Collect. Math. (2018) 69:437–449
https://doi.org/10.1007/s13348-018-0216-z

Disjoint hypercyclic weighted pseudo-shifts on Banach sequence spaces

Ya Wang[1] · **Ze-Hua Zhou**[2]

Received: 17 July 2017 / Accepted: 8 February 2018 / Published online: 22 February 2018

Abstract In this article, we characterize the disjoint hypercyclicity of finite weighted pseudo-shifts on an arbitrary Banach sequence space. Moreover, we obtain some interesting consequences of this characterization.

Keywords Disjoint hypercyclic · Weighted pseudo-shifts · Disjoint Blow-up/Collapse Property

Mathematics Subject Classification 47A16 · 47B38 · 46E15

1 Introduction

Let $\mathbb{N}$ denote the set of non-negative integers, $\mathbb{Z}$ denote the set of all integers. Let X be a separable, infinite dimensional Banach space over the real or complex scalar field $\mathbb{K}$. $L(X)$ denote the space of bounded, linear operators on X. An operator $T \in L(X)$ is said to be *hypercyclic* if there is a vector $x \in X$ such that the orbit $\mathrm{Orb}(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in X. In such a case, x is called a *hypercyclic vector* for T. Recently, Bernal-González [2], Bès and Peris [7] introduced the notion of disjoint hypercyclicity independently. We say that $N \geq 2$, hypercyclic operators $T_1, T_2, \ldots, T_N$ on the space X are called *disjoint hypercyclic* (in short, *d-hypercyclic*) if their direct sum $\oplus_{m=1}^{N} T_m$ has a hypercyclic vector of the form

The work was supported in part by the National Natural Science Foundation of China (Grant Nos. 11771323; 11371276).

✉ Ze-Hua Zhou
zehuazhoumath@aliyun.com; zhzhou@tju.edu.cn

Ya Wang
wangyasjxsy0802@163.com

1 School of Mathematics, Tianjin University, Tianjin 300354, People's Republic of China

2 School of Mathematics, Tianjin University, Tianjin 300350, People's Republic of China

$(x, x, \cdots, x)$ in X^N. Such a vector x is called a *d-hypercyclic vector* for $T_1, T_2, \ldots, T_N$. If the set of d-hypercyclic vectors is dense in X, we say $T_1, T_2, \ldots, T_N$ are *densely d-hypercyclic*.

As it turns out, some well known dynamical properties for a single operator fail to hold true in the disjoint setting. Such as, Bès et al. [4] showed that any d-hypercyclic weighted backward shifts $B_1, B_2, \ldots, B_N$ with $N \geq 2$ satisfy the Disjoint Blow-up/Collapse Property, but they never satisfy the d-Hypercyclicity Criterion. This is in contrast with the fact that the Hypercyclic Criterion and the Blow-up/Collapse Property are equivalent for a single operator; see Bernal-González and Grosse-Erdmann [8] or León-Saavedra [14]. For a single operator $T \in L(X)$, Kitai [13] demonstrated that the existence of just one hypercyclic vector implies that the set of hypercyclic vectors is a dense G_δ subset of X. However, the same cannot be said in the disjoint setting. In [17], Sanders and Shkarin proved that there exist d-hypercyclic operators T_1, T_2 which fail to be densely d-hypercyclic. For more on disjoint hypercyclicity we refer papers [3–6, 11, 15–19]. For more background and examples about hypercyclicity, we refer the books by Bayart and Matheron [1] and by Grosse-Erdmann and Peris Manguillot [9].

In the present note, we consider the disjoint hypercyclicity of finite weighted pseudo-shifts on the same Banach sequence space. We find that the disjoint hypercyclicity of these operators is equivalent to the Disjoint Blow-up/Collapse Property, thus generalizing a result in paper [4]; see Theorem 2.1 and Example 2.2 in Sect. 2. In Sect. 3, we establish some consequences of Theorem 2.1. In particular, applying one of the consequences we obtain that any finite bilateral weighted backward shifts on the space $\ell^2(\mathbb{Z})$ never satisfy the d-Hypercyclicity Criterion.

We now recall some important definitions and results in disjoint hypercyclicity which are used throughout the paper.

Definition 1.1 We say that the operators $T_1, T_2, \ldots, T_N$ in $L(X)$ with $N \geq 2$ are *d-topologically transitive* if for any non-empty open subsets $V_0, V_1, \ldots, V_N$ in X, there exists a positive integer m so that

$$V_0 \cap T_1^{-m}(V_1) \cap T_2^{-m}(V_2) \cap \cdots \cap T_N^{-m}(V_N) \neq \emptyset.$$

Proposition 1.2 [7, Proposition 2.3] *Let $T_1, T_2, \ldots, T_N$ be operators in $L(X)$ with $N \geq 2$. The following statements are equivalent:*

(1) *The operators $T_1, T_2, \ldots, T_N$ are d-topologically transitive;*
(2) *The set of d-hypercyclic vectors for the operators $T_1, T_2, \ldots, T_N$ is a dense G_δ subset of X.*

Definition 1.3 We say that the operators $T_1, T_2, \ldots, T_N$ in $L(X)$ with $N \geq 2$ satisfy the *Disjoint Blow-up/Collapse Property* if for any non-empty open subsets $W, V_0, V_1, \ldots, V_N$ in X with $0 \in W$, there exists a positive integer m so that

$$W \cap T_1^{-m}(V_1) \cap T_2^{-m}(V_2) \cap \cdots \cap T_N^{-m}(V_N) \neq \emptyset \text{ and}$$
$$V_0 \cap T_1^{-m}(W) \cap T_2^{-m}(W) \cap \cdots \cap T_N^{-m}(W) \neq \emptyset.$$

Theorem 1.4 [7, Theorem 2.7] *Let $T_1, T_2, \ldots, T_N$ be operators in $L(X)$ with $N \geq 2$. The following statements are equivalent:*

(a) *The operators $T_1, T_2, \ldots, T_N$ satisfy the d-Hypercyclicity Criterion.*
(b) *For each $r \in \mathbb{N}$, the direct sum operators $\overbrace{T_1 \oplus \cdots \oplus T_1}^{r}, \ldots, \overbrace{T_N \oplus \cdots \oplus T_N}^{r}$ are d-topologically transitive on X^r.*

Proposition 1.5 [7, Proposition 2.4] *If the operators $T_1, T_2, \dots, T_N$ in $L(X)$ with $N \geq 2$ satisfy the Disjoint Blow-up/Collapse Property, then they are d-topologically transitive and hence densely d-hypercyclic.*

For further discussions, we recall some definitions about the sequence space and weighted pseudo-shift. For a comprehensive survey we recommend Grosse-Erdmann's paper [10].

Definition 1.6 (*Sequence Space*) If we allow an arbitrary countably infinite set I as an index set, then a *sequence space over I* is a subspace of the space $\omega(I) = \mathbb{K}^I$ of all scalar families $(x_i)_{i\in I}$. The space $\omega(I)$ is endowed with its natural product topology.

A *topological sequence space X over I* is a sequence space over I that is endowed with a linear topology in such a way that the inclusion mapping $X \hookrightarrow \omega(I)$ is continuous or, equivalently, that every *coordinate functional* $f_i : X \to \mathbb{K}, (x_k)_{k\in I} \mapsto x_i (i \in I)$ is continuous. A *Banach* (Hilbert) *sequence space over* I is a topological sequence space over I that is a Banach (Hilbert) space.

Definition 1.7 (*OP-basis*) By $(e_i)_{i\in I}$ we denote the canonical unit vectors $e_i = (\delta_{ik})_{k\in I}$ in a topological sequence space X over I. We say $(e_i)_{i\in I}$ is an $OP-basis$ or $(Ovsepian\ Pelczyński\ basis)$ if $\operatorname{span}\{e_i : i \in I\}$ is a dense subspace of X and the family of $coordinate\ projections$ $x \mapsto x_i e_i (i \in I)$ on X is equicontinuous.

Definition 1.8 (*Weighted Pseudo-shift*) Let X be a Banach sequence space over I. Then a continuous linear operator $T : X \to X$ is called a *weighted pseudo-shift* if there is a sequence $(b_i)_{i\in I}$ of non-zero scalars and an injective mapping $\varphi : I \to I$ such that

$$T(x_i)_{i\in I} = (b_i x_{\varphi(i)})_{i\in I}$$

for $(x_i) \in X$. We then write $T = T_{b,\varphi}$, and $(b_i)_{i\in I}$ is called the *weight sequence*.

Remark 1.9 (1) If $T = T_{b,\varphi} : X \to X$ is a weighted pseudo-shift, then each $T^n (n \geq 1)$ is also a weighted pseudo-shift as follows

$$T^n (x_i)_{i\in I} = (b_{n,i} x_{\varphi^n(i)})_{i\in I}$$

where

$$\varphi^n(i) = (\varphi \circ \varphi \circ \cdots \circ \varphi)(i) \quad (n-\text{fold})$$

$$b_{n,i} = b_i b_{\varphi(i)} \cdots b_{\varphi^{n-1}(i)} = \prod_{v=0}^{n-1} b_{\varphi^v(i)}.$$

(2) If φ is an invertible mapping with inverse ψ. Then for each $i \in I$ and any integer n with $n \geq 1$,

$$T^n_{b,\varphi} e_i = \prod_{v=1}^{n} b_{\psi^v(i)} e_{\psi^n(i)}.$$

Specifically, if $n = 1$,

$$T_{b,\varphi} e_i = b_{\psi(i)} e_{\psi(i)}.$$

Definition 1.10 Let $\varphi : I \to I$ be an injective mapping. We call $(\varphi^n)_n$ a *run-away sequence* if for each pair of finite subsets $I_0 \subset I$ and $J_0 \subset I$, there exists an $n_0 \in \mathbb{N}$ such that $\varphi^n(J_0) \bigcap I_0 = \emptyset$ for every $n \geq n_0$.

2 Disjoint hypercyclicity of weighted pseudo-shifts

The following theorem is the main result in this section, which provides equivalent conditions for the disjoint hypercyclic weighted pseudo-shifts on an arbitrary Banach sequence space.

Theorem 2.1 *Let X be a Banach sequence space over I in which $(e_i)_{i\in I}$ is an OP-basis. Let $\varphi : I \to I$ be an invertible mapping with inverse ψ. $N \geq 2$, for each $1 \leq l \leq N$, let $T_l = T_{b^{(l)},\varphi} : X \to X$ be a weighted pseudo-shift with weight sequence $b^{(l)} = (b_i^{(l)})_{i\in I}$. For each $i \in I$ and integers n, l with $n \geq 1$ and $2 \leq l \leq N$, define*

$$\alpha_{i,n}^{(l)} = \prod_{v=0}^{n-1} \frac{b_{\varphi^v(i)}^{(l)}}{b_{\varphi^v(i)}^{(1)}}.$$

Then the following assertions are equivalent:

(1) *$T_1, T_2, \ldots, T_N$ are d-hypercyclic.*
(2) *$T_1, T_2, \ldots, T_N$ satisfy the Disjoint Blow-up/Collapse Property.*
(3) *There exists a strictly increasing sequence $(n_k)_{k\geq 0}$ of positive integers such that for every $i \in I$ and integer l with $1 \leq l \leq N$, we have*

$$\begin{cases} \lim\limits_{k\to\infty} \left\| \left(\prod\limits_{v=0}^{n_k-1} b_{\varphi^v(i)}^{(1)} \right)^{-1} e_{\varphi^{n_k}(i)} \right\| = 0, \\ \lim\limits_{k\to\infty} \left\| \left(\prod\limits_{v=1}^{n_k} b_{\psi^v(i)}^{(l)} \right) e_{\psi^{n_k}(i)} \right\| = 0 \end{cases}$$

and the set

$$\left\{ \left((\alpha_{i,n_k}^{(2)})_{i\in I}, (\alpha_{i,n_k}^{(3)})_{i\in I}, \ldots, (\alpha_{i,n_k}^{(N)})_{i\in I} \right) : k \geq 0 \right\}$$

is dense in $\underbrace{\omega(I) \times \cdots \times \omega(I)}_{N-1}$ with respect to the product topology for $\omega(I) = \mathbb{K}^I$.

Proof (1) $\Rightarrow$ (3). Suppose $T_1, T_2, \cdots, T_N$ are d-hypercyclic, then for each $1 \leq l \leq N$, T_l is a hypercyclic operator on X. Referencing the proof of Theorem 5 in paper [10], we get that $(\varphi^n)_n$ is a run-away sequence. Let

$$\left\{ \left((\lambda_{i,k}^{(2)})_{i\in I}, \ldots, (\lambda_{i,k}^{(N)})_{i\in I} \right) \right\}_{k\geq 0}$$

be a countable dense subset of $\underbrace{\omega(I) \times \cdots \times \omega(I)}_{N-1}$ with respect to the product topology. As by the hypothesis I is a countably infinite set, we fix $I = \{i_0, i_1, i_2, \ldots\}$ and set $I_k = \{i_0, i_1, i_2, \ldots, i_k\}$ for each $k \in \mathbb{N}$. For each integer $k \geq 0$, select a positive real number $C_k > 1$ such that $\max\{|\lambda_{i,k}^{(l)}| : 2 \leq l \leq N \text{ and } i \in I_k\} < C_k$.

Let k be any nonnegative integer fixed. Since $(\varphi^n)_n$ is run-away, there exists an integer $m_k \in \mathbb{N}$ such that for every $n \geq m_k$,

$$\varphi^n(I_k) \bigcap I_k = \emptyset. \tag{2.1}$$

By the equicontinuity of the coordinate projections in X, there is some $\delta_k > 0$ so that for $x = (x_i)_{i\in I} \in X$,

$$\|x_i e_i\| < \frac{1}{2^{k+1}} \text{ for all } i \in I, \text{ if } ||x|| < \delta_k. \tag{2.2}$$

Let $\delta'_k = \min\{\delta_k, \frac{1}{2^{k+1}}\}$. Since $T_1, T_2, \cdots, T_N$ are d-hypercyclic, there is a d-hypercyclic vector $g \in X$ for $T_1, T_2, \ldots, T_N$ and a positive integer $r_k > m_k$ such that

$$\|g\| < \delta'_k \text{ and } \left\| T_l^{r_k} g - \sum_{i\in I_k} e_i \right\| < \delta'_k \text{ for each } 1 \le l \le N. \tag{2.3}$$

By the continuous inclusion of X into $\omega(I)$, we can in addition obtain that

$$\sup_{i\in I_k} \left|\left(T_l^{r_k} g\right)_i - 1\right| \le \frac{1}{2} \quad (1 \le l \le N), \tag{2.4}$$

where $\left(T_l^{r_k} g\right)_i$ denotes the scalar at position i of $T_l^{r_k} g$.

Let $A_k = \{\frac{1}{C_k \cdot 2^{k+1}} \cdot \left|\prod_{v=1}^{r_k} b^{(l)}_{\psi^v(i)}\right| \cdot \|e_{\psi^{r_k}(i)}\| : i \in I_k, 1 \le l \le N\}$ and $\varepsilon_k = \min\{A_k, \delta'_k\}$.

Again by the equicontinuity of the coordinate projections in X, there is some δ''_k with $0 < \delta''_k < \delta'_k$ so that for $x = (x_i)_{i\in I} \in X$,

$$\|x_i e_i\| < \varepsilon_k \text{ for all } i \in I, \text{ if } ||x|| < \delta''_k. \tag{2.5}$$

Since g is a d-hypercyclic vector, we can choose a positive integer $n_k > 2r_k$ (indeed, the selection of n_k can be sufficiently large) such that

$$\left\| T_1^{n_k+r_k} g - \sum_{i\in I_k} \left(\prod_{v=1}^{r_k} b^{(1)}_{\psi^v(i)}\right) e_{\psi^{r_k}(i)} \right\| < \delta''_k \tag{2.6}$$

and for each integer l with $2 \le l \le N$,

$$\left\| T_l^{n_k+r_k} g - \sum_{i\in I_k} \lambda^{(l)}_{i,k} \left(\prod_{v=1}^{r_k} b^{(l)}_{\psi^v(i)}\right) e_{\psi^{r_k}(i)} \right\| < \delta''_k. \tag{2.7}$$

By (2.1), $\psi^{n_k}(I_k) \bigcap \psi^{r_k} I_k = \emptyset$. Then we see from (2.5) and (2.6) that for each $i \in I_k$,

$$\left\| (T_1^{r_k} g)_i \prod_{v=1}^{n_k} b^{(1)}_{\psi^v(i)} e_{\psi^{n_k}(i)} \right\| = \left\| (T_1^{n_k+r_k} g)_{\psi^{n_k}(i)} e_{\psi^{n_k}(i)} \right\| < \varepsilon_k \le \frac{1}{2^{k+1}}, \tag{2.8}$$

where $(T_1^{n_k+r_k} g)_{\psi^{n_k}(i)}$ denotes the scalar at position $\psi^{n_k}(i)$ of $T_1^{n_k+r_k} g$.

Similarly, inequality (2.7) implies that for each $i \in I_k$ and $2 \le l \le N$,

$$\left\| (T_l^{r_k} g)_i \prod_{v=1}^{n_k} b^{(l)}_{\psi^v(i)} e_{\psi^{n_k}(i)} \right\| = \left\| (T_l^{n_k+r_k} g)_{\psi^{n_k}(i)} e_{\psi^{n_k}(i)} \right\| < \frac{1}{2^{k+1}}. \tag{2.9}$$

It follows from (2.4) that for any $i \in I_k$ and $1 \le l \le N$

$$\frac{2}{3} \le \frac{1}{(T_l^{r_k} g)_i} \le 2. \tag{2.10}$$

Thus by (2.8), (2.9) and (2.10) we have

$$\left\| \prod_{v=1}^{n_k} b^{(l)}_{\psi^v(i)} e_{\psi^{n_k}(i)} \right\| < \frac{1}{2^k} \quad \text{for all } i \in I_k \text{ and } 1 \le l \le N. \tag{2.11}$$

Next observe that for each $i \in I_k$ and $1 \le l \le N$

$$\begin{aligned}(T_l^{n_k+r_k} g)_{\psi^{r_k}(i)} &= \prod_{v=0}^{n_k+r_k-1} b^{(l)}_{\varphi^v(\psi^{r_k}(i))} g_{\varphi^{n_k}(i)} \\ &= \prod_{v=1}^{r_k} b^{(l)}_{\psi^v(i)} \prod_{v=0}^{n_k-1} b^{(l)}_{\varphi^v(i)} g_{\varphi^{n_k}(i)}.\end{aligned}$$

So by (2.5) and (2.6) we know that for each $i \in I_k$,

$$\begin{aligned}&\left\| \left(\prod_{v=1}^{r_k} b^{(1)}_{\psi^v(i)} \prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)} g_{\varphi^{n_k}(i)} - \prod_{v=1}^{r_k} b^{(1)}_{\psi^v(i)} \right) e_{\psi^{r_k}(i)} \right\| \\ &< \varepsilon_k \le \frac{1}{C_k \cdot 2^{k+1}} \cdot \left| \prod_{v=1}^{r_k} b^{(1)}_{\psi^v(i)} \right| \cdot \|e_{\psi^{r_k}(i)}\|,\end{aligned}$$

which gives

$$\left| \prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)} g_{\varphi^{n_k}(i)} - 1 \right| < \frac{1}{C_k \cdot 2^{k+1}} < \frac{1}{2^{k+1}}. \tag{2.12}$$

Hence $g_{\varphi^{n_k}(i)} \ne 0$ and

$$\left| \frac{1}{\prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)} g_{\varphi^{n_k}(i)}} \right| \le 2,$$

this together with (2.2) and the first inequality of (2.3), we obtain that for each $i \in I_k$,

$$\begin{aligned}\left\| \left(\prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)} \right)^{-1} e_{\varphi^{n_k}(i)} \right\| &= \left\| \left(\prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)} g_{\varphi^{n_k}(i)} \right)^{-1} g_{\varphi^{n_k}(i)} e_{\varphi^{n_k}(i)} \right\| \\ &\le 2 \left\| g_{\varphi^{n_k}(i)} e_{\varphi^{n_k}(i)} \right\| < \frac{1}{2^k}.\end{aligned} \tag{2.13}$$

Likewise, by (2.5) and (2.7) we get that for each $i \in I_k$ and $2 \le l \le N$,

$$\begin{aligned}&\left\| \left(\prod_{v=1}^{r_k} b^{(l)}_{\psi^v(i)} \prod_{v=0}^{n_k-1} b^{(l)}_{\varphi^v(i)} g_{\varphi^{n_k}(i)} - \lambda^{(l)}_{i,k} \prod_{v=1}^{r_k} b^{(l)}_{\psi^v(i)} \right) e_{\psi^{r_k}(i)} \right\| \\ &< \frac{1}{C_k \cdot 2^{k+1}} \cdot \left| \prod_{v=1}^{r_k} b^{(l)}_{\psi^v(i)} \right| \cdot \|e_{\psi^{r_k}(i)}\|.\end{aligned}$$

It follows that

$$\left| \prod_{v=0}^{n_k-1} b^{(l)}_{\varphi^v(i)} g_{\varphi^{n_k}(i)} - \lambda^{(l)}_{i,k} \right| < \frac{1}{2^{k+1}}. \tag{2.14}$$

Noting that inequality (2.12) also implies

$$\frac{1}{\left|\prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)}\right|} \leq \frac{|g_{\varphi^{n_k}(i)}|}{1-\frac{1}{2^{k+1}}}.$$

Therefore by (2.12) and (2.14), for each $i \in I_k$ and $2 \leq l \leq N$ we have

$$\begin{aligned}
|\alpha^{(l)}_{i,n_k} - \lambda^{(l)}_{i,k}| &= \left|\prod_{v=0}^{n_k-1} \frac{b^{(l)}_{\varphi^v(i)}}{b^{(1)}_{\varphi^v(i)}} - \lambda^{(l)}_{i,k}\right| \\
&= \frac{1}{\left|\prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)}\right|}\left|\prod_{v=0}^{n_k-1} b^{(l)}_{\varphi^v(i)} - \lambda^{(l)}_{i,k}\prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)}\right| \\
&\leq \frac{|g_{\varphi^{n_k}(i)}|}{1-\frac{1}{2^{k+1}}}\left|\prod_{v=0}^{n_k-1} b^{(l)}_{\varphi^v(i)} - \lambda^{(l)}_{i,k}\prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)}\right| \\
&\leq \frac{2^{k+1}}{2^{k+1}-1}\left|g_{\varphi^{n_k}(i)}\prod_{v=0}^{n_k-1} b^{(l)}_{\varphi^v(i)} - \lambda^{(l)}_{i,k}\right| \\
&\quad + \frac{2^{k+1}}{2^{k+1}-1}|\lambda^{(l)}_{i,k}|\left|1-\prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)} g_{\varphi^{n_k}(i)}\right| \\
&< \frac{2^{k+1}}{2^{k+1}-1}\left(\frac{1}{2^{k+1}} + C_k\frac{1}{C_k\cdot 2^{k+1}}\right) = \frac{2}{2^{k+1}-1}.
\end{aligned} \tag{2.15}$$

With the above argument, we can define inductively a strictly increasing sequence $(n_k)_{k\geq 0}$ of positive integers by letting n_k be a positive integer satisfying (2.11), (2.13) and (2.15) for each $k \in \mathbb{N}$. It is clear that $(n_k)_{k\geq 0}$ satisfies the conditions in statement (3). Since for each $i \in I$, there exists a positive integer n_0' such that $i \in I_k$ for all $k > n_0'$. Hence for any integer k with $k > n_0'$ we have

$$\begin{cases}
\left\|\left(\prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)}\right)^{-1} e_{\varphi^{n_k}(i)}\right\| < \frac{1}{2^k}, \\
\left\|\prod_{v=1}^{n_k} b^{(l)}_{\psi^v(i)} e_{\psi^{n_k}(i)}\right\| < \frac{1}{2^k} \quad \text{for } 1 \leq l \leq N, \\
|\alpha^{(l)}_{i,n_k} - \lambda^{(l)}_{i,k}| < \frac{2}{2^{k+1}-1} \quad \text{for } 2 \leq l \leq N.
\end{cases}$$

Then the result follows from the fact that

$$\left\|\left(\prod_{v=0}^{n_k-1} b^{(1)}_{\varphi^v(i)}\right)^{-1} e_{\varphi^{n_k}(i)}\right\| \to 0, \quad \left\|\left(\prod_{v=1}^{n_k} b^{(l)}_{\psi^v(i)}\right) e_{\psi^{n_k}(i)}\right\| \to 0 \text{ for } 1 \leq l \leq N,$$

and $|\alpha^{(l)}_{i,n_k} - \lambda^{(l)}_{i,k}| \to 0$ for $2 \leq l \leq N$, as $k \to \infty$.

(3) $\Rightarrow$ (2). Suppose $T_1, T_2, \ldots, T_N$ satisfy conditions in statement (3) with respect to the sequence $(n_k)_{k\geq 0}$. Let $V_0, V_1, \ldots, V_N, W$ be any non-empty open subsets of X with $0 \in W$.

To complete the proof of (2), we show that there is some integer $n_{k_0} \in (n_k)_{k \geq 0}$ such that

$$W \cap T_1^{-n_{k_0}}(V_1) \cap T_2^{-n_{k_0}}(V_2) \cap \cdots \cap T_N^{-n_{k_0}}(V_N) \neq \emptyset \tag{2.16}$$

and

$$V_0 \cap T_1^{-n_{k_0}}(W) \cap T_2^{-n_{k_0}}(W) \cap \cdots \cap T_N^{-n_{k_0}}(W) \neq \emptyset. \tag{2.17}$$

Since $\text{span}\{e_i : i \in I\}$ is dense in X, there exists an integer $r \geq 1$, vectors $h_0, h_1, \ldots, h_N \in \text{span}\{e_i : i \in I_r\}$ and an $\varepsilon > 0$ such that $B(0; \varepsilon) \subset W$ and $B(h_m; \varepsilon) \subset V_m$ for $0 \leq m \leq N$. For each $0 \leq m \leq N$, put $h_m = \sum_{i \in I_r} h_{m,i} e_i$. We further assume that $h_{1,i} \neq 0$ for all $i \in I_r$, this can be done just by adding a small perturbation to h_1.

Now we define the linear map $S : \text{span}\{e_i : i \in I\} \to \text{span}\{e_i : i \in I\}$ by

$$Se_i = (b_i^{(1)})^{-1} e_{\varphi(i)}.$$

An easy calculation gives that, for each $i \in I$, integers n, m with $n \geq 1$ and $2 \leq m \leq N$,

$$S^n(e_i) = \left(\prod_{v=0}^{n-1} b_{\varphi^v(i)}^{(1)} \right)^{-1} e_{\varphi^n(i)}, \tag{2.18}$$

$$T_1^n S^n e_i = e_i \tag{2.19}$$

and

$$T_m^n S^n e_i = \prod_{v=0}^{n-1} \frac{b_{\varphi^v(i)}^{(m)}}{b_{\varphi^v(i)}^{(1)}} e_i = \alpha_{i,n}^{(m)} e_i. \tag{2.20}$$

By equation (2.18) and the fact that $\|h_{1,i} e_i\| \neq 0$ for $i \in I_r$, we can find a positive integer $n_{k_0} \in (n_k)_{k \geq 0}$ such that

$$\|S^{n_{k_0}} h_1\| = \left\| \sum_{i \in I_r} h_{1,i} \left(\prod_{v=0}^{n_{k_0}-1} b_{\varphi^v(i)}^{(1)} \right)^{-1} e_{\varphi^{n_{k_0}}(i)} \right\| < \varepsilon, \tag{2.21}$$

$$\|T_m^{n_{k_0}} h_0\| = \left\| \sum_{i \in I_r} h_{0,i} \left(\prod_{v=1}^{n_{k_0}} b_{\psi^v(i)}^{(m)} \right) e_{\psi^{n_{k_0}}(i)} \right\| < \varepsilon \ \text{ for each } 1 \leq m \leq N \tag{2.22}$$

and for every $i \in I_r$, $2 \leq m \leq N$,

$$|\alpha_{i,n_{k_0}}^{(m)} - \frac{h_{m,i}}{h_{1,i}}| < \frac{\varepsilon}{(r+1) M_1 M_2}$$

where $M_1 = \max\{\|e_i\| : i \in I_r\}$, $M_2 = \max\{|h_{1,i}| : i \in I_r\}$. It follows that

$$|h_{1,i} \alpha_{i,n_{k_0}}^{(m)} - h_{m,i}| = |h_{1,i}| |\alpha_{i,n_{k_0}}^{(m)} - \frac{h_{m,i}}{h_{1,i}}| < \frac{\varepsilon}{(r+1) M_1} \ (i \in I_r, 2 \leq m \leq N). \tag{2.23}$$

Therefore by (2.22) we have that

$$h_0 \in V_0 \cap T_1^{-n_{k_0}}(W) \cap T_2^{-n_{k_0}}(W) \cap \cdots \cap T_N^{-n_{k_0}}(W).$$

By (2.20) and (2.23), for each $2 \le m \le N$ we have

$$\|T_m^{n_{k_0}} S^{n_{k_0}} h_1 - h_m\| = \left\| \sum_{i \in I_r} (h_{1,i} \alpha_{i,n_{k_0}}^{(m)} - h_{m,i}) e_i \right\| \le \sum_{i \in I_r} |h_{1,i} \alpha_{i,n_{k_0}}^{(m)} - h_{m,i}| \|e_i\| < \varepsilon. \tag{2.24}$$

Thus by (2.19), (2.21) and (2.24),

$$S^{n_{k_0}} h_1 \in W \cap T_1^{-n_{k_0}}(V_1) \cap T_2^{-n_{k_0}}(V_2) \cap \cdots \cap T_N^{-n_{k_0}}(V_N).$$

Now we achieve our target in (2.16) and (2.17).

(2) $\Rightarrow$ (1) This implication is immediate from Proposition 1.5.

Remark Let $T_{b^{(1)},\varphi}, T_{b^{(2)},\varphi}, \ldots, T_{b^{(N)},\varphi}$ be $N \ge 2$ weighted pseudo-shifts on Banach sequence space X, generated by the invertible mapping φ and weight sequences $b^{(1)} = (b_i^{(1)})_{i \in I}, b^{(2)} = (b_i^{(2)})_{i \in I}, \ldots, b^{(N)} = (b_i^{(N)})_{i \in I}$. By Theorem 2.1, Proposition 1.5 and Proposition 1.2, $T_{b^{(1)},\varphi}, T_{b^{(2)},\varphi}, \ldots, T_{b^{(N)},\varphi}$ are d-hypercyclic if and only if the set of d-hypercyclic vectors is a dense G_δ subset of X if and only if $T_{b^{(1)},\varphi}, T_{b^{(2)},\varphi}, \ldots, T_{b^{(N)},\varphi}$ are d-topologically transitive if and only if $T_{b^{(1)},\varphi}, T_{b^{(2)},\varphi}, \ldots, T_{b^{(N)},\varphi}$ satisfy the Disjoint Blow-up/Collapse Property.

In a recent paper [4], the authors provided a characterization for the disjoint hypercyclic bilateral weighted backward shifts on $\ell^2(\mathbb{Z})$. In the following, we study it as a special case of Theorem 2.1.

Example 2.2 Let $X = \ell^2(\mathbb{Z})$, $N \ge 2$. For each $1 \le l \le N$, let T_l be the bilateral weighted backward shift on X with weight sequence $\{w_j^{(l)} : j \in \mathbb{Z}\}$; that is, for each $j \in \mathbb{Z}$, $T_l e_j = w_j^{(l)} e_{j-1}$, where $(e_j)_{j \in \mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. In this case X is a Banach sequence space over $I = \mathbb{Z}$ in which $(e_j)_{j \in \mathbb{Z}}$ form an OP-basis. And each T_l is a weighted pseudo-shift $T_{b^{(l)},\varphi}$ with $b_i^{(l)} = w_{i+1}^{(l)}$ and $\varphi(i) = i + 1$ for each $i \in \mathbb{Z}$. For integers i, n and l with $i \in \mathbb{Z}, n \ge 1$ and $2 \le l \le N$, define

$$\alpha_{i,n}^{(l)} = \prod_{j=1}^{n} \frac{w_{i+j}^{(l)}}{w_{i+j}^{(1)}}.$$

Next, by Theorem 2.1, $T_1, T_2, \ldots, T_N$ are d-hypercyclic if and only if $T_1, T_2, \ldots, T_N$ satisfy the Disjoint Blow-up/Collapse Property if and only if there exists a strictly increasing sequence $(n_k)_{k=0}^{\infty}$ of positive integers such that for each $i \in \mathbb{Z}$ and $1 \le l \le N$ we have

$$\left| \prod_{j=1}^{n_k} w_{i+j}^{(1)} \right| \to \infty, \quad \left| \prod_{j=0}^{n_k - 1} w_{i-j}^{(l)} \right| \to 0 \text{ as } k \to \infty,$$

and the set

$$\left\{ (\ldots, \alpha_{-1,n_k}^{(2)}, \ldots, \alpha_{-1,n_k}^{(N)}, \alpha_{0,n_k}^{(2)}, \ldots, \alpha_{0,n_k}^{(N)}, \alpha_{1,n_k}^{(2)}, \ldots, \alpha_{1,n_k}^{(N)}, \ldots) : k \ge 0 \right\}$$

is dense in $\mathbb{K}^{\mathbb{Z}}$ with respect to the product topology. Which are the same with [4, Theorem 2.1].

3 Consequences of Theorem 2.1

In this section, we observe some consequences of Theorem 2.1. First, applying Theorem 2.1 to the special case where each T_l is the direct sum of finite weighted backward shifts on $\ell^2(\mathbb{Z})$, we obtain the following corollary.

Corollary 3.1 *Let $X = \ell^2(\mathbb{Z})$, $N \geq 2$, $m \geq 1$. For each $1 \leq l \leq N$ and $1 \leq u \leq m$, let $T_{l,u}$ be the bilateral weighted backward shift on X with weight sequence $(a^{(l)}_{u,j})_{j\in\mathbb{Z}}$. For any $(u, j) \in \{1, \dots, m\} \times \mathbb{Z}$, and integers n, l with $n \geq 1$ and $2 \leq l \leq N$, define*

$$\alpha^{(l)}_{(u,j),n} = \prod_{v=1}^{n} \frac{a^{(l)}_{u,j+v}}{a^{(1)}_{u,j+v}}.$$

Then the direct sum operators $T_1 = \bigoplus_{u=1}^m T_{1,u}$, $T_2 = \bigoplus_{u=1}^m T_{2,u}, \dots, T_N = \bigoplus_{u=1}^m T_{N,u}$ on X^m are d-hypercyclic if and only if there is an increasing sequence $(n_k)_{k\geq 0}$ of positive integers such that for each $j \in \mathbb{Z}$ and integer l with $1 \leq l \leq N$, we have

$$\min \left\{ \left| \prod_{v=1}^{n_k} a^{(1)}_{u,j+v} \right| : 1 \leq u \leq m \right\} \to \infty \text{ as } k \to \infty,$$

$$\max \left\{ \left| \prod_{v=0}^{n_k-1} a^{(l)}_{u,j-v} \right| : 1 \leq u \leq m \right\} \to 0 \text{ as } k \to \infty$$

and the set

$$\left\{ \left(\left(\alpha^{(2)}_{(u,j),n_k}\right)_{1\leq u\leq m, j\in\mathbb{Z}}, \dots, \left(\alpha^{(N)}_{(u,j),n_k}\right)_{1\leq u\leq m, j\in\mathbb{Z}} \right) : k \geq 0 \right\}$$

is dense in $\underbrace{\mathbb{K}^{m\times\mathbb{Z}} \times \cdots \times \mathbb{K}^{m\times\mathbb{Z}}}_{N-1}$ with respect to the product topology.

Proof In order to apply Theorem 2.1, we show that for each $1 \leq l \leq N$, T_l is a particular weighted pseudo-shift. Let $(x_1, \dots, x_m) = ((x_{1,j})_{j\in\mathbb{Z}}, \dots, (x_{m,j})_{j\in\mathbb{Z}})$ be any vector in X^m. If we identify $(x_1, \dots, x_m)$ with $(x_{u,j})_{1\leq u\leq m, j\in\mathbb{Z}}$, then X^m can be seen as a Hilbert sequence space over $I = \{1, \dots, m\} \times \mathbb{Z}$, in which the canonical unite vectors form an OP-basis. In this interpretation, T_l is the operator given by

$$T_l(x_{u,j})_{1\leq u\leq m, j\in\mathbb{Z}} = (y^{(l)}_{u,j})_{1\leq u\leq m, j\in\mathbb{Z}} \quad \text{where} \quad y^{(l)}_{u,j} = a^{(l)}_{u,(j+1)} x_{u,(j+1)}.$$

Hence each T_l is a weighted pseudo-shift $T_{b^{(l)},\varphi}$ with

$$b^{(l)}_{u,j} = a^{(l)}_{u,j+1} \text{ and } \varphi(u, j) = (u, j+1) \text{ for } (u, j) \in \{1, \dots, m\} \times \mathbb{Z},$$

where $(b^{(l)}_{u,j})_{(u,j)\in I}$ is the weight sequence.

Applying Theorem 2.1 we can complete the proof.

In [4], the authors proved that if B_1, B_2 be the unilateral weighted backward shifts on ℓ^2, then $B_1 \oplus B_1$, $B_2 \oplus B_2$ have no d-hypercyclic vector. Considering the special case $T_{l,1} = T_{l,2} = \cdots = T_{l,m}$ $(1 \leq l \leq N)$ in Corollary 3.1, we achieve a similar phenomenon.

Corollary 3.2 *Let $N \geq 2$. For each $1 \leq l \leq N$, T_l be the bilateral weighted backward shift on $\ell^2(\mathbb{Z})$. Then for any integer $m \geq 2$, the direct sum operators $\overbrace{T_1 \oplus \cdots \oplus T_1}^{m}$, …,*

$\overbrace{T_N \oplus \cdots \oplus T_N}^{m}$ *are not d-hypercyclic on* X^m*. In particular,* $T_1, T_2, \ldots, T_N$ *fail to satisfy the d-Hypercyclicity Criterion.*

Proof It is easy to see that, this is the case $a^{(l)}_{1,j} = a^{(l)}_{2,j} = \cdots = a^{(l)}_{m,j}$ for each $j \in \mathbb{Z}$ and $1 \le l \le N$ in Corollary 3.1. In this case, for any $j \in \mathbb{Z}, n \ge 1$ and $2 \le l \le N$ we have $\alpha^{(l)}_{(1,j),n} = \alpha^{(l)}_{(2,j),n} = \cdots = \alpha^{(l)}_{(m,j),n}$. Thus for any increasing sequence $(n_k)_{k\ge 0}$ the set

$$\left\{\left(\left(\alpha^{(2)}_{(u,j),n_k}\right)_{1\le u\le m, j\in\mathbb{Z}}, \ldots, \left(\alpha^{(N)}_{(u,j),n_k}\right)_{1\le u\le m, j\in\mathbb{Z}}\right) : k \ge 0\right\}$$

can not be dense in $\underbrace{\mathbb{K}^{m\times\mathbb{Z}} \times \cdots \times \mathbb{K}^{m\times\mathbb{Z}}}_{N-1}$. By Corollary 3.1, $\overbrace{T_1 \oplus \cdots \oplus T_1}^{m}$, $\ldots$, $\overbrace{T_N \oplus \cdots \oplus T_N}^{m}$ are not d-hypercyclic.

The conclusion that $T_1, T_2, \ldots, T_N$ not satisfy the d-Hypercyclicity Criterion follows from Proposition 1.2 and Theorem 1.4.

Remark Bès et al. [4, Proposition 2.4] proved that for any integer $N \ge 2$, there exist bilateral weighted backward shifts $T_1, T_2, \ldots, T_N$ on $\ell^2(\mathbb{Z})$ which are d-hypercyclic. So by Example 2.2 and Corollary 3.2, we can also get the assertion that the Disjoint Blow-up/Collapse Property and the d-Hypercyclicity Criterion are not equivalent, which mentioned in the Introduction.

Next, we study the bilateral backward operator weighted shifts as special cases of weighted pseudo-shifts, then applying Theorem 2.1 to give a characterization for their d-hypercyclicity. Before stating the main result, we settle some terminology. One may find details in reference [12].

Let $\mathcal{K}$ be a separable complex Hilbert space with an orthonormal basis $\{f_k\}_{k=0}^{\infty}$. Define a separable Hilbert space

$$\ell^2(\mathbb{Z}, \mathcal{K}) := \left\{x = (\ldots, x_{-1}, [x_0], x_1, \ldots) : x_i \in \mathcal{K} \text{ and } \sum_{i\in\mathbb{Z}} ||x_i||^2 < \infty\right\}$$

under the inner product $\langle x, y\rangle = \sum_{i\in\mathbb{Z}} \langle x_i, y_i\rangle_{\mathcal{K}}$ for $x = (x_i)_{i\in\mathbb{Z}}, y = (y_i)_{i\in\mathbb{Z}}$ in $\ell^2(\mathbb{Z}, \mathcal{K})$.

Let $\{A_n\}_{n=-\infty}^{\infty}$ be a uniformly bounded sequence of invertible operators on $\mathcal{K}$, where the operators $\{A_n\}_{n=-\infty}^{\infty}$ are all diagonal with respect to the basis $\{f_k\}_{k=0}^{\infty}$.

The bilateral backward operator weighted shift T on $\ell^2(\mathbb{Z}, \mathcal{K})$ is defined by

$$T(\ldots, x_{-1}, [x_0], x_1, \ldots) = (\ldots, A_0x_0, [A_1x_1], A_2x_2, \ldots).$$

Corollary 3.3 *Let* $N \ge 2$ *and for each integer* l *with* $1 \le l \le N$*, let* T_l *be a bilateral backward operator weighted shift on* $\ell^2(\mathbb{Z}, \mathcal{K})$ *with weight sequence* $\{A^{(l)}_n\}_{n=-\infty}^{\infty}$*. Suppose* $\{(a^{(l)}_{i,n})_{i\in\mathbb{N}}\}_{n\in\mathbb{Z}}$ *be a uniformly bounded sequence such that for each* $n \in \mathbb{Z}$ *and* $1 \le l \le N$*,*

$$A^{(l)}_n f_i = a^{(l)}_{i,n} f_i \quad \text{and} \quad A_n^{-1} f_i = (a^{(l)}_{i,n})^{-1} f_i \quad \text{for every } i \in \mathbb{N}.$$

For each $(i, j) \in \mathbb{N} \times \mathbb{Z}$*, and integers* n, l *with* $n \ge 1$ *and* $2 \le l \le N$*, define*

$$\alpha^{(l)}_{(i,j),n} = \prod_{v=1}^{n} \frac{a^{(l)}_{i,j+v}}{a^{(1)}_{i,j+v}}.$$

Then the following assertions are equivalent:

(1) $T_1, T_2, \ldots, T_N$ *are d-hypercyclic.*
(2) $T_1, T_2, \ldots, T_N$ *satisfy the Disjoint Blow-up/Collapse Property.*
(3) *There exists a strictly increasing sequence* $(n_k)_{k\geq 0}$ *of positive integers such that for every* $i \in \mathbb{N}$, $j \in \mathbb{Z}$ *and integer* l *with* $1 \leq l \leq N$,

$$\begin{cases} \lim\limits_{k\to\infty} \left\| \prod\limits_{v=1}^{n_k} (A_{j+v}^{(1)})^{-1} f_i \right\| = 0, \\ \\ \lim\limits_{k\to\infty} \left\| \prod\limits_{v=0}^{n_k-1} A_{j-v}^{(l)} f_i \right\| = 0 \end{cases}$$

and the set

$$\left\{ \left(\left(\alpha_{(i,j),n_k}^{(2)} \right)_{i\in\mathbb{N}, j\in\mathbb{Z}}, \ldots, \left(\alpha_{(i,j),n_k}^{(N)} \right)_{i\in\mathbb{N}, j\in\mathbb{Z}} \right) : k \geq 0 \right\}$$

is dense in $\underbrace{\mathbb{K}^{\mathbb{N}\times\mathbb{Z}} \times \cdots \times \mathbb{K}^{\mathbb{N}\times\mathbb{Z}}}_{N-1}$ *with respect to the product topology.*

Proof We start by proving that for each $1 \leq l \leq N$, T_l is a weighted pseudo-shift on the Hilbert sequence space $\ell^2(\mathbb{Z}, \mathcal{K})$.

For any $x = (x_j)_{j\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathcal{K})$, since each x_j is in $\mathcal{K}$, there exist scalars $\{x_{i,j}\}_{i\in\mathbb{N}}$ such that $x_j = \sum_{i=0}^{\infty} x_{i,j} f_i$. If we identify the tuple

$$(\ldots, x_{-1}, [x_0], x_1, \ldots) = (\ldots, (x_{i,(-1)})_{i\in\mathbb{N}}, [(x_{i,0})_{i\in\mathbb{N}}], (x_{i,1})_{i\in\mathbb{N}}, \ldots)$$

with $(x_{i,j})_{i\in\mathbb{N}, j\in\mathbb{Z}}$, the space $\ell^2(\mathbb{Z}, \mathcal{K})$ can be regarded as a Hilbert sequence space over $I := \mathbb{N} \times \mathbb{Z}$.

For each $(i_0, j_0) \in I$, we define $e_{i_0,j_0} := (\ldots, z_{-1}, [z_0], z_1, \ldots)$ in $\ell^2(\mathbb{Z}, \mathcal{K})$, by letting $z_{j_0} = f_{i_0}$ and $z_j = 0$ for $j \neq j_0$. It is easy to see that $(e_{i,j})_{(i,j)\in I}$ is an OP-basis of $\ell^2(\mathbb{Z}, \mathcal{K})$.

In this interpretation, T_l is the operator given by

$$T_l(x_{i,j})_{(i,j)\in I} = (y_{i,j}^{(l)})_{(i,j)\in I} \quad \text{where} \quad y_{i,j}^{(l)} = a_{i,(j+1)}^{(l)} x_{i,(j+1)}.$$

Hence T_l is the weighted pseudo-shift $T_{b^{(l)},\varphi}$ with

$$b_{i,j}^{(l)} = a_{i,j+1}^{(l)} \quad \text{and} \quad \varphi(i,j) = (i, j+1) \qquad \text{for } (i,j) \in I.$$

By Theorem 2.1, we can easily get the proof. Since for each $(i, j) \in I$, $1 \leq l \leq N$ and any positive integer n_k, we have

$$\begin{aligned} \left\| \left(\prod_{v=0}^{n_k-1} b_{\varphi^v(i,j)}^{(1)} \right)^{-1} e_{\varphi^{n_k}(i,j)} \right\| &= \left\| \left(\prod_{v=0}^{n_k-1} b_{i,j+v}^{(1)} \right)^{-1} e_{i,j+n_k} \right\| \\ &= \left\| \left(\prod_{v=1}^{n_k} a_{i,j+v}^{(1)} \right)^{-1} e_{i,j+n_k} \right\| \\ &= \left\| \prod_{v=1}^{n_k} (A_{j+v}^{(1)})^{-1} f_i \right\|, \end{aligned}$$

$$\left\| \left(\prod_{v=1}^{n_k} b^{(l)}_{\psi^v(i,j)} \right) e_{\psi^{n_k}(i,j)} \right\| = \left\| \left(\prod_{v=1}^{n_k} b^{(l)}_{i,j-v} \right) e_{i,j-n_k} \right\|$$
$$= \left\| \left(\prod_{v=0}^{n_k-1} a^{(l)}_{i,j-v} \right) e_{i,j-n_k} \right\|$$
$$= \left\| \prod_{v=0}^{n_k-1} A^{(l)}_{j-v} f_i \right\|.$$

and for each $2 \le l \le N$,

$$\alpha^{(l)}_{(i,j),n_k} = \prod_{v=0}^{n_k-1} \frac{b^{(l)}_{\varphi^v(i,j)}}{b^{(1)}_{\varphi^v(i,j)}} = \prod_{v=1}^{n_k} \frac{a^{(l)}_{i,j+v}}{a^{(1)}_{i,j+v}}.$$

Remark 3.4 (1) Corollary 3.1 turns out to be particular case of Corollary 3.3 when $\dim \mathcal{K} = m < \infty$;
(2) Example 2.2 turns out to be particular case of Corollary 3.1 when $m = 1$.

References

1. Bayart, F., Matheron, É.: Dynamics of Linear Operators. Cambridge Tracts in Mathematics, vol. 179. Cambridge University Press, Cambridge (2009)
2. Bernal-González, L.: Disjoint hypercyclic operators. Studia Math. **182**(2), 113–131 (2007)
3. Bès, J., Martin, Ö.: Compositional disjoint hypercyclicity equals disjoint supercyclicity. Houst. J. Math. **38**(4), 1149–1163 (2012)
4. Bès, J., Martin, Ö., Sanders, R.: Weighted shifts and disjoint hypercyclicity. J. Oper. Theory **72**(1), 15–40 (2014)
5. Bès, J., Martin, Ö., Peris, A.: Disjoint hypercyclic linear fractional composition operators. J. Math. Anal. Appl. **381**, 843–856 (2011)
6. Bès, J., Martin, Ö., Peris, A., Shkarin, S.: Disjoint mixing operators. J. Funct. Anal. **263**(5), 1283–1322 (2012)
7. Bès, J., Peris, A.: Disjointness in hypercyclicity. J. Math. Anal. Appl. **336**, 297–315 (2007)
8. Bernal-González, L., Grosse-Erdmann, K.-G.: The Hypercyclicity Criterion for sequences of operators. J. Studia Math. **157**(1), 17–32 (2003)
9. Grosse-Erdmann, K.-G., Peris Manguillot, A.: Linear Chaos. Universitext, Springer, New York (2011)
10. Grosse-Erdmann, K.-G.: Hypercyclic and chaotic weighted shifts. Studia Math. **139**(1), 47–68 (2000)
11. Han, S.A., Liang, Y.X.: Disjoint hypercyclic weighted translations generated by aperiodic elements. Collect. Math. **67**(3), 347–356 (2016)
12. Hazarika, M., Arora, S.C.: Hypercyclic operator weighted shifts. Bull. Korean Math. Soc. **41**(4), 589–598 (2004)
13. Kitai, C.: Invariant closed sets for linear operators. Ph.D. thesis, Univ. of Toronto (1982)
14. León-Saavedra, F.: Notes about the Hypercyclicity Criterion. Math. Slovaca **53**(3), 313–319 (2003)
15. Martin, Ö. : Disjoint hypercyclic and supercyclic composition operators. Ph.D. thesis, Bowling Green State University (2010)
16. Salas, H.: Dual disjoint hypercyclic weighted shifts. J. Math. Anal. Appl. **374**(1), 106–117 (2011)
17. Sanders, R., Shkarin, S.: Existence of disjoint weakly mixing operators that fail to satisfy the Disjoint Hypercyclicity Criterion. J. Math. Anal Appl. **417**(2), 834–855 (2014)
18. Shkarin, S.: A short proof of existence of disjoint hypercyclic operators. J. Math. Anal. Appl. **367**(2), 713–715 (2010)
19. Salas, H.: The strong disjoint blow-up/collapse property. J. Funct. Spaces Appl. Article ID 146517 (2013)