcal{B}_r \otimes x^{(r,j)}\}) \geq p, \quad j = s+1, \ldots, m. \tag{3.3}$$

Next, we will show that the inequality (3.3) holds strictly for some $j \in \{s + 1, \ldots, m\}$. Otherwise, for any given k $(k > p)$ and $j = s+1, \ldots, m$, there exists a nonzero p-tuple $(\beta_{kj1}, \beta_{kj2}, \ldots, \beta_{kjp}) \in \mathbb{F}^p$ such that

$$\mathcal{B}_k \otimes x^{(k,j)} = \sum_{l=1}^{p} \beta_{kjl} \mathcal{B}_l \otimes x^{(l,j)}.$$

Contracting both sides in $(s+1)$-modes with y satisfying $y^\top x^{(k,j)} = 0$, we have

$$0 = (y^\top x^{(k,j)})\mathcal{B}_k = \sum_{l=1}^{p} \beta_{kjl}(y^\top x^{(l,j)})\mathcal{B}_l.$$

By the linear independence of $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_p$, it follows that $y^\top x^{(k,j)} = 0$ for $\beta_{kjl} \neq 0$. By the choice of y, we have

$$x^{(k,j)} \sim x^{(l,j)} \text{ for all } \ l = 1, \ldots, p \ \text{ and } \ \beta_{kjl} \neq 0. \tag{3.4}$$

That is, if $\beta_{kjl} \neq 0$, there exists $\gamma_{kjl} \neq 0$ such that $x^{(l,j)} = \gamma_{kjl}x^{(k,j)}$. Then $\mathcal{B}_k = \sum_{l=1}^{p} \gamma_{kjl}\beta_{kjl}\mathcal{B}_l$.

Noting that both $\{\mathcal{B}_l \,|\, l = 1, \ldots, p\}$ and $\{\mathcal{B}_l \otimes x^{(l,j)} \,|\, l = 1, \ldots, p\}$ are linearly independent, the coefficients β_{kjl} and $\beta_{kjl}\gamma_{kjl}$ are uniquely determined. For simplicity, $\gamma_{kjl}\beta_{kjl}$ is denoted by τ_{kl}. By construction, if $\tau_{kl} \neq 0$, then $\gamma_{jkl} \neq 0$ for any j. From (3.4), we have

$$\mathcal{B}_k \otimes x^{(k,s+1)} \otimes x^{(k,s+2)} \otimes \cdots \otimes x^{(k,m)} = \sum_{l=1}^{p} \left(1/ \prod_{j=s+1}^{m} \gamma_{kjl} \right) \tau_{kl}\mathcal{B}_l \otimes x^{(l,s+1)} \otimes \cdots \otimes x^{(l,m)}.$$

This contradicts the assumption. Hence, (3.3) holds strictly for some j and the conclusion follows. ∎

LEMMA 3.6. *Let $x_i \in \mathtt{F}^n, i = 1, \ldots, r$, be nonzero vectors and $x_i \not\sim x_j$ for $1 \le i \neq j \le r$. If $r_k := \mathrm{Dim}\mathtt{Span}(\{x_1^{\otimes k}, x_2^{\otimes k}, \ldots, x_r^{\otimes k}\}) < r$, then $r_{k+1} := \mathrm{Dim}\mathtt{Span}(\{x_1^{\otimes (k+1)}, x_2^{\otimes (k+1)}, \ldots, x_r^{\otimes (k+1)}\}) > r_k$.*

Proof. Suppose that the set $\{x_1^{\otimes k}, x_2^{\otimes k}, \ldots, x_p^{\otimes k}\}$ is linearly independent and $p < r$. We now show that the set $\{x_1^{\otimes (k+1)}, x_2^{\otimes (k+1)}, \ldots, x_{p+1}^{\otimes (k+1)}\}$ is also linearly independent. Otherwise, there exists a nonzero p-tuple $(\alpha_1, \ldots, \alpha_p) \in \mathtt{F}^p$ such that

$$x_{p+1}^{\otimes (k+1)} = \sum_{i=1}^{p} \alpha_i x_i^{\otimes (k+1)}.$$

For any nonzero vector $z \in \mathtt{F}^n$ with $z^\top x_{p+1} = 0$, we have by contracting above tensor in 1-modes with z

$$\sum_{i=1}^{p} \alpha_i (z^\top x_i) x_i^{\otimes k} = 0.$$

From the linear independence of $\{x_i^{\otimes k} \,|\, i = 1, \ldots, p\}$, we have $z^\top x_i = 0$ for $\alpha_i \neq 0$. Hence, if $\alpha_i \neq 0$, it holds that $x_i \sim x_{p+1}$ and a contradiction follows. ∎

By Lemma 3.6, Corollary 4.4 of [11] is improved as follows.

COROLLARY 3.7. *Let $x_i \in \mathtt{F}^n, i = 1, \ldots, r$, be nonzero vectors and $x_i \not\sim x_j$ for $1 \le i \neq j \le r$. Then for any $m \ge r - \bar{r} + 1$, $\{x_1^{\otimes m}, x_2^{\otimes m}, \ldots, x_r^{\otimes m}\}$ is linearly independent, where $\bar{r} = \mathrm{Dim}\mathtt{Span}(\{x_1, \ldots, x_r\})$.*

We are ready to present our main result of this section.

THEOREM 3.8. *Let $\mathcal{A} \in \mathcal{S}^m(\mathtt{F}^n)$ and $m \ge 3$. If $r := r(\mathcal{A}) < m$, then any rank decomposition of $\mathcal{A}$ is a symmetric rank decomposition.*

Proof. From Lemma 3.4, the result holds for the case $r \leq 2$. It suffices to consider the case $m > r \geq 3$. Let (3.1) be a rank decomposition of $\mathcal{A}$ with $r \geq 3$.

Set $j_1 = 1$. By Corollary 3.3, $r_1 := \mathtt{DimSpan}(\{x^{(1,1)}, x^{(2,1)}, \ldots, x^{(r,1)}\}) \geq 2$. By Lemma 3.5, there exists a $j_2 \in \{2, \ldots, m\}$ such that $r_2 := \mathtt{DimSpan}(\{x^{(1,1)} \otimes x^{(1,j_2)}, x^{(2,1)} \otimes x^{(2,j_2)}, \ldots, x^{(r,1)} \otimes x^{(r,j_2)}\}) > r_1$. Clearly, $r_2 \geq 3$. Continuing this procedure, an index set $I_s = \{j_1, j_2, \ldots, j_s\}$ $(s < r)$ can be found such that $r = r_s > r_{s-1} > \cdots > r_1 \geq 2$ and $r_k \geq k+1$, $k = 1, 2, \ldots, s$, hold. Here, $r_k = \mathtt{DimSpan}(\{x^{(i,j_1)} \otimes x^{(i,j_2)} \otimes \cdots \otimes x^{(i,j_k)} \,|\, i = 1, 2, \ldots r\}), k = 1, \ldots, s$. It is easy to check $s \leq r-1 \leq m-2$ from the fact $r = r_s \geq s+1$.

For simplicity, let $J_s = \{1, \ldots, m\} \backslash I_s := \{j_{s+1}, \ldots, j_m\}$ and denote for $k = 1, 2, \ldots, s$

$$\left\{ \begin{array}{lcl} \mathcal{A}_{i,k} & = & x^{(i,j_1)} \otimes x^{(i,j_2)} \otimes \cdots \otimes x^{(i,j_k)}, \\ \mathcal{B}_{i,k} & = & x^{(i,j_{k+1})} \otimes x^{(i,j_{k+2})} \otimes \cdots \otimes x^{(i,j_m)}. \end{array} \right.$$

Based on these notations, $\mathcal{A}$ can be rewritten as

$$\mathcal{A} = \sum_{i=1}^{r} \mathcal{A}_{i,s} \otimes \mathcal{B}_{i,s}.$$

From the linear independence of $\{\mathcal{A}_{1,s}, \ldots, \mathcal{A}_{r,s}\}$, a set of covectors $\{\phi_i\}_{i=1,\ldots,r}$ can be found, which is dual to $\{\mathcal{A}_{1,s}, \ldots, \mathcal{A}_{r,s}\}$. That is, $\phi_j(\mathcal{A}_{i,s}) = \delta_{ij}$ for $i, j = 1, 2, \ldots, r$. Contracting $\mathcal{A}$ in I_s-modes with ϕ_j $(j = 1, 2, \ldots, s)$, we have

$$\mathcal{A} \cdot_{I_s} \phi_j = \sum_{i=1}^{r} \phi_j(\mathcal{A}_{i,s}) \mathcal{B}_{i,s} = \mathcal{B}_{j,s},$$

which is a rank-1 symmetric tensor. By Lemma 3.4, $x^{(i,j_k)} \sim x^{(i,j_m)}$, $k = s+1, \ldots, m$. Hence, for $i = 1, 2, \ldots, r$, $\mathcal{B}_{i,s}$ has the form $\mathcal{B}_{i,s} = \alpha_i y_i^{\otimes(m-s)}$ with $y_i \in \mathtt{F}^n$ and $\alpha_i \neq 0$. We thus have the following rank decomposition:

$$\mathcal{A} = \sum_{i=1}^{r} \alpha_i \mathcal{A}_{i,s} \otimes y_i^{\otimes(m-s)} = \sum_{i=1}^{r} \alpha_i \mathcal{A}_{i,s-1} \otimes x^{(i,j_s)} \otimes \mathcal{B}_{i,s}. \tag{3.5}$$

By Lemma 3.1, $\{\mathcal{A}_{i,s-1} \otimes y_i^{\otimes(m-s)} \,|\, i = 1, 2, \ldots, r\}$ is linearly independent. Applying Lemma 3.5 to vectors $\mathcal{A}_{i,s-1} \otimes y_i^{\otimes(m-s)} (i = 1, 2, \ldots, r)$ several times, we have the linear independence of $\{\mathcal{A}_{i,s-1} \otimes y_i^{\otimes(r-r_{s-1})} \,|\, i = 1, 2, \ldots, r\}$. Therefore, there exists a set of covectors $\{\nu_i\}_{i=1,\ldots,r}$ which is dual to $\{\mathcal{A}_{i,s-1} \otimes y_i^{\otimes(r-r_{s-1})} \,|\, i = 1, 2, \ldots, r\}$. That is, $\nu_i(\mathcal{A}_{j,s-1} \otimes y_j^{\otimes(r-r_{s-1})}) = \delta_{ij}$, $i, j = 1, \ldots, r$. Hence,

$$\begin{aligned} \mathcal{A} \cdot_{(I_{s-1} \bigcup \{j_{s+1}\})} \nu_i &= \sum_{j=1}^{r} (\nu_i(\mathcal{A}_{j,s-1} \otimes y_j^{\otimes(r-r_{s-1})})) y_j^{\otimes(m-s-(r-r_{s-1}))} \otimes x^{(j,j_s)} \\ &= y_i^{\otimes(m-s-(r-r_{s-1}))} \otimes x^{(i,j_s)} \end{aligned}$$

is a rank-1 symmetric tensor. Here, $m - s - (r - r_{s-1}) = m - r + (r_{s-1} - s) \geq m - r \geq 1$. Therefore, $x^{(i,j_s)} \sim y_i$ and $\mathcal{A} = \sum_{i=1}^{r} \beta_i \mathcal{A}_{i,s-1} \otimes y_i^{\otimes(m-s+1)}$ for some $\beta_i \in \mathtt{F}$, $i = 1, 2, \ldots, r$.

Repeating the above procedure from $s-1$ to 2, we can show

$$\mathcal{A} = \sum_{i=1}^{r} \beta_i x^{(i,1)} \otimes y_i^{\otimes(m-1)} = \sum_{i=1}^{r} \beta_i y_i^{\otimes(m-1)} \otimes x^{(i,1)}.$$

Form Lemma 3.1, $y_1^{\otimes(m-1)}, \ldots, y_r^{\otimes(m-1)}$ are linearly independent. Moreover, $y_i \not\sim y_j$ for any $1 \leq i \neq j \leq r$. Otherwise, $y_i \sim y_j$ will lead to $y_i^{\otimes(m-1)} \sim y_j^{\otimes(m-1)}$. Applying Corollary 3.7 to vectors $y_1, \ldots, y_r$, $\{y_1^{\otimes(r-1)}, \ldots, y_r^{\otimes(r-1)}\}$ is then linearly independent. So there is a set of covectors $\{\bar{\nu}_i\}_{i=1,\ldots,r}$ dual to $\{y_i^{\otimes(r-1)}\}_{i=1,\ldots,r}$ such that $\bar{\nu}_i(y_j^{\otimes(r-1)}) = \delta_{ij}$ for $i, j = 1, 2, \ldots, r$. Contracting $\mathcal{A}$ in $\bar{I}$-modes with $\bar{\nu}_i$ ($\bar{I} = \{1, \ldots, r-1\}$), we have

$$\mathcal{A} \cdot_{\bar{I}} \bar{\nu}_i = \sum_{j=1}^{r} \beta_i(\bar{\nu}_i(y_j^{\otimes r-1}))y_i^{\otimes(m-r)} \otimes x^{(j,1)} = \beta_i y_i^{\otimes(m-r)} \otimes x^{(i,1)},$$

which is a rank-1 symmetric tensor. So we can assert that $y_i \sim x^{(i,1)}$ and the decomposition (3.1) is symmetric. ∎

The following corollaries give a positive answer to Comon's Conjecture.

Corollary 3.9. *Let $\mathcal{A} \in \mathcal{S}^m(\mathtt{F}^n)$ and $r(\mathcal{A}) < m$. Then Comon's Conjecture is true.*

Corollary 3.10. *Let $\mathcal{A} \in \mathcal{S}^m(\mathtt{F}^n)$ and $r_S(\mathcal{A}) \leq m$. Then Comon's Conjecture is true.*

The following lemma can be found in [12, 20].

Lemma 3.11. *For any binary symmetric tensor of order m, its symmetric rank is not larger than m.*

By Corollary 3.10 and Lemma 3.11, we have the following result.

Corollary 3.12. *For any binary symmetric tensor, Comon's Conjecture is true.*

To end this section, we consider the following example.

Example 3.13. Consider a 3-order two-dimensional symmetric tensor $\mathcal{A}$ with non-zero elements

$$\mathcal{A}_{111} = -1, \quad \mathcal{A}_{122} = 1.$$

Shown in [11], $r_S(\mathcal{A}) = 3$ over the real field $\mathtt{R}$ with a symmetric rank decomposition

$$\mathcal{A} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\otimes 3} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}^{\otimes 3} - 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes 3}, \tag{3.6}$$

and $r_S(\mathcal{A}) = 2$ over the complex field $\mathtt{C}$ with a symmetric rank decomposition

$$\mathcal{A} = \frac{\sqrt{-1}}{2}\begin{pmatrix} -\sqrt{-1} \\ 1 \end{pmatrix}^{\otimes 3} - \frac{\sqrt{-1}}{2}\begin{pmatrix} \sqrt{-1} \\ 1 \end{pmatrix}^{\otimes 3}. \tag{3.7}$$

We shall show below that $r_S(\mathcal{A}) = r(\mathcal{A})$ over both $\mathtt{C}$ and $\mathtt{R}$.

In fact, from Lemma 3.1, it is easy to see that $r(\mathcal{A}) \neq 1$ over $\mathtt{C}$ and $\mathtt{R}$. This implies that $r(\mathcal{A}) = 2$ over $\mathtt{C}$. Hence, Comon's Conjecture is true for $\mathcal{A}$ over $\mathtt{C}$. On the other hand, since $3 = r_S(\mathcal{A}) \geq r(\mathcal{A}) \geq 2$ over $\mathtt{R}$, it suffices to show that $r(\mathcal{A}) \neq 2$ over $\mathtt{R}$ by contradiction. By a proof similar to that of Lemma 3.4, we know that a rank decomposition is a symmetric rank decomposition for tensor $\mathcal{A}$. Hence, it is established that $r(\mathcal{A}) = 3$, and Comon's Conjecture is true for $\mathcal{A}$ over $\mathtt{R}$.

Remark 3.14. It is well known that the rank and the symmetric rank of a (symmetric) tensor may be different in different fields; see section 3.1 in [19]. However, under the conditions of Corollaries 3.9 and 3.10, the rank and the symmetric rank of a symmetric tensor coincide regardless of the fields $\mathtt{F}$.

4. Comon's Conjecture for the case that the rank is equal to the order. For any symmetric tensor with the rank being less than its order, any rank decomposition is shown in the above section to be a symmetric rank decomposition. This section is concerned with the case that the rank is equal to the order. To begin with, let us consider the following example.

EXAMPLE. Let $a, b \in \mathtt{F}^n$ be nonzero vectors and $a \not\sim b$. For $m \geq 3$, introduce the following m-order n-dimensional symmetric tensor:

$$\mathcal{A} = a \otimes b \otimes b \otimes \cdots \otimes b + b \otimes a \otimes b \otimes b \otimes \cdots \otimes b + \cdots + b \otimes b \otimes b \otimes \cdots \otimes a. \tag{4.1}$$

Then by Proposition 5.6 of [11], we have $r_S(\mathcal{A}) = m$. From Corollary 3.10, $r(\mathcal{A}) = m$. Hence, (4.1) is a rank decomposition of $\mathcal{A}$ and is not a symmetric rank decomposition of $\mathcal{A}$.

Therefore, for a symmetric tensor whose rank is larger than or equal to its order, a rank decomposition need not be a symmetric rank decomposition. To proceed, we ask, what is the symmetric rank of a symmetric tensor with its rank being its order? To answer this question, we need the following lemma.

LEMMA 4.1. *Let $\mathcal{A} \in \mathtt{S}^m(\mathtt{F}^n)$ with $r(\mathcal{A}) = m \geq 3$. If* (3.1) *is a rank decomposition of $\mathcal{A}$ with $r = m$ and* $\mathrm{Dim}\mathtt{Span}(\{x^{(1,1)}, x^{(2,1)}, \ldots, x^{(m,1)}\}) \geq 3$*, then* (3.1) *is a symmetric rank decomposition.*

Proof. Similarly to the proof of Theorem 3.8, an index set $I = \{j_1, j_2, \ldots, j_s\} \subset \{1, 2, \ldots, m\}$ can be found such that $m = r_s > r_{s-1} > \cdots > r_1 \geq 3$ and $r_k \geq k + 2$. Here, $r_k := \mathrm{Dim}\mathtt{Span}(\{\mathcal{A}_{i,k} = x^{(i,j_1)} \otimes x^{(i,j_2)} \otimes \cdots \otimes x^{(i,j_k)} | i = 1, 2, \ldots r\})$ for $k = 1, 2, \ldots, s$. It is easy to check that $s \leq m - 2$. The rest of the proof is similar to that of Theorem 3.8 and is omitted here. ∎

THEOREM 4.2. *Let $\mathcal{A} \in \mathtt{S}^m(\mathtt{F}^n)$ and $r(\mathcal{A}) = m$. Then $r_S(\mathcal{A}) = m$.*

Proof. From Lemma 4.1, it suffices to consider the case $\mathrm{Dim}\mathtt{Span}(\{x^{(1,1)}, x^{(1,2)}, \ldots, x^{(m,m)}\}) = 2$. To simplify the notation, we denote $\mathtt{Span}(\{x^{(1,1)}, x^{(1,2)}, \ldots, x^{(m,m)}\}) = \mathtt{Span}(\{p, q\})$ for two nonzero vectors $p, q \in \mathtt{F}^n$. Let $x^{(i,j)} = Pa^{(i,j)}$, where $P = [p, q]$ is a matrix and $a^{(i,j)}$ are two-dimensional vectors. Then, it holds that

$$\mathcal{A}x^m = \sum_{i=1}^{m} \prod_{j=1}^{m} (x^\top x^{(i,j)}) = \sum_{i=1}^{m} \prod_{j=1}^{m} ((P^\top x)^\top a^{(i,j)}) = \mathcal{T} y^m.$$

Here, $x \in \mathtt{F}^n$, $y = P^\top x \in \mathtt{F}^2$, and

$$\mathcal{T} = \sum_{i=1}^{m} a^{(i,1)} \otimes a^{(i,2)} \otimes \cdots \otimes a^{(i,m)} = (P^\top, P^\top, \ldots, P^\top) \cdot \mathcal{A}$$

is an m-order two-dimensional symmetric tensor. From Proposition 3.1.3.1 of [20] and its symmetric version, we have $r(\mathcal{A}) = r(\mathcal{T})$ and $r_S(\mathcal{A}) = r_S(\mathcal{T})$.

In view of binary tensor $\mathcal{T}$ and Lemma 3.11, we have $r_S(\mathcal{T}) \leq m$, which implies $r_S(\mathcal{A}) \leq m$. Together with the fact $r_S(\mathcal{A}) \geq r(\mathcal{A}) = m$, one obtains immediately that $r_S(\mathcal{A}) = m$, and the statement is established now. ∎

Based on this result, Corollaries 3.9 and 3.10 can be improved as follows.

COROLLARY 4.3. *For $\mathcal{A} \in \mathtt{S}^m(\mathtt{F}^n)$, Comon's Conjecture is true if $r(\mathcal{A}) \leq m$ or $r_S(\mathcal{A}) \leq m + 1$.*

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