A NOTE ON THE 4-GIRTH-THICKNESS OF $K_{n,n,n}$

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ABSTRACT. The 4-girth-thickness $\theta(4, G)$ of a graph G is the minimum number of planar subgraphs of girth at least four whose union is G. In this paper, we obtain that the 4-girth-thickness of complete tripartite graph $K_{n,n,n}$ is $\lceil \frac{n+1}{2} \rceil$ except for $\theta(4, K_{1,1,1}) = 2$. And we also show that the 4-girth-thickness of the complete graph K_{10} is three which disprove the conjecture posed by Rubio-Montiel concerning to $\theta(4, K_{10})$.

1. Introduction

The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G. It was defined by W.T.Tutte [10] in 1963. Then, the thicknesses of some graphs have been obtained when the graphs are hypercube [7], complete graph [1, 2, 11], complete bipartite graph [3] and some complete multipartite graphs [5, 12, 13].

In 2017, Rubio-Montiel [8] define the g-girth-thickness $\theta(g, G)$ of a graph G as the minimum number of planar subgraphs whose union is G with the girth of each subgraph is at least g. It is a generalization of the usual thickness in which the 3-girth-thickness $\theta(3, G)$ is the usual thickness $\theta(G)$. He also determined the 4-girth-thickness of the complete graph K_n except K_{10} and he conjecture that $\theta(4, K_{10}) = 4$. Let $K_{n,n,n}$ denote a complete tripartite graph in which each part contains n ($n \geq 1$) vertices. In [13], Yang obtained $\theta(K_{n,n,n}) = \left\lceil \frac{n+1}{3} \right\rceil$ when $n \equiv 3 (\mathrm{mod}\ 6)$.

In this paper, we determine $\theta(4, K_{n,n,n})$ for all values of n and we also give a decomposition of K_{10} with three planar subgraphs of girth at least four, which shows $\theta(4, K_{10}) = 3$.

2. THE 4-GIRTH-THICKNESS OF $K_{n,n,n}$

Lemma 1. [4] *A planar graph with n vertices and girth g has at most $\frac{g}{g-2}(n-2)$ edges.*

Theorem 2. *The 4-girth-thickness of $K_{n,n,n}$ is*

$$\theta(4, K_{n,n,n}) = \left\lceil \frac{n+1}{2} \right\rceil$$

except for $\theta(4, K_{1,1,1}) = 2$.

2010 *Mathematics Subject Classification.* 05C10.

Key words and phrases. thickness; 4-girth-thickness; complete tripartite graph.

This work was supported by NNSF of China under Grant No. 11401430.

Proof. It is trivial for $n = 1$, $\theta(4, K_{1,1,1}) = 2$. When $n > 1$, because $|E(K_{n,n,n})| = 3n^2$, $|V(K_{n,n,n})| = 3n$, from Lemma 1, we have

$$\theta(4, K_{n,n,n}) \geq \left\lceil \frac{3n^2}{2(3n-2)} \right\rceil = \left\lceil \frac{n}{2} + \frac{1}{3} + \frac{2}{3(3n-2)} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil.$$

In the following, we give a decomposition of $K_{n,n,n}$ into $\lceil \frac{n+1}{2} \rceil$ planar subgraphs of girth at least four to complete the proof. Let the vertex partition of $K_{n,n,n}$ be (U, V, W), where $U = \{u_1, \ldots, u_n\}$, $V = \{v_1, \ldots, v_n\}$ and $W = \{w_1, \ldots, w_n\}$. In this proof, all the subscripts of vertices are taken modulo $2p$.

Case 1. When $n = 2p$ $(p \geq 1)$.

Let $G_1, \ldots, G_{p+1}$ be the graphs whose edge set is empty and vertex set is the same as $V(K_{2p,2p,2p})$.

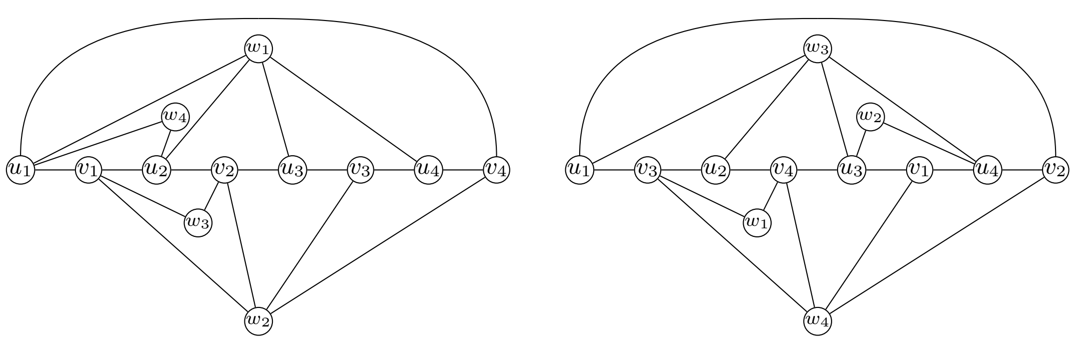
Step 1: For each G_i $(1 \leq i \leq p)$, arrange all the vertices $u_1, v_{3-2i}, u_2, v_{4-2i}, u_3, v_{5-2i}, \ldots, u_{2p}, v_{2p-2i+2}$ on a circle and join u_j to v_{j+2-2i} and v_{j+1-2i}, $1 \leq j \leq 2p$. Then we get a cycle of length $4p$, denote it by G_i^1 $(1 \leq i \leq p)$.

Step 2: For each G_i^1 $(1 \leq i \leq p)$, place the vertex w_{2i-1} inside the cycle and join it to $u_1, \ldots, u_{2p}$, place the vertex w_{2i} outside the cycle and join it to $v_1, \ldots, v_{2p}$. Then we get a planar graph G_i^2 $(1 \leq i \leq p)$.

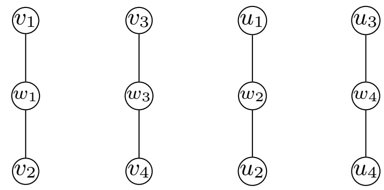
Step 3: For each G_i^2 $(1 \leq i \leq p)$, place vertices w_{2j} for $1 \leq j \leq p$ and $j \neq i$, inside of the quadrilateral $w_{2i-1}u_{2i-1}v_1u_{2i}$ and join each of them to vertices u_{2i-1} and u_{2i}. Place vertices w_{2j-1}, for $1 \leq j \leq p$ and $j \neq i$, inside of the quadrilateral $w_{2i}v_{2i-1}u_kv_{2i}$, in which u_k is some vertex from U. Join each of them to vertices v_{2i-1} and v_{2i}. Then we get a planar graph $\overline{G}_i$ $(1 \leq i \leq p)$.

Step 4: For G_{p+1}, join w_{2i-1} to both v_{2i-1} and v_{2i}, join w_{2i} to both u_{2i-1} and u_{2i}, for $1 \leq i \leq p$, then we get a planar graph $\overline{G}_{p+1}$.

For $\overline{G}_1 \cup \cdots \cup \overline{G}_{p+1} = K_{n,n,n}$, and the girth of $\overline{G}_i$ $(1 \leq i \leq p+1)$ is at least four, we obtain a 4-girth planar decomposition of $K_{2p,2p,2p}$ with $p+1$ planar subgraphs. Figure 1 shows a 4-girth planar decomposition of $K_{4,4,4}$ with three planar subgraphs.



(a) The graph $\overline{G}_1$ (b) The graph $\overline{G}_2$



(c) The graph $\overline{G}_3$

FIGURE 1. A 4-girth planar decomposition of $K_{4,4,4}$

Case 2. When $n = 2p + 1$ $(p > 1)$.

Base on the 4-girth planar decomposition $\{\overline{G}_1, \cdots, \overline{G}_{p+1}\}$ of $K_{2p,2p,2p}$, by adding vertices and edges to each $\overline{G}_i$ $(1 \le i \le p+1)$ and some other modifications on it, we will get a 4-girth planar decomposition of $K_{2p+1,2p+1,2p+1}$ with $p+1$ subgraphs.

Step 1: (Add u to $\overline{G}_i, 1 \le i \le p$) For each $\overline{G}_i$ $(1 \le i \le p)$, we notice that the order of the $p-1$ interior vertices w_{2j}, $1 \le j \le p$, and $j \ne i$ in the quadrilateral $w_{2i-1}u_{2i-1}v_1u_{2i}$ of $\overline{G}_i$ has no effect on the planarity of $\overline{G}_i$. We adjust the order of them, such that $w_{2i-1}u_{2i-1}w_{2p-2i+2}u_{2i}$ is a face of a plane embedding of $\overline{G}_i$. Place the vertex u in this face and join it to both w_{2i-1} and $w_{2p-2i+2}$. We denote the planar graph we obtain by $\widehat{G}_i$ $(1 \le i \le p)$.

Step 2: (Add v and w to $\widehat{G}_1$) Delete the edge v_1u_2 in $\widehat{G}_1$, put both v and w in the face $w_ku_1v_1w_tv_2u_2$ in which w_k is some vertex from $\{w_{2j} \mid 1 < j \le p\}$ and w_t is some vertex from $\{w_{2j-1} \mid 1 < j \le p\}$. Join v to w, join v to u_1, u_2, and join w to v_1, v_2, we get a planar graph $\widetilde{G}_1$.

Step 3: (Add v and w to $\widehat{G}_i, 2 \le i \le p$) For each $\widehat{G}_i$ $(2 \le i \le p)$, place the vertex v in the face $w_ku_{2i-1}v_1u_{2i}$ in which w_k is some vertex from $\{w_{2j} \mid 1 \le j \le p \text{ and } j \ne i\}$, and join it to u_{2i-1} and u_{2i}. Place the vertex w in the face $w_kv_{2i-1}u_tv_{2i}$ in which w_k is some vertex from $\{w_{2j-1} \mid 1 \le j \le p \text{ and } j \ne i\}$ and u_t is some vertex from U. Join w to both v_{2i-1} and v_{2i}, we get a planar graph $\widetilde{G}_i$ $(2 \le i \le p)$.

Step 4: (Add u, v and w to $\overline{G}_{p+1}$) We add u, v and w to $\overline{G}_{p+1}$. For $1 \le i \le 2p$, join u to each v_i, join v to each w_i, join w to each u_i, join u to both v and w, and join v_1 to u_2, then we get a planar graph $\widetilde{G}_{p+1}$. Figure 2 shows a plane embedding of $\widetilde{G}_{p+1}$.

For $\widetilde{G}_1 \cup \cdots \cup \widetilde{G}_{p+1} = K_{n,n,n}$, and the girth of $\widetilde{G}_i$ $(1 \le i \le p+1)$ is at least four, we obtain a 4-girth planar decomposition of $K_{2p+1,2p+1,2p+1}$ with $p+1$ planar subgraphs. Figure 3 shows a 4-girth planar decomposition of $K_{5,5,5}$ with three planar subgraphs.

Case 3. When $n = 3$, Figure 4 shows a 4-girth planar decomposition of $K_{3,3,3}$ with two planar subgraphs.

Summarizing the above, the theorem is obtained. □

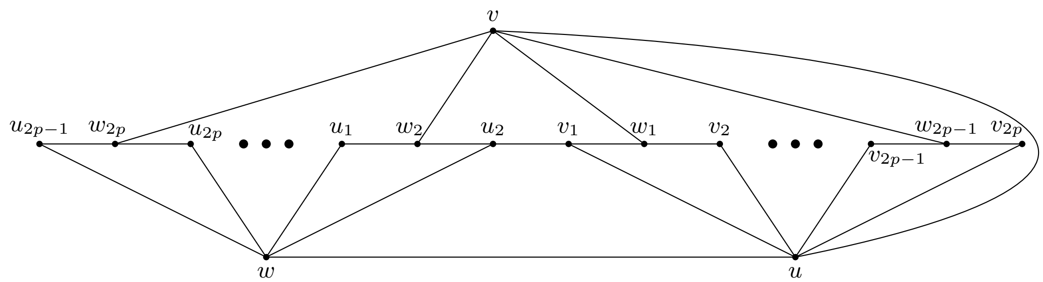
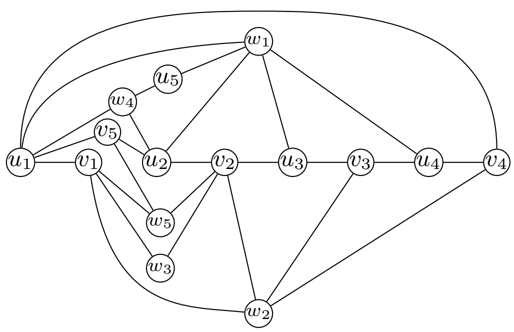
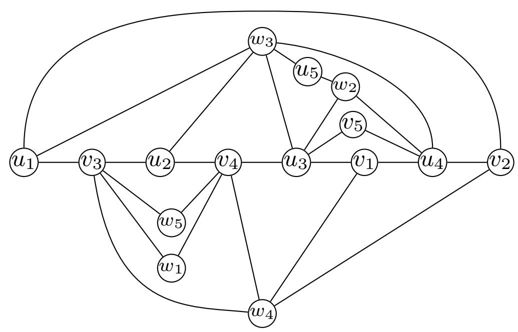


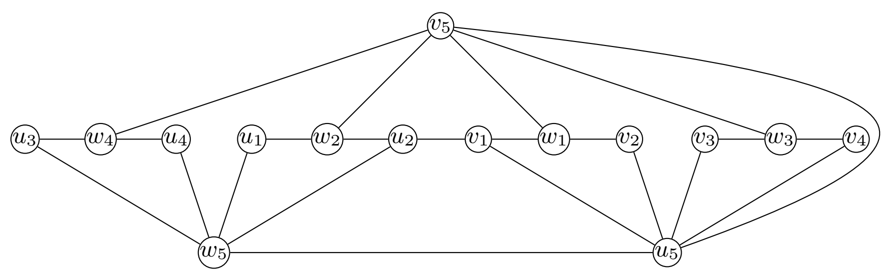
FIGURE 2. The graph $\widetilde{G}_{p+1}$



(a) The graph $\widetilde{G}_1$



(b) The graph $\widetilde{G}_2$



(c) The graph $\widetilde{G}_3$

FIGURE 3. A 4-girth planar decomposition of $K_{5,5,5}$

3. THE 4-GIRTH-THICKNESS OF K_{10}

In [8], the author posed the question whether $\theta(4, K_{10}) = 3$ or 4, and conjectured that it is four. We disprove his conjecture by showing $\theta(4, K_{10}) = 3$.

Theorem 3. *The* 4*-girth-thickness of* K_{10} *is three.*

Proof. From [8], we have $\theta(4, K_{10}) \geq 3$. We draw a 4-girth planar decomposition of K_{10} with three planar subgraphs in Figure 5, which shows $\theta(4, K_{10}) \leq 3$. The theorem follows.

□

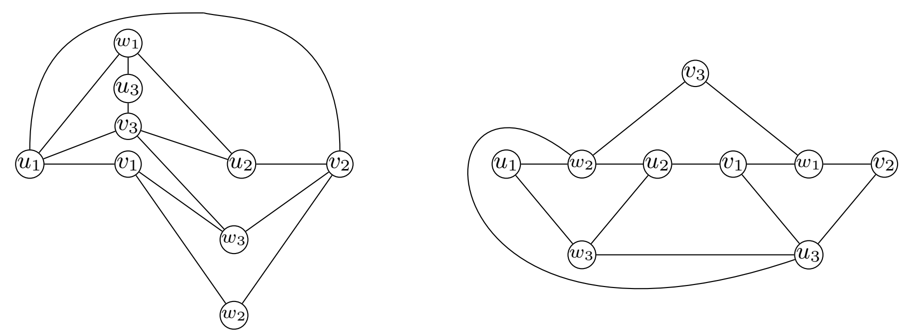


FIGURE 4. A 4-girth planar decomposition of $K_{3,3,3}$

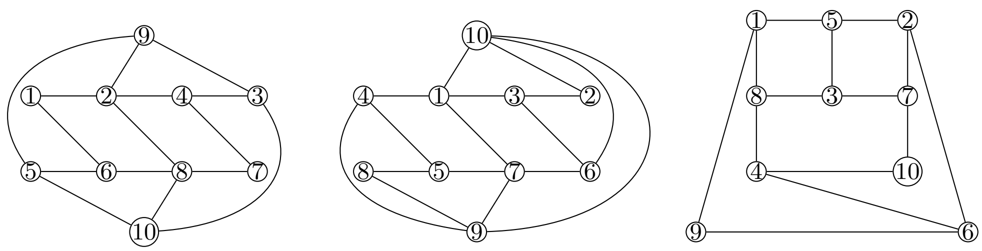


FIGURE 5. A 4-girth planar decomposition of K_{10}

We would like to state that after submitting this paper for review, we notice that there exist two results regarding the 4-girth-thickness of $K_{2p,2p,2p}$ and K_{10}. Rubio-Montiel [9] obtained the exact value of the 4-girth-thickness of the complete multipartite graph when each part has an even number of vertices. And by computer, Castañeda-López et al. [6] found the other two decompositions of K_{10} into three planar subgraphs of girth at least four. In this paper, we give these results in a constructive way.

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