The relationships between some types of partial differential equations and ordinary differential equations as well as their applications

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Abstract

In this article, we establish some relationships between several types of partial differential equations and ordinary differential equations. One application of these relationships is that we can get the exact values of the blowup time and the blowup rate of the solution to a partial differential equation by solving an ordinary differential equation. Another application of these relationships is that we can give the estimates for the spatial integration(or mean value) of the solution to a partial differential equation. We also obtain the lower and upper bounds for the blowup time of the solution to a parabolic equation with weighted function and space-time integral in the nonlinear term.

Keywords: Parabolic equation; Wave equation; Blowup rate; Bounds for blowup time; Spatial integration of the solution.

1 Introduction

In this paper, we are concerned with the blowup times of the solutions to some types of partial differential equation problems.

The first type of problems can be written as

$$\begin{cases} u_t = div\big(c(x)\nabla G(u)\big) + h\Big(x, \int_0^t \int_\Omega u \mathrm{d}x \mathrm{d}s\Big), \quad x \in \Omega,\ t > 0 \\ a_1 \frac{\partial u}{\partial \nu} + b_1 u = g(x), \quad x \in \partial\Omega,\ t > 0 \\ u(x, 0) = u_0(x) \geq 0,\ x \in \Omega, \end{cases} \tag{1.1}$$

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where $c(x) \geq 0$, $G'(u) \geq 0$, and $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a smooth bounded domain, h is a given smooth function, (a_1, b_1) and $g(x)$ are defined by (1.3) and (1.4) respectively.

The second type of problems can be written as

$$\begin{cases} u_t = div\big(c(x)\nabla G(u)\big) + d_1 u + f\Big(x, \int_\Omega u \mathrm{d}x\Big), & x \in \Omega,\ t > 0 \\ a_1 \frac{\partial u}{\partial \nu} + b_1 u = g(x), \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0,\ x \in \Omega, \end{cases} \tag{1.2}$$

where $d_1 \geq 0$, $c(x) \geq 0$, $G'(u) \geq 0$, $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a smooth bounded domain, f is a given smooth function, while

$$(a_1, b_1) = \begin{cases} (1,0) & \text{if } G'(0) \neq 0 \\ (0,1) & \text{if } G'(0) = 0 \end{cases} \tag{1.3}$$

and g satisfies

$$\begin{cases} g(x) \equiv 0 & \text{if } G(u) \neq u \\ g(x) \geq 0 & \text{if } G(u) = u. \end{cases} \tag{1.4}$$

Besides obtaining the exact blowup times of the solutions to these types of problems above, we also give the estimate for the bound of the blowup time to solution of the following problem

$$\begin{cases} u_t = \triangle u + a(x)\Big[\int_0^t \int_\Omega \beta(x) u(x,s) \mathrm{d}x \mathrm{d}s\Big]^p, & x \in \Omega,\ t > 0 \\ u(x,t) = 0 \quad \text{or} \quad \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_0(x) \geq 0,\ x \in \Omega, \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a smooth bounded domain, ν is the outward norm vector, $u_0(x)$ is a continuous nonnegative function and satisfies the compatible condition $u_0(x) = 0$ or $\frac{\partial u_0}{\partial \nu} = 0$ on $\partial\Omega$, $\beta(x) \in C(\bar{\Omega})$, $\beta(x) \geq 0$, $\beta(x) \not\equiv 0$, the weighted function $a(x) \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies

(a_1) $\quad a(x) \geq c > 0$ for some constant c

or

(a_2) $\quad a(x) > 0$ in Ω, and $a(x) = 0$ on $\partial\Omega$.

These models often appear in physical theory and engineering applications. Since the equation has the nonlocal nonlinear term in each model, we call it nonlocal partial differential equation. There is an extensive literature on nonlocal parabolic equation or nonlocal wave equation, we can refer to [1, 2, 6, 15, 9, 10, 16, 13, 20, 25, 29, 30, 31, 32, 33, 34, 35] and the references therein.

There are many interesting topics on these problems, such as the conditions on global existence and blowup in finite time, estimates for the blowup rate and blowup time of the solutions. By the results of [4, 5, 7, 8, 11, 12, 14, 21], we know that the solution to (1.2) (or (1.1), or (1.5)) will blow up in finite time under some assumptions, one of the essential conditions for (1.2)(or (1.1), or (1.5)) is that the function f (or h)

satisfies $f(x,\theta) \geq c\theta^{p_1} > 0$ (or $h(x,\theta)$) with some $p_1 > 1$ for θ large and any $x \in \Omega$. Yet we don't care about the conditions on the blowup in finite time and global existence of the solution in this paper, we focus on the lower and upper bounds for the blowup time of the solutions.

Our first result is about the exact value of the blowup time t_1^* of the blowup solution to (1.1).

Theorem 1. *Assume that the solution to (1.1) will blow up in finite time. Then the exact value of the blowup time t_1^* is*

$$t_1^* = \int_0^\infty \frac{\mathrm{d}\eta}{\sqrt{2\Big(\tilde{H}(\eta) - \tilde{H}(0)\Big) + 2\eta \int_{\partial\Omega} c(x)g(x)dS + \Big(\int_\Omega u_0 \mathrm{d}x\Big)^2}}. \tag{1.6}$$

Here $\tilde{H}(l) = \int_\Omega \tilde{h}(x,l)\mathrm{d}x$ *and* $\tilde{h}(\vartheta) = \int_\Omega h(x,\vartheta)\mathrm{d}x$ *for* $\vartheta > 0$. *And the blowup rate can be written as*

$$\int_\Omega u(x,t)dx = \frac{2}{t_2^* - t}. \tag{1.7}$$

Our second result is about the exact value of the blowup time t_2^* of the blowup solution to (1.2).

Theorem 2. *Assume that the solution to (1.2) will blow up in finite time. Then the exact value of the blowup time t_2^* is*

$$t_2^* = \int_{T(0)}^\infty \frac{d\xi}{\tilde{f}(\xi) + d_1\xi + \int_{\partial\Omega} c(x)g(x)dS}. \tag{1.8}$$

Here $T(t) = \int_\Omega u\mathrm{d}x$ *and* $\tilde{f}(\theta) = \int_\Omega f(x,\theta)\mathrm{d}x$ *for* $\theta > 0$.

For (1.5), we can establish the lower bounds for the blowup time of the solution to it as follows.

Theorem 3. *Assume that u is a nonnegative solution to (1.5) which becomes unbounded in L^{k+1}-norm at $t = t^*$. Then a lower bound for blowup time of the solution is given by*

$$t^* \geq \int_{\phi(0)}^\infty \frac{\mathrm{d}\eta}{k\eta + K_1\varepsilon^{-(k+1)p}\eta^p}. \tag{1.9}$$

Here

$$\phi(t) = \int_\Omega u^{k+1}\mathrm{d}x, \quad K_1 = \int_\Omega \big(a(x)\big)^{k+1}\mathrm{d}x\Big(\int_\Omega \big(\beta(x)\big)^{\frac{k+1}{k}}\mathrm{d}x\Big)^{kp} \tag{1.10}$$

and the constant $k > 0$

We would like to compare our methods with others. There are many results about the topic on the bounds for blowup time of the solution to a parabolic equation, we can refer to [3, 19, 22, 23, 24, 26, 27, 28] and the references therein. Differing from the methods in these references, in order to establish the lower and upper bounds for the blowup time of the solutions to these problems above, we will establish some relationships between these partial differential equations and some ordinary differential equations. Using these relationships, we can obtain the exact values of the blowup time and the blowup rate of the solutions.

Using Fourier transform, or Laplace transform, or other transform, we may change a partial differential equation into an ordinary differential equation, but we must make its inverse transform in order to obtain the behavior for the solution of the partial differential equation. However, we needn't to make inverse transform and can use our methods to directly deal with these types of nonlocal parabolic equations and nonlocal wave equations in this paper.

The rest of the paper is organized as follows. In Section 2, we will establish some relationships between partial differential equations and ordinary differential equations and get the exact value of the blowup time of the solution. In Section 3, we will deal with the integration $\int_\Omega u(x,t)$ of the partial differential equation. In Section 4, we will apply the method of constructing the sub-solution of (1.5) to obtain the upper bound and use another method to get the lower bound for blowup time of the solution.

2 The exact values of blowup time and blowup rate

To illustrate our idea, we discuss a model as follows:

$$\begin{cases} u_t = \triangle u + a(x)h\Big(\int_0^t \int_\Omega u(x,s)\mathrm{d}x\mathrm{d}s\Big), \quad x \in \Omega,\ t>0 \\ \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t>0 \\ u(x,0) = u_0(x) \geq 0,\ x \in \Omega, \end{cases} \tag{2.1}$$

where $h(\theta)$ is a nonnegative function which is increasing in θ. By the result of [8, 14], if $\lim_{\theta\to\infty} \frac{h(\theta)}{\theta^p} \geq c > 0$ for some $p>1$, then the solution to (2.1) will blow up in finite time.

Let

$$J(t) = \int_0^t \int_\Omega u(x,s)\mathrm{d}x\mathrm{d}s. \tag{2.2}$$

Integrating the first equation of (2.1) and using Green's formula, we get

$$\Big(\int_\Omega u\mathrm{d}x\Big)_t = \int_\Omega u_t \mathrm{d}x = \int_\Omega a(x)\mathrm{d}x h\Big(\int_0^t \int_\Omega u(x,s)\mathrm{d}x\Big) := Ah\big(J(t)\big). \tag{2.3}$$

Then by (2.2), we can obtain the following problem

$$\begin{cases} J''(t) = Ah\big(J(t)\big) \\ \\ J(0) = 0,\ J'(0) = \int_\Omega u_0(x)\mathrm{d}x. \end{cases} \tag{2.4}$$

Multiplying the first equation of (2.4) by $J^{'}(t)$ and integrating it with respect to t, we get

$$\int_0^t J^{''}(s)J^{'}(s)\mathrm{d}s = A\int_0^t h\big(J(s)\big)J^{'}(s)\mathrm{d}s. \tag{2.5}$$

After some elementary computations, we finally obtain

$$J^{'}(t) = \sqrt{2A\Big(H\big(J(t)\big) - H(0)\Big) + \Big(\int_\Omega u_0\mathrm{d}x\Big)^2}, \tag{2.6}$$

where $H(\theta) = \int_\Omega h(x,\theta)\mathrm{d}x$. Integrating (2.6) from 0 to t^*, we get the blow up time

$$t^* = \int_0^\infty \frac{\mathrm{d}\eta}{\sqrt{2A\Big(H(\eta) - H(0)\Big) + \Big(\int_\Omega u_0\mathrm{d}x\Big)^2}}. \tag{2.7}$$

Now we have established the following proposition:

Propsition 2.1. *Assume that u is a nonnegative solution to (2.1) which becomes unbounded in L^1-norm at $t = t^*$. Then the exact blowup time of the solution is given by (2.7). Especially, $h(\theta) = \theta^p$ and $p > 1$, the blowup time of the solution is*

$$t^* = \int_0^\infty \frac{\mathrm{d}\eta}{\sqrt{\frac{2A}{1+p}\eta^{p+1} + \Big(\int_\Omega u_0\mathrm{d}x\Big)^2}}. \tag{2.8}$$

Similar to the discussions above, we give the proof of Theorem 1.

Proof of Theorem 1: It is easy to verify the relationship between (1.1) and the following problem

$$\begin{cases} J^{''}(t) = \tilde{h}\big(J(t)\big) + \int_{\partial\Omega} c(x)g(x)dS \\ \\ J(0) = 0,\ J^{'}(0) = \int_\Omega u_0(x)\mathrm{d}x. \end{cases} \tag{2.9}$$

Here $J(t) = \int_0^t\int_\Omega u\mathrm{d}x\mathrm{d}s$ and $\tilde{h}(\vartheta) = \int_\Omega h(x,\vartheta)\mathrm{d}x$ for $\vartheta > 0$. Assume the solution will blow up in finite time. Then using (2.9), we can obtain the exact value of the blowup time t_2^* of the solution to (1.1)

$$t_2^* = \int_0^\infty \frac{\mathrm{d}\eta}{\sqrt{2\Big(\tilde{H}(\eta) - \tilde{H}(0)\Big) + 2\eta\int_{\partial\Omega} c(x)g(x)dS + \Big(\int_\Omega u_0\mathrm{d}x\Big)^2}} \tag{2.10}$$

with $\tilde{H}(l) = \int_\Omega \tilde{h}(x,l)\mathrm{d}x$. Here we have used the fact that $\int_\Omega u(x,t)dx \to \infty$ if and only if $\int_0^t\int_\Omega u(x,s)dxds \to \infty$(see [30]).

Next, we can integrate (2.9) from t to t_2^* and get

$$
\begin{aligned}
t_2^* - t &= \int_t^{t_2^*} \frac{J'(t)}{\sqrt{2\Big(\tilde{H}\big(J(t)\big) - \tilde{H}(0)\Big) + 2J(t)\int_{\partial\Omega} c(x)g(x)dS + \Big(\int_\Omega u_0 \mathrm{d}x\Big)^2}} \\
&= \int_{J(t)}^{\infty} \frac{d\eta}{\sqrt{2\Big(\tilde{H}(\eta) - \tilde{H}(0)\Big) + 2\eta\int_{\partial\Omega} c(x)g(x)dS + \Big(\int_\Omega u_0 \mathrm{d}x\Big)^2}} \\
&:= \Psi(J)(t). \qquad (2.11)
\end{aligned}
$$

Since $\Psi(J(t))$ is decreasing in J, we know that Ψ^{-1} exists and it is also a decreasing function. Consequently, we have

$$\int_0^t \int_\Omega u(x,s)dxds = J(t) = \Psi^{-1}(t_2^* - t). \qquad (2.12)$$

Using (2.11) and (2.12), we can obtain the blowup rate of the solution to (1.1) which will blow up at t_2^* in $L^1(\Omega)$. For example, if $h(\eta) = \frac{\left(\int_\Omega u_0(x)dx\right)^2}{2|\Omega|^2} e^\eta - \frac{\int_{\partial\Omega} c(x)g(x)dS}{|\Omega|^2}\eta$, we have

$$t_2^* - t = \frac{1}{\int_\Omega u_0(x)dx} \int_{J(t)}^{\infty} e^{-\frac{\eta}{2}} \mathrm{d}\eta$$

and

$$\int_0^t \int_\Omega u(x,s)dxds = J(t) = -2\ln\Big[\frac{\int_\Omega u_0(x)dx}{2}(t_2^* - t)\Big],$$

which means that

$$\int_\Omega u(x,t)dx = \frac{2}{t_2^* - t}. \qquad (2.13)$$

Theorem 1 is proved. □

Similarly, we can prove Theorem 2.

The proof of Theorem 2: It is easy to get the relationship between (1.2) and the following problem

$$
\begin{cases}
T'(t) = \tilde{f}\big(T(t)\big) + d_1 T(t) + \int_{\partial\Omega} c(x)g(x)dS \\
T(0) = \int_\Omega u_0(x)\mathrm{d}x.
\end{cases} \qquad (2.14)
$$

Here $T(t) = \int_\Omega u\mathrm{d}x$ and $\tilde{f}(\theta) = \int_\Omega f(x,\theta)\mathrm{d}x$. Assume that the solution will blow up in finite time. Then using (2.14), we can obtain the exact value of the blowup time t_1^* of the solution to (1.2)

$$t_1^* = \int_{T(0)}^{\infty} \frac{d\xi}{\tilde{f}(\xi) + d_1\xi + \int_{\partial\Omega} c(x)g(x)dS}. \qquad (2.15)$$

We will establish the blowup rate of the solution to (1.2) by (2.14). In fact, we can integrate (2.14) from t to t_1^* and get

$$\begin{aligned} t_1^* - t &= \int_t^{t_1^*} \frac{T'(t)}{\tilde{f}\big(T(t)\big) + d_1 T(t) + \int_{\partial\Omega} c(x)g(x)dS} \\ &= \int_{T(t)}^{\infty} \frac{d\eta}{\tilde{f}\big(\eta\big) + d_1\eta + \int_{\partial\Omega} c(x)g(x)dS} \\ &:= \Phi(T)(t). \end{aligned} \tag{2.16}$$

Noticing that $\Phi(T(t))$ is decreasing in T, we know that Φ^{-1} exists and it is also a decreasing function. Consequently, (2.16) means that

$$T(t) = \Phi^{-1}(t_1^* - t), \tag{2.17}$$

which gives the blowup rate of the solution to (1.2) which will blow up at t_1^* in $L^1(\Omega)$. An interesting phenomenon is that the blowup rate only depends on the nonlinearity but is independent of the diffusion. For example, if $d_1 = 0$, $f(\tau) = c\tau^p$ and $g(x) = 0$ with $c > 0$, $p > 1$, we can obtain

$$\int_\Omega u(x,t)dx = T(t) = [c(p-1)|\Omega|(t_1^* - t)]^{\frac{1}{1-p}}. \tag{2.18}$$

If $d_1 = 0$, $f(\tau) = ce^{a\tau}$ and $g(x) = 0$ with $c, a > 0$, we get

$$\int_\Omega u(x,t)dx = T(t) = -\frac{1}{a}\ln[ac|\Omega|(t_1^* - t)]. \tag{2.19}$$

Theorem 2 is proved. □

Remark 2.1. By the discussions above, we see that if the nonlinear term $f(x,t,u)$ in a parabolic equation satisfies that $\int_\Omega f(x,t,u)dx$ is a function of $\int_\Omega udx$, then $\int_\Omega udx$ satisfies an ordinary equation. And we can use this fact to obtain the exact values of the blowup time and blowup rate of the solution.

3 Some relationships between partial differential equations and ordinary differential equations

In this section, we focus on the spatial integration(or mean value) of the solutions to some partial differential equations. We would like to illustrate our idea by some examples.

Example 3.1. We can establish the relationship between a parabolic equation and an ordinary differential equation.

$$\begin{cases} u_t = \Delta u^m + d_2 u + k(x,t) & x \in \Omega,\ t > 0 \\ \frac{\partial u}{\partial \nu} = 0 \text{ or } u = 0, & x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_0(x) \geq 0,\ x \in \Omega. \end{cases} \tag{3.1}$$

Here $m > 1$, $d_2 \in \mathbb{R}$ and $k(x,t) \in L^1(\Omega)$ for any $t > 0$. It is well known that the solution is global existence. Integrating the first equation of (3.1) over Ω, we have

$$\left(\int_\Omega u(x,t)dx\right)_t = d_2 \int_\Omega u(x,t)dx + \int_\Omega k(x,t)dx.$$

Letting $\int_\Omega u(x,t)dx = I(t)$ and $\int_\Omega k(x,t)dx = K(t)$, we obtain an ordinary differential equation

$$I'(t) = d_2 I(t) + K(t).$$

Especially, if $K(t) = 0$ for all $t \geq 0$, we can get

$$I(t) = I(0)e^{d_2 t},$$

which implies that

$$\frac{\int_\Omega u(x,t)dx}{|\Omega|} = \frac{\int_\Omega u_0(x)dx}{|\Omega|} e^{d_2 t}. \tag{3.2}$$

Physically, u often represents temperature(or density) in the model of (3.1). (3.2) illustrates the link between the mean value of the temperature(or density) at time t and that of the initial temperature(or density).

Example 3.2. Consider the following problem

$$\begin{cases} u_t = \Delta u^m + d_3 u + l(x,t), & x \in \mathbb{R}^N,\ t > 0 \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbb{R}^N. \end{cases} \tag{3.3}$$

Here $m > 1$, $d_3 \in \mathbb{R}$, $l(x,t) \in L^1(\mathbb{R}^N)$ for any $t > 0$ and $u_0(x)$ is a continuous function which has compact support set in $\mathbb{R}^N$. It is well known that the solution is global existence. Integrating the first equation of (3.1) over $\mathbb{R}^N$, we have

$$\left(\int_{\mathbb{R}^N} u(x,t)dx\right)_t = d_3 \int_{\mathbb{R}^N} u(x,t)dx + \int_{\mathbb{R}^N} l(x,t)dx.$$

Letting $\int_{\mathbb{R}^N} u(x,t)dx = Y(t)$ and $\int_{\mathbb{R}^N} l(x,t)dx = L(t)$, we obtain an ordinary differential equation

$$Y'(t) = d_3 Y(t) + L(t).$$

Especially, if $L(t) = 0$ for all $t \geq 0$, we can get

$$Y(t) = Y(0)e^{d_3 t},$$

which means that

$$\int_{\mathbb{R}^N} u(x,t)dx = e^{d_3 t} \int_{\mathbb{R}^N} u_0(x)dx. \tag{3.4}$$

Remark 3.1. From Examples 3.1 and 3.2, we can see that if $\int_\Omega f(x,t,u)dx$(or $\int_{\mathbb{R}^N} f(x,t,u)dx$) is a function of $\int_\Omega udx$(or $\int_{\mathbb{R}^N} udx$), then $\int_\Omega udx$(or $\int_{\mathbb{R}^N} udx$) satisfies an ordinary equation.

Example 3.3. We will claim that, whether the solution to a wave solution will blow up in finite time or exist globally, there exists the relationship between a wave equation and an ordinary partial equation.

$$\begin{cases} au_{tt} + bu_t = div\big(c(x)\nabla u\big) + d_4 u + f\Big(x, \int_\Omega u\mathrm{d}x\Big)\big(\ or\ h\Big(x, \int_0^t \int_\Omega u\mathrm{d}x\mathrm{d}s\Big)\big),\ x \in \Omega,\ t > 0 \\ \frac{\partial u}{\partial \nu} = g(x), \quad x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x),\ x \in \Omega, \end{cases} \tag{3.5}$$

where $a > 0,\ b \geq 0,\ c(x) \geq 0$. In fact, from (3.5), we can find that $T(t) = \int_\Omega udx$ satisfies the following problem:

$$\begin{cases} aT''(t) + bT'(t) = \int_{\partial\Omega} c(x)g(x)\mathrm{d}S + d_4 T(t) + \tilde{f}\big(T(t)\big) \\ \\ T(0) = \int_\Omega u_0(x)\mathrm{d}x,\ T'(0) = \int_\Omega v_0(x)\mathrm{d}x, \end{cases} \tag{3.6}$$

or $J(t) = \int_0^t \int_\Omega udxd\tau$ satisfies

$$\begin{cases} aJ^{(3)}(t) + bJ''(t) = \int_{\partial\Omega} c(x)g(x)\mathrm{d}S + d_4 J(t) + \tilde{h}\big(J(t)\big) \\ \\ J(0) = 0,\ J'(0) = \int_\Omega u_0(x)\mathrm{d}x,\ J''(0) = \int_\Omega v_0(x)\mathrm{d}x. \end{cases} \tag{3.7}$$

Example 3.4. Considering the following Cauchy problem

$$\begin{cases} au_{tt} + bu_t = div\big(c(x)\nabla u\big) + d_5 u + f\Big(x, \int_{\mathbb{R}^N} u\mathrm{d}x\Big), \quad x \in \mathbb{R}^N,\ t > 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = v_0(x),\ x \in \mathbb{R}^N, \end{cases} \tag{3.8}$$

where $a > 0$, $b \geq 0$, $c(x) \geq 0$, we can also obtain the estimate for $\int_{\mathbb{R}^N} u(x,t)$. We omit the details here.

4 Bounds for blowup time of the solution to (1.5)

There are many literature on (1.5), we can refer to [17, 18, 30] and the references therein. Since we hope to obtain the bounds for the blowup time t^* of $u(x,t)$, we are only concerned with (1.5) in the case of $p > 1$. Similar to [3, 19], we will obtain the upper bound for the blowup time to the solution by constructing sub-solution. Let $\lambda > 0$ be the first eigenvalue of

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0 \quad \text{in } \Omega \\ \varphi|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

and φ be the corresponding eigenfunction satisfying that $\varphi(x) > 0$ in Ω and $\max\limits_{x\in\ \Omega} \varphi(x) = 1$. Assume that there exist two positive constants c_1 and c_2 such that $u_0(x) \geq c_1 \exp(1)\varphi(x)$ and $c_2 a(x) \geq \varphi(x)$. Letting

$$\varepsilon = \left(\frac{c_1}{2}\right)^{\frac{p-1}{p}} c_2^{\frac{1}{p}} \int_\Omega \beta(x)\varphi(x)\mathrm{d}x, \tag{4.2}$$

and constructing

$$\underline{u}(x,t) = c_1\varphi(x)\left[\frac{1}{(1-\varepsilon t)^2}\exp\left(\frac{1}{1-\varepsilon t}\right) + \exp(A - 2\lambda t)\right], \tag{4.3}$$

where A is large enough such that $\underline{u}_t - \Delta\underline{u} < 0$ at $t = 0$. Since $\underline{u}_t - \Delta\underline{u}$ is increasing in t, we can consider the first time value t_0 such that $\underline{u}_t - \Delta\underline{u} = 0$ in Ω. That is,

$$\left\{\frac{\varepsilon}{(1-\varepsilon t_0)^4} + \frac{2\varepsilon}{(1-\varepsilon t_0)^3} + \frac{\lambda}{(1-\varepsilon t_0)^2}\right\}\exp\left(2\lambda t_0 + \frac{1}{(1-\varepsilon t_0)}\right) = \lambda e^A. \tag{4.4}$$

Obviously, we can choose A large such that $t_0 \geq \frac{1}{\varepsilon}\cdot\frac{r_0-1}{r_0} \geq \frac{1}{\varepsilon}\cdot\frac{\ln 2}{\ln 2+1}$, where

$$r_0 = \inf_{r\in\mathbb{R}^+}\{r \geq 1 + \ln 2, \quad \exp\big((p-1)r\big) \geq 2(3\varepsilon+\lambda)r^4\}.$$

Now we will compare $\underline{u}_t - \Delta\underline{u}$ with $a(x)\Big[\int_0^t\int_\Omega \beta(x)\underline{u}(x,s)\mathrm{d}x\mathrm{d}s\Big]^p$ respectively in the time interval $[0, t_0]$ and $(t_0, \frac{1}{\varepsilon})$. By the discussions above, we have

$$\underline{u}_t - \Delta\underline{u} \leq 0 \leq a(x)\Big[\int_0^t\int_\Omega \beta(x)\underline{u}(x,s)\mathrm{d}x\mathrm{d}s\Big]^p$$

if $t \in [0, t_0]$. Meanwhile, we get that

$$\begin{aligned}\underline{u}_t - \Delta\underline{u} &\leq c_1\varphi\left[\frac{2\varepsilon}{(1-\varepsilon t)^3} + \frac{\varepsilon}{(1-\varepsilon t)^4} + \frac{\lambda}{(1-\varepsilon t)^2}\right]\exp\left(\frac{1}{1-\varepsilon t}\right)\\ &\leq c_1\varphi\frac{3\varepsilon+\lambda}{(1-\varepsilon t)^4}\exp\left(\frac{1}{1-\varepsilon t}\right)\\ &\leq \frac{1}{2}c_1\varphi\exp\left(\frac{p}{1-\varepsilon t}\right)\end{aligned}$$

and

$$\begin{aligned}&a(x)\Big[\int_0^t\int_\Omega \beta(x)\underline{u}(x,s)\mathrm{d}x\mathrm{d}s\Big]^p\\ &\geq c_1^p a(x)\Big(\int_\Omega \beta(x)\varphi(x)\mathrm{d}x\Big)^p\varepsilon^{-p}\left(\exp\left(\frac{1}{1-\varepsilon t}\right) - \exp(1)\right)^p\\ &\geq 2^{-p}c_1^p a(x)\Big(\int_\Omega \beta(x)\varphi(x)\mathrm{d}x\Big)^p\varepsilon^{-p}\exp\left(\frac{p}{1-\varepsilon t}\right).\end{aligned}$$

if $t \in (t_0, \frac{1}{\varepsilon})$. Consequently,

$$\underline{u}_t - \Delta \underline{u} \leq a(x)\Big[\int_0^t \int_\Omega \beta(x)\underline{u}(x,s)\mathrm{d}x\mathrm{d}s\Big]^p$$

if $t \in (t_0, \frac{1}{\varepsilon})$. Using comparison principle, we can verify that $\underline{u}(x,t)$ is a sub-solution of (1.5). As a result, the solution $u(x,t)$ of (1.5) will blow up at some $t^* < \frac{1}{\varepsilon}$ for any $x \in \Omega$. Therefore, we have established the following propsition:

Proposition 4.1. *Assume that u is the blowup solution to (1.5) with $p > 1$. Then the upper bound for the blowup time of the solution to (1.5) is $\frac{1}{\varepsilon}$.*

Now, we will give the proof of Theorem 3 and establish the lower bound estimates for the blowup time of the solution to (1.5) if $p > 1$.

The proof of Theorem 3:

Let

$$\phi(t) = \int_\Omega u^{k+1}\mathrm{d}x \tag{4.5}$$

with $k > 0$. Multiplying the first equation of (1.5) by u^k and integrating it by parts, we get

$$\begin{aligned}\int_\Omega u^k u_t \mathrm{d}x &= \int_\Omega u^k \triangle u \mathrm{d}x + \int_\Omega a(x)u^k\Big[\int_0^t \int_\Omega \beta(x)u(x,s)\mathrm{d}x\mathrm{d}s\Big]^p \mathrm{d}x \\ &= \int_{\partial\Omega} u^k \frac{\partial u}{\partial \nu}\mathrm{d}S - k\int_\Omega u^{k-1}|\nabla u|^2 \mathrm{d}x + \int_\Omega a(x)u^k\Big[\int_0^t \int_\Omega \beta(x)u(x,s)\mathrm{d}x\mathrm{d}s\Big]^p \mathrm{d}x.\end{aligned} \tag{4.6}$$

Using Young's inequality, we obtain

$$\begin{aligned}&a(x)u^k\Big[\int_0^t \int_\Omega \beta(x)u(x,s)\mathrm{d}x\mathrm{d}s\Big]^p \\ &\leq \frac{k}{k+1}u^{k+1} + \frac{1}{k+1}\big(a(x)\big)^{k+1}\Big[\int_0^t \int_\Omega \beta(x)u(x,s)\mathrm{d}x\mathrm{d}s\Big]^{(k+1)p}.\end{aligned} \tag{4.7}$$

Then it follows that

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} \leq k\phi + \int_\Omega \big(a(x)\big)^{k+1}\Big[\int_0^t \int_\Omega \beta(x)u(x,s)\mathrm{d}x\mathrm{d}s\Big]^{(k+1)p}\mathrm{d}x. \tag{4.8}$$

Note that

$$\int_\Omega \beta(x)u(x,s)\mathrm{d}x \leq \Big(\int_\Omega u^{k+1}\mathrm{d}x\Big)^{\frac{1}{k+1}}\Big(\int_\Omega \big(\beta(x)\big)^{\frac{k+1}{k}}\mathrm{d}x\Big)^{\frac{k}{k+1}} \tag{4.9}$$

by Hölder's inequality. Then we obtain

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}t} &\leq k\phi + K_1\Big(\int_0^t \phi^{\frac{1}{k+1}}\mathrm{d}s\Big)^{(k+1)p} \\ &\leq k\phi + K_1\phi^p(t)(t^*)^{(k+1)p} \\ &\leq k\phi + K_1\varepsilon^{-(k+1)p}\phi^p,\end{aligned} \tag{4.10}$$

where ε is taken as (4.2). And here

$$K_1 = \int_\Omega \big(a(x)\big)^{k+1} \mathrm{d}x \Big(\int_\Omega \big(\beta(x)\big)^{\frac{k+1}{k}} \mathrm{d}x \Big)^{kp}. \tag{4.11}$$

Integrating (4.10) from 0 to t^*, we have

$$t^* \geq \int_{\phi(0)}^{\infty} \frac{\mathrm{d}\eta}{k\eta + K_1 \varepsilon^{-(k+1)p} \eta^p}. \tag{4.12}$$

Theorem 3 is proved. □

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