

THE HILBERT-SCHMIDT NORM OF A COMPOSITION OPERATOR ON THE BERGMAN SPACE

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ABSTRACT. We use a generalized Nevanlinna counting function to compute the Hilbert-Schmidt norm of a composition operator on the Bergman space $L_a^2(\mathbb{D})$ and weighted Bergman spaces $L_a^1(dA_\alpha)$ when α is a nonnegative integer.

1. INTRODUCTION

1.1. Background. Let \mathbb{D} denote the unit disc in the complex plane \mathbb{C} and let φ be a holomorphic function on \mathbb{D} with $\varphi(\mathbb{D}) \subset \mathbb{D}$. For every function f analytic in \mathbb{D} , the composition operator C_φ is a linear operator defined by $C_\varphi(f) = f \circ \varphi$.

In the last decades, lots of properties about composition operators on a variety kinds of analytic function spaces has been investigated widely. [1, 5, 8, 10] are good references for composition operators. One of the classical spaces is the Hardy space H^2 , which is the space consisting of the analytic functions f on \mathbb{D} such that

$$\|f\|_{H^2}^2 = \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

Another one is the Bergman space $L_a^2(\mathbb{D})$, which is the space consisting of those holomorphic functions f on \mathbb{D} satisfying

$$\|f\|_{L_a^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty,$$

where

$$dA(z) = \frac{1}{\pi} dx dy = \frac{r}{\pi} dr d\theta$$

is the normalized area measure on \mathbb{D} . It is well known that C_φ is always bounded on both H^2 and $L_a^2(\mathbb{D})$.

In [7], J. Shapiro computes the essential norm of C_φ acting on H^2 in terms of the Nevanlinna counting function of φ . Recall that the essential norm of a bounded operator T on a Banach space X , denoted by $\|T\|_{e,X}$, is the distance from T to the subspace of all compact operators acting on

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X in the operator norm. Also, for a self-map φ on \mathbb{D} , the Nevanlinna counting function N_φ is defined on $\mathbb{D} \setminus \{\varphi(0)\}$ and given by

$$N_\varphi(w) = \sum_{\varphi(z)=w} \log \frac{1}{|z|},$$

where multiplicities are counted and $N_\varphi(w)$ is taken to be zero if w is not in the range of φ . The fundamental work of Shapiro ([7, Theorem 2.3]) asserts that

$$\|C_\varphi\|_{e, H^2} = \limsup_{|w| \rightarrow 1} \frac{N_\varphi(w)}{\log(1/|w|)}.$$

Later, in [4], D. Luecking and K. Zhu proved that for $0 < p < \infty$, C_φ is in the Schatten class \mathcal{S}_p of H^2 if and only if

$$\int_{\mathbb{D}} \left(\frac{N_\varphi(z)}{\log \frac{1}{|z|}} \right)^{p/2} d\lambda(z) < \infty,$$

where $d\lambda(z) = dA(z)/(1 - |z|^2)^2$ is the Möbius invariant measure on \mathbb{D} .

On the Bergman space $L_a^2(\mathbb{D})$, P. Poggi-Corradini verified in [6] that

$$\|C_\varphi\|_{e, L_a^2(\mathbb{D})} = \limsup_{|w| \rightarrow 1} \frac{N_{\varphi,2}(w)}{(\log(1/|w|))^2},$$

where

$$N_{\varphi,2}(w) = \sum_{\varphi(z)=w} \left(\log \frac{1}{|z|} \right)^2, \quad w \in \mathbb{D} \setminus \{\varphi(0)\}.$$

Moreover, it is shown in [4] that for $0 < p < \infty$, C_φ is in the Schatten class \mathcal{S}_p of $L_a^2(\mathbb{D})$ if and only if

$$\int_{\mathbb{D}} \left(\frac{N_{\varphi,2}(z)}{(\log \frac{1}{|z|})^2} \right)^{p/2} d\lambda(z) < \infty.$$

1.2. Overview. From the line stated above, we know that the Schatten p -class membership of composition operators are closely related to the Nevanlinna counting functions. In particular, the Schatten 1-class \mathcal{S}_1 is usually called the trace class, and \mathcal{S}_2 is usually called the Hilbert-Schmidt class.

For any $T \in \mathcal{S}_1$ of a separable Hilbert space H , let $\{e_k\}$ be any orthonormal basis of H , the trace of T is given by

$$\text{tr}(T) = \sum_{k=0}^{\infty} \langle T e_k, e_k \rangle.$$

It is known that the sum is independent of the choice of the orthonormal basis. The Hilbert-Schmidt norm of T is defined by

$$\|T\|_{HS}^2 = \text{tr}(T^*T).$$

In this paper, we will compute the Hilbert-Schmidt norm of a composition operator on $L_a^2(\mathbb{D})$. The following assertion is established:

Theorem 1. For an analytic self-map φ of \mathbb{D} , let

$$\tilde{N}_\varphi(w) = 2N_\varphi(w) - \sum_{\varphi(z)=w} (1 - |z|^2)$$

be the general counting function of φ . Then

(i)

$$\int_{\mathbb{D}} f(\varphi(z))\overline{g(\varphi(z))}dA(z) = f(\varphi(0))\overline{g(\varphi(0))} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\tilde{N}_\varphi(z)dA(z)$$

for $f, g \in L_a^2(\mathbb{D})$.

(ii) If C_φ is in the Hilbert-Schmidt class of $L_a^2(\mathbb{D})$, then

$$\|C_\varphi\|_{HS}^2 = 1 + \frac{|\varphi(0)|^2(2 - |\varphi(0)|^2)}{(1 - |\varphi(0)|^2)^2} + \int_{\mathbb{D}} \frac{\tilde{N}_\varphi(z)(2 + 4|z|^2)}{(1 - |z|^2)^4} dA(z).$$

2. PROOF OF THEOREM 1(I)

Proof of Theorem 1(i). The argument is inspired by [3]. Let $f, g \in L_a^2(\mathbb{D})$, we can use the Littlewood-Paley formula ([2, page 228]) to deduce that

$$\begin{aligned} & \int_{\mathbb{D}} f(\varphi(z))\overline{g(\varphi(z))}dA(z) \\ &= \int_0^1 \left(\frac{1}{\pi} \int_0^{2\pi} f(\varphi(re^{i\theta}))\overline{g(\varphi(re^{i\theta}))}d\theta \right) r dr \\ &= \int_0^1 2 \left(f(\varphi(0))\overline{g(\varphi(0))} + r^2 \int_{\mathbb{D}} f'(\varphi(rw))\overline{g'(\varphi(rw))}|\varphi'(rw)|^2 \log \frac{1}{|w|^2} dA(w) \right) r dr \\ &= f(\varphi(0))\overline{g(\varphi(0))} + 2 \int_0^1 \left(\int_{\mathbb{D}} f'(\varphi(rw))\overline{g'(\varphi(rw))}|\varphi'(rw)|^2 \log \frac{1}{|w|^2} dA(w) \right) r^3 dr. \end{aligned}$$

Letting $u = rw$, then

$$\begin{aligned} & \int_{\mathbb{D}} f'(\varphi(rw))\overline{g'(\varphi(rw))}|\varphi'(rw)|^2 \log \frac{1}{|w|^2} dA(w) \\ &= \frac{2}{r^2} \int_{r\mathbb{D}} f'(\varphi(u))\overline{g'(\varphi(u))}|\varphi'(u)|^2 \log \frac{r}{|u|} dA(u) \\ &= \frac{2}{\pi r^2} \int_0^{2\pi} \int_0^r f'(\varphi(se^{it}))\overline{g'(\varphi(se^{it}))}|\varphi'(se^{it})|^2 \log \frac{r}{s} s ds dt. \end{aligned}$$

The Fubini's theorem implies that

$$\begin{aligned} & 2 \int_0^1 \left(\int_{\mathbb{D}} f'(\varphi(rw))\overline{g'(\varphi(rw))}|\varphi'(rw)|^2 \log \frac{1}{|w|^2} dA(w) \right) r^3 dr \\ &= \frac{4}{\pi} \int_0^{2\pi} \int_0^1 \int_0^r f'(\varphi(se^{it}))\overline{g'(\varphi(se^{it}))}|\varphi'(se^{it})|^2 \log \frac{r}{s} s ds r dr dt \\ &= \frac{4}{\pi} \int_0^{2\pi} \int_0^1 f'(\varphi(se^{it}))\overline{g'(\varphi(se^{it}))}|\varphi'(se^{it})|^2 \int_s^1 r \log \frac{r}{s} dr ds dt. \end{aligned}$$

Using the identity

$$\int_s^1 r \log \frac{r}{s} dr = \frac{1}{2} \log \frac{1}{s} - \frac{1}{4}(1 - s^2), \quad (1)$$

we have

$$\begin{aligned} & \frac{4}{\pi} \int_0^{2\pi} \int_0^1 f'(\varphi(se^{it})) \overline{g'(\varphi(se^{it}))} |\varphi'(se^{it})|^2 \int_s^1 r \log \frac{r}{s} dr s ds dt \\ &= \frac{4}{\pi} \int_0^{2\pi} \int_0^1 f'(\varphi(se^{it})) \overline{g'(\varphi(se^{it}))} |\varphi'(se^{it})|^2 \left(\frac{1}{2} \log \frac{1}{s} - \frac{1}{4}(1 - s^2) \right) s ds dt \\ &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 f'(\varphi(se^{it})) \overline{g'(\varphi(se^{it}))} |\varphi'(se^{it})|^2 \left(\log \frac{1}{s^2} - (1 - s^2) \right) s ds dt \\ &= \int_{\mathbb{D}} f'(\varphi(w)) \overline{g'(\varphi(w))} |\varphi'(w)|^2 \left(\log \frac{1}{|w|^2} - (1 - |w|^2) \right) dA(w) \\ &= \int_{\mathbb{D}} f'(z) \overline{g'(z)} \tilde{N}_\varphi(z) dA(z). \end{aligned}$$

The proof is completed. \square

Let $\varphi(z) = z$ be the identity map of \mathbb{D} in Theorem 4 (i), then we have the following corollary.

Corollary 2. *If $f \in L_a^2(\mathbb{D})$, then*

$$\|f\|_{L_a^2(\mathbb{D})}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \left(\log \frac{1}{|z|^2} - (1 - |z|^2) \right) dA(z).$$

3. THE HILBERT-SCHMIDT NORM

Proof of Theorem 1(ii). It is well known that an orthonormal basis for $L_a^2(\mathbb{D})$ is

$$e_n(z) = \sqrt{n+1} z^n, \quad n \geq 0.$$

Thus

$$\begin{aligned} \|C_\varphi\|_{HS}^2 &= \text{tr}(C_\varphi^* C_\varphi) \\ &= \sum_{k=0}^{\infty} \langle C_\varphi^* C_\varphi e_k, e_k \rangle \\ &= \sum_{k=0}^{\infty} \langle C_\varphi e_k, C_\varphi e_k \rangle \\ &= \sum_{k=0}^{\infty} \langle \sqrt{k+1} \varphi(z)^k, \sqrt{k+1} \varphi(z)^k \rangle \\ &= \sum_{k=0}^{\infty} (k+1) \langle \varphi(z)^k, \varphi(z)^k \rangle \\ &= 1 + \sum_{k=1}^{\infty} (k+1) \int_{\mathbb{D}} \varphi(z)^k \overline{\varphi(z)^k} dA(z) \end{aligned}$$

Now we can use Theorem 1(i) to deduce that

$$\begin{aligned} \|C_\varphi\|_{HS}^2 &= 1 + \sum_{k=1}^{\infty} (k+1) \left(|\varphi(0)|^{2k} + \int_{\mathbb{D}} k^2 |z|^{2k-2} \tilde{N}_\varphi(z) dA(z) \right) \\ &= 1 + \frac{|\varphi(0)|^2 (2 - |\varphi(0)|^2)}{(1 - |\varphi(0)|^2)^2} + \int_{\mathbb{D}} \sum_{k=1}^{\infty} (k+1) k^2 |z|^{2k-2} \tilde{N}_\varphi(z) dA(z) \\ &= 1 + \frac{|\varphi(0)|^2 (2 - |\varphi(0)|^2)}{(1 - |\varphi(0)|^2)^2} + \int_{\mathbb{D}} \tilde{N}_\varphi(z) \frac{(2 + 4|z|^2)}{(1 - |z|^2)^4} dA(z). \end{aligned}$$

This completes the proof. \square

4. COMPOSITION OPERATORS ON THE WEIGHTED BERGMAN SPACE

For $\alpha > -1$, the weighted Bergman space $L_a^2(dA_\alpha)$ is the space of analytic functions in \mathbb{D} satisfying

$$\|f\|_{L_a^2(dA_\alpha)}^2 = \int_{\mathbb{D}} |f(z)|^2 dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$. In some sense, H^2 can be treated as $L_a^2(dA_{-1})$. We have the following corollary.

Corollary 3. *If C_φ is in the Hilbert-Schmidt class of H^2 , then*

$$\|C_\varphi\|_{HS, H^2}^2 = 1 + \frac{|\varphi(0)|^2}{1 - |\varphi(0)|^2} + \int_{\mathbb{D}} \frac{1 + |z|^2}{(1 - |z|^2)^3} N_\varphi(z) dA(z).$$

Proof. An orthonormal basis for H^2 can be given as

$$e_n(z) = z^n, \quad n \geq 0.$$

Thus

$$\begin{aligned} \|C_\varphi\|_{HS, H^2}^2 &= 1 + \sum_{n=1}^{\infty} \left(|\varphi(0)|^{2n} + \int_{\mathbb{D}} n^2 |z|^{2n-2} N_\varphi(z) dA(z) \right) \\ &= 1 + \frac{|\varphi(0)|^2}{1 - |\varphi(0)|^2} + \int_{\mathbb{D}} \sum_{n=1}^{\infty} n^2 |z|^{2n-2} N_\varphi(z) dA(z) \\ &= 1 + \frac{|\varphi(0)|^2}{1 - |\varphi(0)|^2} + \int_{\mathbb{D}} \frac{1 + |z|^2}{(1 - |z|^2)^3} N_\varphi(z) dA(z). \end{aligned}$$

\square

When α is an arbitrary nonnegative integer, we can extend the results of Theorem 1 to the weighted Bergman space case. In the rest of this section, we discuss the cases when $\alpha = 1$ and 2.

Theorem 4. *For an analytic self-map φ of \mathbb{D} , let*

$$\tilde{N}_\varphi^1(w) = 2N_\varphi(w) - \frac{1}{2} \sum_{\varphi(z)=w} (3 - 4|z|^2 + |z|^4)$$

be the general 1-order counting function of φ . Then

(i)

$$\int_{\mathbb{D}} f(\varphi(z)) \overline{g(\varphi(z))} dA_1(z) = f(\varphi(0)) \overline{g(\varphi(0))} + \int_{\mathbb{D}} f'(z) \overline{g'(z)} \tilde{N}_\varphi^1(z) dA(z)$$

for $f, g \in L_a^2(dA_1)$.

(ii) *If C_φ is in the Hilbert-Schmidt class of $L_a^2(dA_1)$, then*

$$\|C_\varphi\|_{HS, L_a^2(dA_1)}^2 = 1 + \frac{|\varphi(0)|^2(3 - 3|\varphi(0)|^2 + |\varphi(0)|^4)}{(1 - |\varphi(0)|^2)^3} + \int_{\mathbb{D}} \frac{3(3|z|^2 + 1)}{(1 - |z|^2)^5} \tilde{N}_\varphi^1(z) dA(z).$$

Proof. (i). The argument is parallel to the proof of Theorem 1. It is worth to notice that equation (1) should be substituted by

$$\int_s^1 r(1 - r^2) \log \frac{r}{s} dr = \frac{1}{4} \log \frac{1}{s} - \frac{1}{16} (3 - 4s^2 + s^4).$$

(ii). Let

$$e_n(z) = \sqrt{\frac{(n+1)(n+2)}{2}} z^n, \quad n \geq 0.$$

Then, according to [10, page 78], $\{e_n\}$ forms an orthonormal basis for $L_a^2(dA_1)$. Thus

$$\begin{aligned} \|C_\varphi\|_{HS, L_a^2(dA_1)}^2 &= \sum_{n=0}^{\infty} \left\langle \sqrt{\frac{(n+1)(n+2)}{2}} (\varphi(z))^n, \sqrt{\frac{(n+1)(n+2)}{2}} (\varphi(z))^n \right\rangle_{L_a^2(dA_1)} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2} \int_{\mathbb{D}} \varphi(z)^n \overline{\varphi(z)^n} dA_1(z) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2} \left(|\varphi(0)|^{2n} + \int_{\mathbb{D}} n^2 |z|^{2n-2} \tilde{N}_\varphi^1(z) dA(z) \right) \\ &= 1 + \frac{|\varphi(0)|^2(3 - 3|\varphi(0)|^2 + |\varphi(0)|^4)}{(1 - |\varphi(0)|^2)^3} + \int_{\mathbb{D}} \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2} n^2 |z|^{2n-2} \tilde{N}_\varphi^1(z) dA(z) \\ &= 1 + \frac{|\varphi(0)|^2(3 - 3|\varphi(0)|^2 + |\varphi(0)|^4)}{(1 - |\varphi(0)|^2)^3} + \int_{\mathbb{D}} \frac{3(3|z|^2 + 1)}{(1 - |z|^2)^5} \tilde{N}_\varphi^1(z) dA(z). \end{aligned}$$

□

Corollary 5. *If $f \in L_a^2(dA_1)$, then*

$$\|f\|_{L_a^2(dA_1)}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \left(\log \frac{1}{|z|^2} - \frac{1}{2} (3 - 4|z|^2 + |z|^4) \right) dA(z).$$

Theorem 6. *For an analytic self-map φ of \mathbb{D} , let*

$$\tilde{N}_\varphi^2(w) = 2N_\varphi(w) - \frac{1}{6} \sum_{\varphi(z)=w} (11 - 18|z|^2 + 9|z|^4 - 2|z|^6)$$

be the general 2-order counting function of φ . Then

(i)

$$\int_{\mathbb{D}} f(\varphi(z))\overline{g(\varphi(z))}dA_2(z) = f(\varphi(0))\overline{g(\varphi(0))} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\tilde{N}_\varphi^2(z)dA(z)$$

for $f, g \in L_a^2(dA_2)$.(ii) If C_φ is in the Hilbert-Schmidt class of $L_a^2(dA_2)$, then

$$\|C_\varphi\|_{HS, L_a^2(dA_2)}^2 = 1 + \frac{(2 - |\varphi(0)|^2)(2 - 2|\varphi(0)|^2 + |\varphi(0)|^4)}{|\varphi(0)|^{-2}(1 - |\varphi(0)|^2)^4} + \int_{\mathbb{D}} \frac{6(6|z|^2 + 1)}{(1 - |z|^2)^6} \tilde{N}_\varphi^2(z)dA(z).$$

Proof. (i). In this case, the equation (1) should be substituted by

$$\int_s^1 r(1 - r^2)^2 \log \frac{r}{s} dr = \frac{1}{6} \log \frac{1}{s} - \frac{1}{72}(11 - 18s^2 + 9s^4 - 2s^6).$$

(ii). Let

$$e_n(z) = \sqrt{\frac{(n+1)(n+2)(n+3)}{6}} z^n, \quad n \geq 0.$$

Then $\{e_n\}$ is an orthonormal basis for $L_a^2(dA_2)$. Thus

$$\begin{aligned} \|C_\varphi\|_{HS, L_a^2(dA_2)}^2 &= 1 + \sum_{n=1}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} \int_{\mathbb{D}} \varphi(z)^n \overline{\varphi(z)^n} dA_2(z) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} \left(|\varphi(0)|^{2n} + \int_{\mathbb{D}} n^2 |z|^{2n-2} \tilde{N}_\varphi^2(z) dA(z) \right) \\ &= 1 + \frac{|\varphi(0)|^2(2 - |\varphi(0)|^2)(2 - 2|\varphi(0)|^2 + |\varphi(0)|^4)}{(1 - |\varphi(0)|^2)^4} \\ &\quad + \int_{\mathbb{D}} \sum_{n=1}^{\infty} \frac{(n+1)(n+2)(n+3)}{6} n^2 |z|^{2n-2} \tilde{N}_\varphi^2(z) dA(z) \\ &= 1 + \frac{|\varphi(0)|^2(2 - |\varphi(0)|^2)(2 - 2|\varphi(0)|^2 + |\varphi(0)|^4)}{(1 - |\varphi(0)|^2)^4} + \int_{\mathbb{D}} \frac{6(6|z|^2 + 1)}{(1 - |z|^2)^6} \tilde{N}_\varphi^2(z) dA(z). \end{aligned}$$

□

Corollary 7. If $f \in L_a^2(dA_2)$, then

$$\|f\|_{L_a^2(dA_2)}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \left(\log \frac{1}{|z|^2} - \frac{1}{6}(11 - 18|z|^2 + 9|z|^4 - 2|z|^6) \right) dA(z).$$

Remark 8. We can use *Maple* to compute that

$$\begin{aligned} \int r(1 - r^2)^n \log \frac{r}{s} dr &= -\frac{n}{16} r^4 {}_3F_2([2, 2, -n+1]; [3, 3]; r^2) \\ &\quad + \frac{r^2}{4} \left((2 \log \frac{r}{s} - 1) {}_2F_1([1, -n]; [2]; r^2) \right), \end{aligned}$$

where ${}_mF_l([\alpha_1, \dots, \alpha_m]; [\beta_1, \dots, \beta_l]; x)$ is the hypergeometric function given by

$${}_mF_l([\alpha_1, \dots, \alpha_m]; [\beta_1, \dots, \beta_l]; x) = \sum_{k=0}^{\infty} \frac{x^k \prod_{j=1}^m (\alpha_j)_k}{k! \prod_{j=1}^l (\beta_j)_k},$$

and $(\alpha)_k$ is the Pochhammer symbol defined by

$$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)}.$$

It is easy to check that if α_j is a negative integer for some $j \in \{1, \dots, m\}$, then the hypergeometric function ${}_mF_l([\alpha_1, \dots, \alpha_m]; [\beta_1, \dots, \beta_l]; x)$ is a polynomial. In particular, for $n \geq 2$ is a positive integer, ${}_3F_2([2, 2, -n + 1]; [3, 3]; x)$ and ${}_2F_1([1, -n]; [2]; x)$ are polynomials. Thus, if n is a positive integer with $n \geq 2$,

$$\begin{aligned} & \int_s^1 r(1-r^2)^n \log \frac{r}{s} dr \\ &= -\frac{n}{16} {}_3F_2([2, 2, -n + 1]; [3, 3]; 1) + \frac{1}{4} (2 \log \frac{1}{s} - 1) {}_2F_1([1, -n]; [2]; 1) \\ &+ \frac{n}{16} s^4 {}_3F_2([2, 2, -n + 1]; [3, 3]; s^2) + \frac{s^2}{4} {}_2F_1([1, -n]; [2]; s^2). \end{aligned}$$

The corresponding result similar to Theorem 1 can be obtained.

5. THE HILBERT-SCHMIDT NORM OF C_φ^*

It is well known that T is in the Schatten- p class \mathcal{S}_p on a Hilbert space H if and only if T^* is in \mathcal{S}_p . Moreover, $\|T^*\|_{\mathcal{S}_p} = \|T\|_{\mathcal{S}_p}$.

According to Theorem 6.4 in [10], the trace of a positive operator T on $L_a^2(dA_\alpha)$ can be expressed as

$$\mathrm{tr}(T) = (\alpha + 1) \int_{\mathbb{D}} \tilde{T}(z) d\lambda(z),$$

where

$$\tilde{T}(z) = \langle T k_z, k_z \rangle_{L_a^2(dA_\alpha)}, \quad z \in \mathbb{D}$$

is the Berezin transform of T and

$$k_z(w) = \frac{(1 - |z|^2)^{(2+\alpha)/2}}{(1 - w\bar{z})^{2+\alpha}}$$

is the normalized reproducing kernel of $L_a^2(dA_\alpha)$. The reproducing kernel of $L_a^2(dA_\alpha)$ is given by

$$K_\alpha(w, z) = \frac{1}{(1 - w\bar{z})^{2+\alpha}}, \quad z, w \in \mathbb{D}.$$

Obviously,

$$k_z(w) = K_\alpha(w, z) / \sqrt{K_\alpha(z, z)}.$$

For the composition operator C_φ on $L_a^2(dA_\alpha)$, it is easy to check that

$$C_\varphi^* K_\alpha(w, z) = K_\alpha(w, \varphi(z))$$

and

$$\widetilde{C_\varphi C_\varphi^*}(z) = \|C_\varphi^* k_z\|^2 = \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{2+\alpha}.$$

Combine these facts together, we have

$$\|C_\varphi^*\|_{HS, L_a^2(dA_\alpha)}^2 = \text{tr}(C_\varphi C_\varphi^*) = (\alpha + 1) \int_{\mathbb{D}} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{2+\alpha} d\lambda(z).$$

In particular, if we let $\alpha = 0, 1, 2$ respectively, we have

Corollary 9. *Let φ an analytic self-map of \mathbb{D} . Then*

(i)

$$\int_{\mathbb{D}} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^2 d\lambda(z) = 1 + \frac{|\varphi(0)|^2(2 - |\varphi(0)|^2)}{(1 - |\varphi(0)|^2)^2} + \int_{\mathbb{D}} \frac{\tilde{N}_\varphi(z)(2 + 4|z|^2)}{(1 - |z|^2)^4} dA(z).$$

(ii)

$$\begin{aligned} 2 \int_{\mathbb{D}} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^3 d\lambda(z) &= 1 + \frac{|\varphi(0)|^2(3 - 3|\varphi(0)|^2 + |\varphi(0)|^4)}{(1 - |\varphi(0)|^2)^3} \\ &\quad + \int_{\mathbb{D}} \frac{3(3|z|^2 + 1)}{(1 - |z|^2)^5} \tilde{N}_\varphi^1(z) dA(z). \end{aligned}$$

(iii)

$$\begin{aligned} 3 \int_{\mathbb{D}} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^4 d\lambda(z) &= 1 + \frac{(2 - |\varphi(0)|^2)(2 - 2|\varphi(0)|^2 + |\varphi(0)|^4)}{|\varphi(0)|^{-2}(1 - |\varphi(0)|^2)^4} \\ &\quad + \int_{\mathbb{D}} \frac{6(6|z|^2 + 1)}{(1 - |z|^2)^6} \tilde{N}_\varphi^2(z) dA(z). \end{aligned}$$

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