Large Local and Global Dynamic Bifurcations of Nonlinear Evolution Equations

DESHENG LI & ZHI-QIANG WANG

ABSTRACT. We present new local and global dynamic bifurcation results for nonlinear evolution equations of the form $u_t + Au = f_\lambda(u)$ on a Banach space X, where A is a sectorial operator, and $\lambda \in \mathbb{R}$ is the bifurcation parameter. Suppose the equation has a trivial solution branch $\{(0,\lambda) \mid \lambda \in \mathbb{R}\}$. Denote Φ_λ the local semiflow generated by the initial value problem of the equation. It is shown that if the crossing number n at a bifurcation value $\lambda = \lambda_0$ is nonzero, and moreover, if $S_0 = \{0\}$ is an isolated invariant set of Φ_{λ_0}, then either there is a one-sided neighborhood I_1 of λ_0 such that Φ_λ bifurcates a topological sphere $\mathbb{S}^{n-1}$ for each $\lambda \in I_1 \setminus \{\lambda_0\}$, or there is a two-sided neighborhood I_2 of λ_0 such that the system Φ_λ bifurcates from the trivial solution an isolated nonempty compact invariant set K_λ with $0 \notin K_\lambda$ for each $\lambda \in I_2 \setminus \{\lambda_0\}$. We also prove that the bifurcating invariant set has nontrivial Conley index. Building upon this fact, we establish a global dynamical bifurcation theorem. Roughly speaking, we prove that for any given neighborhood Ω of the bifurcation point $(0,\lambda_0)$, the connected bifurcation branch Γ from $(0,\lambda_0)$ either meets the boundary $\partial\Omega$ of Ω, or meets another bifurcation point $(0,\lambda_1)$. This result extends the well-known Rabinowitz's Global Bifurcation Theorem to the setting of dynamic bifurcations of evolution equations without requiring the crossing number to be odd.

As an illustration example, we consider the well-known Cahn-Hilliard equation. Some global features on dynamical bifurcations of the equation are discussed.

1. INTRODUCTION

Dynamic bifurcation concerns the changes in the qualitative or topological structures of limiting motions such as equilibria, periodic solutions, homoclinic orbits, heteroclinic orbits, invariant tori, and so on for nonlinear evolution equations, as some relevant parameters in the equations vary. Historically, the subject can be traced back to the early work of Poincaré [30] around 1892. It is now a fundamental tool to study nonlinear problems in mathematical physics and mechanics [5, 11, 27], and it enables us to understand how and when a system organizes new states and patterns near the original "trivial" one when the control parameters cross some critical values.

A relatively simpler case for dynamic bifurcation is that of the bifurcations from equilibria. Generally speaking, there are two typical such bifurcations in the classical bifurcation theory. One is the bifurcation from equilibria to equilibria (static bifurcation), and the other is from equilibria to periodic solutions (Hopf bifurcation). The former usually requires a "crossing odd-multiplicity" condition: namely, that the linearized equation of a system has an odd number of eigenvalues (counting with multiplicity) crossing the imaginary axis when the control parameter crosses a critical value (the Krasnosel'skii's Bifurcation Theorem). We also know that in such a case the bifurcation has some global features, which fact is addressed by the well-known Rabinowitz's Global Bifurcation Theorem. Situations become very complicated if one drops the "crossing odd-multiplicity" condition mentioned above. If the system under consideration is a gradient one, then by a classical bifurcation theorem on potential operator equations due to Krasnosel'skii (see [11, Chapter II, Section 7] or [12]), one can still have local bifurcation results, whereas the global bifurcation remains an open problem. To deal with general systems, Ma and Wang [20] proved some new local and global static bifurcation theorems by using higher-order nondegenerate singularities of nonlinearities. The Hopf bifurcation theory has a long history and, to some extent, forms the central part of the classical dynamic bifurcation theory. It focuses on the case when a pair of conjugate eigenvalues of the linearized equation cross the imaginary axis, and was fully developed in the 20th century. There has been a vast body of literature on how to determine Hopf bifurcation for nonlinear systems arising from applications. One can also find some nice results concerning global results in [1, 39], etc.

This present work is mainly concerned with the general case of the bifurcations from equilibria in terms of invariant-set bifurcation, where the number of eigenvalues of the linearized equation crossing the imaginary axis might be even and greater than two. A particular but very important case in this line is the theory of attractor bifurcation, which was first introduced by Ma and Wang in 2003 [19] and was further developed by the authors into a dynamic transition theory [25]. Roughly speaking, it states that if the trivial equilibrium solution θ of a system changes from an attractor to a repeller on the local center manifold when the bifurcation parameter λ crosses a critical value λ_0, then the system bifurcates a compact invariant set K which is an attractor of the system on the center manifold. It is also known that K has the *shape* of an n-dimensional sphere, where n denotes the *crossing number* at $\lambda = \lambda_0$ (the number of eigenvalues of the linearized equation crossing the imaginary axis); see Theorem 1 in [35] or [22, Theorem 6.1]. Note that a fundamental assumption of this theory is that the trivial equilibrium θ is an attractor (repeller) of the system on the center manifold at $\lambda = \lambda_0$. Hence, it is no longer applicable when $S_0 = \{\theta\}$ is only an isolated invariant set when $\lambda = \lambda_0$. Fortunately in such a case, we know that dynamic bifurcation still occurs as long as there are eigenvalues crossing the imaginary axis. This has already been addressed in the literature (see, e.g., Rybakowski [34, pp. 101–102] and Ward [38]).

An abstract global dynamic bifurcation theorem was also proved in Ward [38] in terms of semiflows on complete metric spaces. Let Φ_λ be a family of dynamical systems on a complete metric space X, where $\lambda \in \mathbb{R}$. Suppose that θ is an equilibrium solution for each Φ_λ. Let $[a, b]$ be a compact interval which contains exactly one bifurcation value $\lambda_0 \in [a, b]$. Ward's global bifurcation theorem states that if

$$h(\Phi_a, \{\theta\}) \neq h(\Phi_b, \{\theta\}),$$

then a continua $\Gamma \subset X \times \mathbb{R}$ of bounded solutions bifurcates from (θ, λ_0), where $h(\Phi_\lambda, \{\theta\})$ denotes the Conley index of $\{\theta\}$ with respect to Φ_λ. Moreover, either Γ is unbounded in $X \times [a, b]$, or it intersects $X \times \{a, b\}$. Note that, because of the requirement on the uniqueness of bifurcation values in $[a, b]$, the theorem mentioned above may fail to work when a λ-interval contains multiple bifurcation values. This is somewhat different from the situation of the Rabinowitz Global Bifurcation Theorem.

In this paper, we consider the abstract evolution equation

$$u_t + Au = f_\lambda(u) \tag{1.1}$$

on a Banach space X, where A is a sectorial operator on X with *compact resolvent*, $f_\lambda(u)$ is a locally Lipschitz continuous mapping from $X^\alpha \times \mathbb{R}$ to X for some $0 \leq \alpha < 1$, and $\lambda \in \mathbb{R}$ is the bifurcation parameter. Our main goal is to establish new local and global dynamic bifurcation results.

Suppose that

$$f_\lambda(0) \equiv 0, \quad \lambda \in \mathbb{R}.$$

Thus, $u = 0$ is always a trivial solution of (1.1) for all λ. It is also assumed that $f_\lambda(u)$ is differentiable in u with $\mathrm{D}f_\lambda(u)$ being continuous in (u, λ).

First, as one of our main purposes here, we give some more precise and general results on local dynamic bifurcations in terms of invariant sets. We show that if the crossing number n at a bifurcation value $\lambda = \lambda_0$ is nonzero, and if, moreover, $S_0 = \{0\}$ is an isolated invariant set of the system, then either there is a one-sided neighborhood I_1 of λ_0 such that the system bifurcates an $(n-1)$-dimensional topological sphere $\mathbb{S}^{n-1}$ for each $\lambda \in I_1 \setminus \{\lambda_0\}$, or there is a two-sided neighborhood I_2 of λ_0 such that the system bifurcates from the trivial solution an isolated nonempty compact invariant set K_λ with $0 \notin K_\lambda$ for each $\lambda \in I_2 \setminus \{\lambda_0\}$.

Then, we prove that the invariant set K_λ from bifurcation has nontrivial Conley index. This result plays a key role in establishing our global dynamic bifurcation theorem. However, it may be of independent interest in its own right.

Finally, as our main goal in this work, we establish a global dynamic bifurcation theorem, extending Rabinowitz's Global Bifurcation Theorem on operator equations to dynamical systems without assuming the "crossing odd-multiplicity" condition and the uniqueness of bifurcation values in parameter intervals. Roughly speaking, given a neighborhood $\Omega \subset X^\alpha \times \mathbb{R}$ of the bifurcation point $(0, \lambda_0)$, we prove that the connected bifurcation branch Γ from $(0, \lambda_0)$ either meets the boundary $\partial\Omega$ of Ω, or meets another bifurcation point $(0, \lambda_1)$.

As an example, we consider the homogeneous Neumann boundary value problem of the Cahn-Hilliard equation

$$u_t + \Delta(\kappa\Delta u - f(u)) = 0$$

on a bounded domain $\Omega \subset R^d$ $(d \leq 3)$ with sufficiently smooth boundary, where

$$f(u) = a_1 u + a_2 u^2 + a_3 u^3, \; a_3 > 0.$$

The local attractor bifurcation and phase transition of the problem have been extensively studied in Ma and Wang [22–24]. Other results related to bifurcation of the problem can be found in [2, 26], and so on. Here, by applying the theoretical results obtained above, we give some more precise local dynamic bifurcation results and demonstrate global features of the bifurcations.

This paper is organized as follows. In Section 2 we make some preliminaries, and in Section 3 we present results on local invariant manifolds of the equation (1.1) and give a slightly modified version of a reduction theorem for Conley index in [34]. In Section 4 we prove some local dynamic bifurcation results. Section 5 is concerned with the nontriviality of the Conley indices of bifurcating invariant sets. Section 6 is devoted to the global dynamic bifurcation theorem. Section 7 consists of an example mentioned above.

2. PRELIMINARIES

This section is concerned with some preliminaries.

2.1. Basic topological notions and facts. Let X be a complete metric space with metric $d(\cdot, \cdot)$. For convenience, we always identify a singleton $\{x\}$ with the point x for any $x \in X$.

Let A and B be nonempty subsets of X. The *distance* $d(A, B)$ between A and B is defined as

$$d(A, B) = \inf\{d(x, y) \mid x \in A, \; y \in B\},$$

and the *Hausdorff semi-distance* and *Hausdorff distance* of A and B are defined, respectively, as

$$d_{\mathrm{H}}(A,B) = \sup_{x\in A} d(x,B),$$
$$\delta_{\mathrm{H}}(A,B) = \max\{d_{\mathrm{H}}(A,B), d_{\mathrm{H}}(B,A)\}.$$

We also assign $d_{\mathrm{H}}(\emptyset, B) = 0$.

The closure, interior, and boundary of A are denoted, respectively, by $\bar{A}$, int$A$, and ∂A. A subset U of X is called a *neighborhood* of A, if $\bar{A} \subset \mathrm{int}\, U$. The ε-*neighborhood* $\mathrm{B}(A,\varepsilon)$ of A is defined to be the set $\{y \in X \mid d(y,A) < \varepsilon\}$.

Let A_λ $(\lambda \in \Lambda)$ be a family of nonempty subsets of X, where Λ is a metric space. When we say that A_λ is *upper semicontinuous* in λ at $\lambda_0 \in \Lambda$, this means

$$d_{\mathrm{H}}(A_\lambda, A_{\lambda_0}) \to 0 \quad \text{as } \lambda \to \lambda_0.$$

Lemma 2.1 ([31]). *Let X be a compact metric space, and let A and B be two disjoint closed subsets of X. Then, either there exists a subcontinuum C of X such that*

$$A \cap C \neq \emptyset \neq B \cap C,$$

or $X = X_A \cup X_B$, where X_A and X_B are disjoint compact subsets of X containing A and B, respectively.

Lemma 2.2 ([3, p. 41]). *Let X be a compact metric space. Denote $\mathcal{K}(X)$ the family of compact subsets of X which is equipped with the Hausdorff metric $\delta_{\mathrm{H}}(\ ,\ )$. Then, $\mathcal{K}(X)$ is a compact metric space.*

2.2. Criteria on homotopy equivalence. We denote "$\simeq$" and "$\cong$" the *homotopy equivalence* and *homeomorphism*, respectively, between topological spaces.

Let X be a topological space, and $A \subset X$ be closed. The following result can be found in many textbooks on general topology.

Lemma 2.3. *If A is a strong deformation retract of X, then $X \simeq A$.*

Let $i_A : A \to X$ be the inclusion. Denote

$$M_{i_A} = (X \times \{0\}) \cup (A \times I),$$
$$C_{i_A} = M_{i_A}/(A \times \{1\}).$$

M_{i_A} and C_{i_A} are called the *mapping cylinder* and *mapping cone* of i_A, respectively.

The pair (X,A) is said to have *homotopy extension property* (*H.E.P.* for short) if, for any space Y, any mapping $f : M_{i_A} \to Y$ can be extended to a mapping $F : X \times I \to Y$.

Lemma 2.4 ([9, p. 14]). (X,A) *has the H.E.P. if and only if M_{i_A} is a retract of $X \times I$.*

Lemma 2.5 ([9, Theorem 0.17]). *Suppose (X,A) has the H.E.P. If A is contractible, then $X/A \simeq X$.*

As a consequence of Lemma 2.5, we have the following result.

Corollary 2.6. *Suppose (X,A) has the H.E.P. Let B be a closed subset of A. If B is a strong deformation retract of A, then $X/A \simeq X/B$.*

Proof. We observe $X/A \cong (X/B)/\tilde{A}$, where $\tilde{A} = \pi_B(A)$, and $\pi_B : X \to X/B$ is the projection. In the following, we verify that $(X/B)/\tilde{A} \simeq X/B$, thus completing the proof of what we desired.

Since (X,A) has the H.E.P., M_{i_A} is a retract of $X \times I$. Let $f : X \times I \to M_{i_A}$ be a retraction,

$$f(x,t) = (\varphi(x,t), \xi(x,t)), \quad (x,t) \in X \times I,$$

where $\varphi(x,t) \in X$, and $\xi(x,t) \in I$. Define

$$h : X \times I \to M_{i_{\tilde{A}}} = ((X/B) \times \{0\}) \cup (\tilde{A} \times I)$$

as $h(x,t) = (\pi_B \circ \varphi(x,t), \xi(x,t))$ for $(x,t) \in X \times I$. Let

$$Q(x,t) = (\pi(x), t), \quad (x,t) \in X \times I.$$

Then, $Q : X \times I \to (X/B) \times I$ is a quotient mapping. Observing that

$$h(x,t) = (\pi_B \circ \varphi(x,t), \xi(x,t)) = (\pi_B(x), t), \quad (x,t) \in M_{i_A},$$

we find that h remains constant on $B \times \{t\}$ for each $t \in I$. Consequently, we have $h \equiv$ const. on $Q^{-1}(y,t)$ for each $(y,t) \in (X/B) \times I$. Thus, by the basic knowledge in the theory of general topology (see, e.g., [29, Chapter 2, Theorem 11.1]), there is a mapping $g : (X/B) \times I \to M_{i_{\tilde{A}}}$ such that $h = g \circ Q$. It is trivial to verify that g is a retraction from $(X/B) \times I$ to $M_{i_{\tilde{A}}}$. Thus, the pair $(X/B, \tilde{A})$ has the H.E.P.

Since B is a strong deformation retract of A, the singleton $\{[B]\}$ is a strong deformation retract of $\tilde{A}$; that is, $\tilde{A}$ is contractible. Lemma 2.5 then asserts that $(X/B)/\tilde{A} \simeq X/B$. ◻

2.3. Wedge/smash product of pointed spaces. Let (X,x_0) and (Y,y_0) be two pointed spaces. The *wedge product* $(X,x_0) \vee (Y,y_0)$ and *smash product* $(X,x_0) \wedge (Y,y_0)$ are defined, respectively, as follows:

$$(X,x_0) \vee (Y,y_0) = (\mathcal{W}, (x_0,y_0)),$$
$$(X,x_0) \wedge (Y,y_0) = ((X \times Y)/\mathcal{W}, [\mathcal{W}]),$$

where $\mathcal{W} = X \times \{y_0\} \cup \{x_0\} \times Y$.

We denote $[(X, x_0)]$ the *homotopy type* of a pointed space (X, x_0). Since the operations "$\vee$" and "$\wedge$" preserve homotopy equivalence relations, they can be naturally extended to the homotopy types of pointed spaces. Specifically,

$$[(X, x_0)] \vee [(Y, y_0)] = [(X, x_0) \vee (Y, y_0)],$$
$$[(X, x_0)] \wedge [(Y, y_0)] = [(X, x_0) \wedge (Y, y_0)].$$

Denote $\bar{0}$ and Σ^0 the homotopy types of the pointed spaces $(\{p\}, p)$ and $(\{p, q\}, q)$, respectively, where p and q are two distinct points. Let Σ^m be the homotopy type of a pointed m-dimensional sphere. We easily verify that

$$[(X, x_0)] \vee \bar{0} = [(X, x_0)],$$

and

$$\Sigma^m \wedge \Sigma^n = \Sigma^{m+n}, \quad \forall\, m, n \geq 0.$$

2.4. Local semiflows and basic dynamical concepts. In this subsection, we briefly recall some dynamical concepts and facts that will be used throughout the paper.

Let X be a complete metric space.

A *local semiflow* Φ on X is a continuous map from an open subset $\mathcal{D}_\Phi$ of $\mathbb{R}^+ \times X$ to X satisfying the following:

(i) For all $x \in X$, there exists $T_x \in (0, \infty]$ such that

$$(t, x) \in \mathcal{D}_\Phi \Longleftrightarrow t \in [0, T_x).$$

(ii) $\Phi(0, \ ) = \mathrm{id}_X$.

Furthermore,

$$\Phi(s + t, x) = \Phi(t, \Phi(s, x)), \quad \forall\, x \in X,\ s, t \geq 0$$

as long as $(s + t, x) \in \mathcal{D}_\Phi$. The number T_x in the above definition is called the *escape time* of $\Phi(t, x)$.

Let Φ be a given local semiflow on X. For notational simplicity, we will rewrite $\Phi(t, x)$ as $\Phi(t)x$.

A *trajectory* on an interval J is a continuous mapping $\gamma : J \to X$ such that

$$\gamma(t) = \Phi(t - s)\gamma(s), \quad \forall\, t, s \in J,\ t \geq s.$$

If $J = \mathbb{R}$, then we simply call γ a *complete trajectory*. The *ω-limit set* $\omega(\gamma)$ and *α-limit set* $\alpha(\gamma)$ of a complete trajectory γ are defined, respectively, as

$$\omega(\gamma) = \{y \mid \exists\, x_n \in A \text{ and } t_n \to \infty \text{ such that } \gamma(t_n) \to y\},$$
$$\alpha(\gamma) = \{y \mid \exists\, x_n \in A \text{ and } t_n \to -\infty \text{ such that } \gamma(t_n) \to y\}.$$

Let $S \subset X$. S is said to be *positively invariant* (respectively, *invariant*), if $\Phi(t)S \subset S$ (respectively $\Phi(t)S = S$) for all $t \geq 0$. A compact invariant set $\mathcal{A}$ is called an *attractor* if it attracts a neighborhood U of itself, namely,

$$\lim_{t\to\infty} d_H(\Phi(t)U, \mathcal{A}) = 0.$$

The *attraction basin* of an attractor $\mathcal{A}$, denoted by $\mathcal{U}(\mathcal{A})$, is defined as

$$\mathcal{U}(\mathcal{A}) = \{x \mid \lim_{t\to\infty} d(\Phi(t)x, \mathcal{A}) = 0\}.$$

Remark 2.7. By definition one easily verifies that the attraction basin $\mathcal{U}(\mathcal{A})$ of an attractor $\mathcal{A}$ is open. Furthermore, for any trajectory $\gamma : J \to X$ of Φ (where J is an interval), it holds that

$$\text{either } \gamma(J) \subset \mathcal{U}(\mathcal{A}), \text{ or } \gamma(J) \cap \mathcal{U}(\mathcal{A}) = \emptyset.$$

2.5. Conley index. In this subsection we recall briefly some basic notions and results in the Conley index theory. (See [6, 28] and [34] for details.)

Let Φ be a given local semiflow on X, and let M be a subset of X. We say that Φ *does not explode* in M if $T_x = \infty$ whenever $\Phi([0, T_x))x \subset M$.

M is said to be *admissible* (see [34, p. 13]) if, for any sequences $x_n \in M$ and $t_n \to \infty$ with $\Phi([0, t_n])x_n \subset M$ for all n, the sequence $\Phi(t_n)x_n$ has a convergent subsequence. Also, M is said to be *strongly admissible* if it is admissible and, moreover, Φ does not explode in M.

Definition 2.8. Φ is said to be asymptotically compact on X if each bounded subset B of X is strongly admissible.

From now on, we always assume that

(AC) Φ *is asymptotically compact on* X.

This requirement is fulfilled by a large number of examples from applications.

A compact invariant set S of Φ is said to be *isolated* if there exists a bounded closed neighborhood N of S such that S is the maximal invariant set in N. Consequently, N is called an *isolating neighborhood* of S.

Let there be given an isolated compact invariant set S. A pair of bounded closed subsets (N, E) is called an *index pair* of S if the following hold:

(i) $N \setminus E$ is an isolating neighborhood of S.

(ii) E is N-invariant; specifically, for any $x \in E$ and $t \geq 0$,

$$\Phi([0, t])x \subset N \Longrightarrow \Phi([0, t])x \subset E.$$

(iii) E is an exit set of N. That is, for any $x \in N$, if $\Phi(t_1)x \notin N$ for some $t_1 > 0$, then there exists $0 \leq t_0 \leq t_1$ such that $\Phi(t_0)x \in E$.

Remark 2.9. Index pairs in the terminology of [34] need not be bounded. However, the bounded ones are sufficient for our purposes here.

Definition 2.10 (homotopy index). Let (N, E) be an index pair of S. Then, the homotopy Conley index of S is defined to be the homotopy type $[(N/E, [E])]$ of the pointed space $(N/E, [E])$, denoted by $h(\Phi, S)$.

Remark 2.11. Denote H_* and H^* the singular homology and cohomology theories with coefficient group $\mathbb{Z}$, respectively. Applying H_* and H^* to $h(\Phi, S)$, we obtain the *homology* and *cohomology Conley index* $CH_*(\Phi, S)$ and $CH^*(\Phi, S)$, respectively.

An important property of the Conley index is its continuation property. Here, we state a result in this line for the reader's convenience, which is actually a particular case of [34, Chapter 1, Theorem 12.2].

Let Φ_λ be a family of semiflows with parameter $\lambda \in \Lambda$, where Λ is a connected compact metric space. Assume $\Phi_\lambda(t)x$ is continuous in (t, x, λ). Denote $\bar{\Phi}$ the *skew-product flow* of the family Φ_λ on $X \times \Lambda$ defined as follows:

$$\bar{\Phi}(t)(x, \lambda) = (\Phi_\lambda(t)x, \lambda), \quad (x, \lambda) \in X \times \Lambda.$$

Theorem 2.12. *Suppose $\bar{\Phi}$ satisfies the assumption* (AC) *on $X \times \Lambda$. Let S be a compact isolated invariant set of $\bar{\Phi}$. Then, $h(\Phi_\lambda, S_\lambda)$ is constant for $\lambda \in \Lambda$, where $S_\lambda = \{x \mid (x, \lambda) \in S\}$ is the λ-section of S.*

Proof. Take a bounded closed isolating neighborhood $\mathcal{U}$ of S in $X \times \Lambda$. Then, the λ-section $\mathcal{U}_\lambda$ of $\mathcal{U}$ is an isolating neighborhood of S_λ. Since S is compact in $X \times \Lambda$, one easily verifies that S_λ is upper semicontinuous in λ, specifically, $d_{\mathrm{H}}(S_{\lambda'}, S_\lambda) \to 0$ as $\lambda' \to \lambda$. Consequently, for each fixed $\lambda \in \Lambda$, $\mathcal{U}_\lambda$ is also an isolating neighborhood of $S_{\lambda'}$ for λ' near λ. Now the conclusion directly follows from [34, Chapter 1, Theorem 12.2]. □

Finally, let us also recall the concept of an isolating block.

Let $B \subset X$ be a bounded closed set and $x \in \partial B$ be a boundary point. x is called a *strict egress* (respectively, *strict ingress*, *bounce-off*) point of B if, for every trajectory $\gamma : [-\tau, s] \to X$ with $\gamma(0) = x$, where $\tau \geq 0$, $s > 0$, the following two properties hold:

(1) There exists $0 < \varepsilon < s$ such that

$$\gamma(t) \notin B \text{ (respectively, } \gamma(t) \in IntB, \text{ respectively } \gamma(t) \notin B), \quad \forall\, t \in (0, \varepsilon).$$

(2) If $\tau > 0$, then there exists $0 < \delta < \tau$ such that

$$\gamma(t) \in \operatorname{int} B \text{ (respectively, } \gamma(t) \notin B, \text{ respectively } \gamma(t) \notin B), \quad \forall\, t \in (-\delta, t).$$

Denote B^e (respectively, B^i, B^b) the set of all strict egress (respectively, strict ingress, bounce-off) points of the closed set B, and set $B^- = B^e \cup B^b$.

A closed set $B \subset X$ is called an *isolating block* if B^- is closed and $\partial B = B^i \cup B^-$. It is well known that if B is a bounded isolating block, then (B, B^-) is an index pair of the maximal compact invariant set S (possibly empty) in B.

For convenience, if B is an isolating block, we call B^- the *boundary exit set*.

3. LOCAL INVARIANT MANIFOLDS

In this section, we present some fundamental results on local invariant manifolds of (1.1). We also state a slightly modified version of a reduction property of the Conley index given in [34].

It is well known that under the hypotheses in Section 1, the initial value problem of (1.1) is well posed in X^α. That is, for each $u_0 \in X^\alpha$ the problem has a unique solution $u(t)$ in X^α with $u(0) = u_0$ on some maximal existence interval $[0, T)$ (see, e.g., [10, Theorem 3.3.3]).

Denote Φ_λ the local semiflow generated by the problem on X^α.

For convenience in the statement, given $Z \subset \mathbb{C}$ and $\alpha \in \mathbb{R}$, we will write $\mathrm{Re}(Z) < \alpha$ $(> \alpha)$, which means that $\mathrm{Re}(\mu) < \alpha$ $(> \alpha)$ for all $\mu \in Z$.

Let $L_\lambda = A - \mathrm{D}f_\lambda(0)$. Suppose then that there exists a neighborhood of λ_0, $J_0 = [\lambda_0 - \eta, \lambda_0 + \eta]$, and $\delta > 0$ such that the following hypotheses are fulfilled:

(H1) The spectral $\sigma(L_\lambda)$ has a decomposition $\sigma(L_\lambda) = \bigcup_{1 \le i \le 3} \sigma_\lambda^i$ with

$$\mathrm{Re}(\sigma_\lambda^1) < -\alpha_1 < -\alpha_2 \le \mathrm{Re}(\sigma_\lambda^2) < \alpha_3 < \alpha_4 < \mathrm{Re}(\sigma_\lambda^3)$$

for $\lambda \in J_0$, where α_i $(1 \le i \le 4)$ are positive constants independent of λ.

(H2) For each $\lambda \in J_0$, X has a decomposition $X = X_\lambda^1 \oplus X_\lambda^2 \oplus X_\lambda^3$ corresponding to the spectral decomposition in (H1), where X_λ^i $(i = 1, 2, 3)$ are L_λ-invariant subspaces of X. Moreover, $\dim(X_\lambda^1), \dim(X_\lambda^2) < \infty$.

(H3) There is a family of invertible bounded linear operators $T = T_\lambda$ on X depending continuously on λ such that, when $\lambda \in J_0$, we have

$$TX_\lambda^i = X_{\lambda_0}^i := X^i, \quad i = 1, 2, 3. \tag{3.1}$$

Remark 3.1. Instead of (H3), a more natural hypothesis is to assume that

(H3)' The projection operators $P_\lambda^i : X \to X_\lambda^i$ $(i = 1, 2)$ are continuous in λ.

Indeed, when (H3)' is fulfilled, it can be shown that there is a family of invertible bounded linear operators $T = T_\lambda$ on X such that (3.1) holds true (see Appendix A in [15] for details).

We rewrite $E = X^\alpha$ and set $E^i = E \cap X^i$, $E^{ij} = E \cap (X^i \oplus X^j)$, where $i, j = 1, 2, 3$ $(i \ne j)$. Then, $E = E^2 \oplus E^{13} = E^3 \oplus E^{12}$.

Remark 3.2. Since $\dim(X_\lambda^1), \dim(X_\lambda^2) < \infty$, we have $E^1 = X^1$, $E^2 = X^2$.

Lemma 3.3. *Assume* (H1)–(H3) *are fulfilled. Then, we have the following:*

(1) *There exist an open convex neighborhood W of 0 in E^2 and a mapping $\xi = \xi_\lambda(w)$ from $W \times J_0$ to E^{13} which is continuous in (w, λ) and differentiable in w, such that for each $\lambda \in J_0$,*

$$\mathcal{M}_\lambda^2 := T^{-1} M_\lambda^2, \quad \text{where } M_\lambda^2 := \{w + \xi_\lambda(w) \mid w \in W\},$$

is a local invariant manifold of the system (1.1).

(2) *There exist an open convex neighborhood V of 0 in E^{12} and a mapping $\zeta = \zeta_\lambda(v)$ from $V \times J_0$ to E^3 which is continuous in (v, λ) and differentiable in v, such that for each $\lambda \in J_0$,*

$$\mathcal{M}_\lambda^{12} := T^{-1}M_\lambda^{12}, \quad \text{where } M_\lambda^{12} := \{v + \zeta_\lambda(v) \mid v \in V\},$$

is a local invariant manifold of the system (1.1).

Proof. The above results are just slight modifications of the existing ones in the literature (see, e.g., [34, Chapter II, Theorem 2.1]). Here, we give a sketch of the proof for the reader's convenience.

Let $B_\lambda = TL_\lambda T^{-1}$, and define

$$g_\lambda(v) = T(f_\lambda(T^{-1}v) - \mathrm{D}f_\lambda(0)(T^{-1}v)), \quad v \in E.$$

Setting $u = T^{-1}v$, the system (1.1) can be transformed into an equivalent one:

$$v_t + B_\lambda v = g_\lambda(v). \tag{3.2}$$

It is trivial to check that $\|\mathrm{D}g_\lambda(v)\| \to 0$ as $\|v\|_\alpha \to 0$ uniformly with respect to $\lambda \in J_0$. Further, by the Mean-value Theorem one easily verifies that for any $\varepsilon > 0$, there exists a neighborhood U of 0 in E such that

$$\|g_\lambda(u) - g_\lambda(v)\| \le \varepsilon\|u - v\|_\alpha, \quad \forall\, u, v \in U,\ \lambda \in J_0.$$

We observe that

$$B_\lambda - \mu I = TL_\lambda T^{-1} - \mu I = T(L_\lambda - \mu I)T^{-1},$$

where $I = \mathrm{id}_X$ is the identity mapping on X, from which it can be easily seen that $\mu \in \mathbb{C}$ is a regular value of B_λ if and only if it is a regular value of L_λ. Hence, one concludes that

$$\sigma(B_\lambda) = \sigma(L_\lambda).$$

Since X_λ^i $(i = 1, 2, 3)$ are L_λ-invariant, it follows by (3.1) that X^i are B_λ-invariant for all $\lambda \in J_0$. Now, using some standard arguments in the geometric theory of PDEs (see Henry [10, Section 6] and Hale Appendix [8],) and the uniform contraction principle, it can be shown that there exist an open convex neighborhood W of 0 in E^2 and a mapping $\xi = \xi_\lambda(w)$ from $W \times J_0$ to E^{13} which is continuous in (w, λ) and differentiable in w, such that for each $\lambda \in J_0$,

$$M_\lambda^2 := \{w + \xi_\lambda(w) \mid w \in W\} \tag{3.3}$$

is a local invariant manifold of the system (3.2). Consequently, $\mathcal{M}_\lambda^2 = T^{-1}M_\lambda^2$ is a local invariant manifold of (1.1).

The proof of part (2) follows a fully analogous argument. □

Let $\mathcal{M}_\lambda^2$ and $\mathcal{M}_\lambda^{12}$ be the local invariant manifolds given in Lemma 3.3, and Φ_λ^2 and Φ_λ^{12} be the restrictions of Φ_λ on $\mathcal{M}_\lambda^2$ and $\mathcal{M}_\lambda^{12}$, respectively, where Φ_λ is the local semiflow generated by (1.1).

The following result is a parameterized version of Theorem 3.1, Chapter II in [34], and can be proved in the same manner as in [34]. We omit the details.

Lemma 3.4. *Assume* (H1)–(H3). *Then, there exist a neighborhood U of 0 in E and a number $\varepsilon > 0$ such that, for every $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$, the following hold:*

(1) *$K \subset U$ is a compact invariant set of Φ_λ if and only if it is a compact invariant set of Φ_λ^2 (respectively, Φ_λ^{12}) on $\mathcal{M}_\lambda^2$ (respectively, $\mathcal{M}_\lambda^{12}$).*

(2) *$K \subset U$ is an isolated invariant set of Φ_λ if and only if it is an isolated invariant set of Φ_λ^2 (respectively, Φ_λ^{12}) on $\mathcal{M}_\lambda^2$ (respectively, $\mathcal{M}_\lambda^{12}$); furthermore,*

$$h(\Phi_\lambda, K) = h(\Phi_\lambda^{12}, K) = \Sigma^m \wedge h(\Phi_\lambda^2, K),$$

where $m = \dim(X^1)$ is the dimension of X^1.

4. LOCAL DYNAMIC BIFURCATION

In this section, we state and prove some local dynamic bifurcation results concerning (1.1) in terms of invariant sets, so we always assume $n := \dim(X^2) \geq 1$.

In what follows, by a *k-dimensional topological sphere* we mean the boundary ∂D of any contractible open subset D of a $(k+1)$-dimensional manifold $\mathcal{M}$ without boundary. We use the notation $\mathbb{S}^k$ to denote any k-dimensional topological sphere.

Definition 4.1. $\mu \in \mathbb{R}$ is called a (dynamic) bifurcation value of (1.1) if, for any neighborhood U of 0 and $\varepsilon > 0$, there exists $\lambda \in (\mu - \varepsilon, \mu + \varepsilon)$ such that Φ_λ has a compact invariant set $K_\lambda \subset U$ with $K_\lambda \setminus \{0\} \neq \emptyset$.

If μ is a bifurcation value, then we call $(0, \mu)$ a (dynamic) bifurcation point.

We are basically interested in the bifurcation phenomena of the system (1.1) near a bifurcation value $\lambda = \lambda_0$. Thus, in addition to (H1)–(H3), we also assume

(H4) There exists $\varepsilon_0 > 0$ such that for $\lambda \in [\lambda_0 - \varepsilon_0, \lambda_0 + \varepsilon_0]$,

$$\begin{aligned} \operatorname{Re}(\sigma_\lambda^2) &< 0 \quad (\text{if } \lambda < \lambda_0), \\ \operatorname{Re}(\sigma_\lambda^2) &> 0 \quad (\text{if } \lambda > \lambda_0). \end{aligned}$$

Let $\mathcal{M}_\lambda^2$ and $\mathcal{M}_\lambda^{12}$ be the local invariant manifolds given in Lemma 3.3, and Φ_λ^2 and Φ_λ^{12} be the restrictions of Φ_λ on $\mathcal{M}_\lambda^2$ and $\mathcal{M}_\lambda^{12}$, respectively.

Convention. *For simplicity in statement, from now on we set $\lambda_0 = 0$.*

4.1. Attractor/repeller bifurcation. We give here an attractor/repeller-bifurcation theorem, which slightly generalizes some fundamental results in Ma and Wang [22, Theorem 6.1] and [21, Theorem 4.3]. For the reader's convenience, we also present a self-contained proof for the theorem.

Theorem 4.2. *Assume* (H1)–(H4) *are fulfilled* (*with* $\lambda_0 = 0$).

Suppose 0 *is an attractor* (*respectively, repeller*) *of* Φ_0^2. *Then, there exists a closed neighborhood* U *of* 0 *in* E *and a number* $\varepsilon > 0$ *such that, for each* $\lambda \in [-\varepsilon, 0)$ (*respectively,* $(0, \varepsilon]$), *the system* Φ_λ *bifurcates from* 0 *a maximal compact invariant set* $K_\lambda \neq \emptyset$ *in* $U \setminus \{0\}$ *which contains an invariant topological sphere* $\mathbb{S}^{n-1}$. *Furthermore,*

$$\lim_{\lambda \to 0} d_H(K_\lambda, S_0) = 0.$$

Proof.
Case 1: 0 is an attractor of Φ_0^2. We first consider the equivalent system (3.2) for $\lambda \in J_0$. When (3.2) is restricted to the local center manifold M_λ^2 defined by (3.3), it reduces to an ODE system on an open neighborhood W (independent of λ) of 0 in E^2:

$$w_t = -B_\lambda^2 w + P^2 g_\lambda(w + \xi_\lambda(w)) := F_\lambda(w), \tag{4.1}$$

where $B_\lambda^2 = P^2 B_\lambda$, and P^2 is the projection from $E = X^\alpha$ to E^2. Applying Lemma 3.4 to (3.2), one deduces that there exist a neighborhood $\mathcal{U}$ of 0 in E and $\varepsilon_0 > 0$ such that for $\lambda \in [-\varepsilon_0, \varepsilon_0]$, S is an isolated invariant set of (3.2) in $\mathcal{U}$ if and only if it is an isolated invariant set of the system restricted to the manifold M_λ^2.

Denote φ_λ the local semiflow on W generated by (4.1). Since 0 is an attractor of Φ_0^2, we find that $S_0 := \{0\}$ is an attractor of φ_0. Let $\Omega = \mathcal{U}(S_0)$ be the attraction basin of S_0 in W with respect to φ_0. Then, by the converse Lyapunov theorem on attractors (see, e.g., [14, Theorems 3.1 and 3.2]), one can find a function $V \in C^\infty(\Omega)$ with $V(0) = 0$ and $\lim_{x \to \partial\Omega} V(x) = +\infty$ such that

$$\nabla V(x) F_0(x) \leq -v(x), \quad \forall\, x \in \Omega, \tag{4.2}$$

where $v \in C(\Omega)$ and $v(x) > 0$ for $x \neq 0$. Let

$$N = V_a := \{x \in \Omega \mid V(x) \leq a\}.$$

Then, N is a compact neighborhood of 0 in E^2. Pick two numbers $a, \rho > 0$ sufficiently small so that

$$\tilde{U} := N \times \mathrm{B}_{E^{13}}(\xi_0(N), \rho) \subset \mathcal{U}, \tag{4.3}$$

where ξ_0 is the mapping determining the local center manifold M_0^2 given in Lemma 3.3, and $\mathrm{B}_{E^{13}}(\xi_0(N), \rho)$ denotes the ρ-neighborhood of $\xi_0(N)$ in E^{13}.

By (4.2), we have

$$\nabla V(x) F_0(x) \leq -\mu, \quad \forall\, x \in \partial N,$$

where $\mu = \min_{x\in\partial N} v(x) > 0$, and ∂N is the boundary of N in E^2. Further, by the continuity of F_λ in λ, there exists $0 < \varepsilon_1 \le \varepsilon_0$ such that

$$\nabla V(x)F_\lambda(x) \le -\frac{\mu}{2}, \quad \forall\, x \in \partial N \tag{4.4}$$

for $\lambda \in [-\varepsilon_1, \varepsilon_1]$, which implies that N is a positively invariant set of φ_λ.

It can be assumed that ε_1 is sufficiently small so that

$$\xi_\lambda(N) \subset \mathrm{B}_{E^{13}}(\xi_0(N), \rho), \quad \lambda \in [-\varepsilon_1, \varepsilon_1]. \tag{4.5}$$

Now, assume $\lambda \in [-\varepsilon_1, 0)$. Consider the inverse flow φ_λ^- of φ_λ generated by the system

$$w_t = -F_\lambda(w) := B_\lambda^2 w - P^2 g_\lambda(w + \xi_\lambda(w)). \tag{4.6}$$

By (H4), we find that $\mathrm{Re}(\sigma(B_\lambda^2)) < 0$, which implies that S_0 is an attractor of φ_λ^-. Let $G_\lambda = \mathcal{U}(S_0)$ be the attraction basin of S_0 in W with respect to φ_λ^-. We infer from (4.4) that each $x \in \partial N$ is a strict ingress point of φ_λ, and hence is a strict egress point of φ_λ^-. Thus, one necessarily has $G_\lambda \subset N$. Therefore, the boundary ∂G_λ of G_λ in E^2 is contained in N (see Figure 4.1).

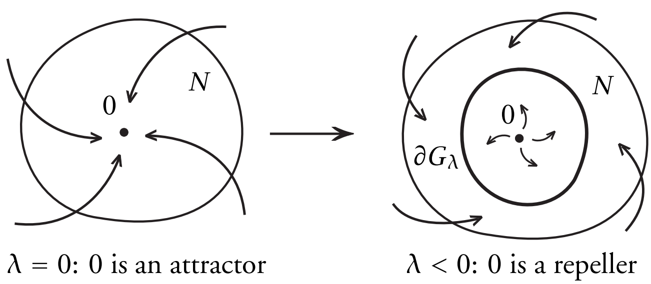


FIGURE 4.1. Attractor-bifurcation

We prove that ∂G_λ is an invariant set of φ_λ^-. For this purpose, it suffices to show that for each $x_0 \in \partial G_\lambda$, there is a complete trajectory $w(t)$ of φ_λ^- (i.e., a solution of (4.6)) with $w(0) = x_0$ such that $w(t) \in \partial G_\lambda$ for all $t \in \mathbb{R}$.

Note that (4.6) always has a unique solution $w(t)$ defined on a maximal existence interval J such that $w(0) = x_0$. Since $x_0 \notin G_\lambda$, by Remark 2.7 we deduce that $w(t) \notin G_\lambda$ for all $t \in J$. We claim that $w(t) \in \partial G_\lambda$ for $t \in J$, and consequently one also has $J = \mathbb{R}$, thus completing the proof of the invariance of ∂G_λ. We argue by contradiction and suppose the claim is false. Then, there would exist $t_0 \in J$ such that $w(t_0) \notin \bar{G}_\lambda$. Hence, $d(w(t_0), \bar{G}_\lambda) > 0$. Take a sequence

$x_k \in G_\lambda$ such that $x_k \to x_0$. Let $w_k(t)$ by the solution of (4.6) with $w_k(0) = x_k$. Then, by continuity properties on ODEs, we know that t_0 belongs to the maximal existence interval J_k of $w_k(t)$ if k is sufficiently large; furthermore, $w_k(t_0) \notin \bar{G}_\lambda$. But by Remark 2.7, this leads to a contradiction since $w_k(0) = x_k \in G_\lambda$.

Denote A_λ the maximal compact invariant set of φ_λ in $N \setminus G_\lambda$. Clearly, $\partial G_\lambda \subset A_\lambda$. It is trivial to check that A_λ is the maximal compact invariant set of φ_λ in $N \setminus S_0$. Since N is an isolating neighborhood of S_0 with respect to φ_0, by a simple argument via contradiction it can be shown that

$$\lim_{\lambda \to 0} d_H(A_\lambda, S_0) = 0. \tag{4.7}$$

We claim that ∂G_λ is an $(n-1)$-dimensional topological sphere. Indeed, define

$$H(s,x) = \begin{cases} \varphi_\lambda^-\left(\dfrac{s}{1-s}\right)x, & s \in [0,1),\ x \in G_\lambda, \\ 0, & s = 1,\ x \in G_\lambda. \end{cases}$$

Then, H is a strong deformation retraction shrinking G_λ to the point 0. This shows that G_λ is contractible, and proves our claim.

Now, we define

$$\begin{aligned} \tilde{K}_\lambda &= \{w + \xi_\lambda(w) \mid w \in A_\lambda\}, \\ \tilde{\mathbb{S}} &= \{w + \xi_\lambda(w) \mid w \in \partial G_\lambda\}, \end{aligned}$$

where ξ_λ is the mapping in (4.1) given by Lemma 3.3. By (4.5) and (4.3), we find that $\tilde{K}_\lambda \subset \tilde{U} \subset \mathcal{U}$. $\tilde{K}_\lambda$ is the maximal compact invariant set of the system (3.2) in $\tilde{U} \setminus \{0\}$. It follows by (4.7) that $\lim_{\lambda\to 0} d_H(\tilde{K}_\lambda, S_0) = 0$.

Finally, let $U_\lambda = T^{-1}\tilde{U}$, where $T = T_\lambda$ is the linear operator in (H3). Then, one can find a closed neighborhood U of 0 in E and a number $0 < \varepsilon \le \varepsilon_1$ such that $U \subset U_\lambda$ for all $\lambda \in [-\varepsilon, 0)$. Set $K_\lambda = T^{-1}\tilde{K}_\lambda$. Then, $\lim_{\lambda\to 0} d_H(K_\lambda, S_0) = 0$. Thus, we may assume that ε is chosen sufficiently small so that $K_\lambda \subset \operatorname{int} U$ for all $\lambda \in [-\varepsilon, 0)$. It is easy to see that U and K_λ fulfill all the requirements of the theorem.

Case 2: The equilibrium 0 is a repeller of Φ_0^2. This case can be treated by replacing (4.1) and (4.6) with each other and repeating the above argument. We omit the details. □

4.2. Invariant-set bifurcation. We now state and prove a general local invariant-set bifurcation theorem.

Theorem 4.3. *Assume that* (H1)–(H4) *are fulfilled. Suppose $S_0 = \{0\}$ is an isolated invariant set of Φ_0. Then, one of the following assertions holds:*

(1) *S_0 is an attractor* (*respectively, repeller*) *of Φ_0^2. In such a case, the system undergoes an attractor-bifurcation* (*respectively repeller-bifurcation*) *in Theorem 4.2.*

(2) *There exist a closed neighborhood U of 0 in E and a two-sided neighborhood I_2 of λ_0 such that Φ_λ has a nonempty maximal compact invariant set K_λ in $U \setminus S_0$ for each $\lambda \in I_2 \setminus \{\lambda_0\}$.*

Furthermore, in both cases the bifurcating invariant set K_λ is upper semicontinuous in λ with $\lim_{\lambda\to 0} d_H(K_\lambda, 0) = 0$.

Proof. Let us first verify the bifurcation results in (1) and (2). For this purpose, it suffices to assume S_0 is neither an attractor nor a repeller of Φ_0^2, and to prove that the second assertion (2) holds true.

Let us start with the local semiflow φ_λ generated by the bifurcation equation (4.1) on W. Since $S_0 = \{0\}$ is an isolated invariant set of Φ_0, by Lemma 3.4 it is isolated for Φ_0^2. Because Φ_λ^2 and φ_λ are conjugate, one concludes that S_0 is an isolated invariant set of φ_0.

Note that (H4) implies

$$\operatorname{Re}(\sigma(B_\lambda^2)) < 0 \quad (\lambda < 0),$$
$$\operatorname{Re}(\sigma(B_\lambda^2)) > 0 \quad (\lambda > 0),$$

where B_λ^2 is the linear operator in (4.1). Hence, S_0 is a repeller of φ_λ when $\lambda < 0$, and an attractor when $\lambda > 0$. By Lemma 3.4 we also have for some $\varepsilon_1 > 0$ that

$$\begin{aligned} h(\varphi_\lambda, S_0) &= \Sigma^n \quad (\lambda \in [-\varepsilon_1, 0)),\\ h(\varphi_\lambda, S_0) &= \Sigma^0 \quad (\lambda \in (0, \varepsilon_1]). \end{aligned} \tag{4.8}$$

Pick a closed neighborhood W_0 of S_0 in E^2 such that it is an isolating neighborhood of S_0 with respect to φ_0. Then, by a simple argument via contradiction, we deduce that W_0 is also an isolating neighborhood of the maximal compact invariant set S_λ of φ_λ in W_0 provided λ is sufficiently small; furthermore,

$$\lim_{\lambda\to 0} d_H(S_\lambda, S_0) = 0. \tag{4.9}$$

Fix a positive number $\varepsilon < \varepsilon_1$ such that W_0 is an isolating neighborhood of S_λ for all $\lambda \in [-\varepsilon, \varepsilon]$. Then, Theorem 2.12 asserts that

$$h(\varphi_\lambda, S_\lambda) \equiv \text{const.}, \quad \lambda \in [-\varepsilon, \varepsilon]. \tag{4.10}$$

In what follows, we show that

$$h(\varphi_0, S_0) \neq \Sigma^0. \tag{4.11}$$

Since S_0 is an isolated invariant set of φ_0, by [7, Theorem 1.5], one can find a connected isolating block B of S_0 (with respect to φ_0) with smooth boundary ∂B.

We claim that $B^- \neq \emptyset$, where B^- is the boundary exit set of B with respect to the flow φ_0. Indeed, if $B^- = \emptyset$ then B is positively invariant under the system φ_0. Because S_0 is the maximal compact invariant set of φ_0 in B, one easily deduces that it is an attractor of φ_0, which contradicts the assumption that S_0 is not an attractor of Φ_0^2 (recall that Φ_λ^2 and φ_λ are conjugate).

Denote H_* the singular homology theories with coefficient group $\mathbb{Z}$. Then, $h(\varphi_0, S_0) = [(B/B^-, [B^-])]$, and therefore,

$$H_0(h(\varphi_0, S_0)) = H_0((B/B^-, [B^-])) = H_0(B, B^-).$$

As B is path-connected and $B^- \neq \emptyset$, by the basic knowledge in the theory of algebraic topology we find $H_0(B, B^-) = 0$. Consequently, $H_0(h(\varphi_0, S_0)) = 0$. On the other hand, recalling that Σ^0 is the homotopy type of any pointed space $(\{p, q\}, q)$ consisting of exactly two distinct points p and q, we have

$$H_0(\Sigma^0) = H_0((\{p, q\}, q)) = \mathbb{Z}.$$

Hence, we see that (4.11) holds true.

Now assume $\lambda \in (0, \varepsilon]$. Combining (4.8) and (4.10) yields

$$h(\varphi_\lambda, S_\lambda) = h(\varphi_0, S_0) \neq h(\varphi_\lambda, S_0),$$

which implies that $S_\lambda \setminus S_0 \neq \emptyset$. Recall that S_0 is an attractor of φ_λ. Let

$$R_\lambda = \{x \in S_\lambda \mid \omega(x) \cap S_0 = \emptyset\}.$$

Then, R_λ is a nonempty compact invariant set of φ_λ with (R_λ, S_0) being a repeller-attractor pair of S_λ (see [34, p. 141]). Because S_λ is maximal in W_0, it can be easily seen that R_λ is precisely the maximal compact invariant set in $W_0 \setminus S_0$.

Consider the inverse flow φ_λ^- of φ_λ on W. Then, we have

$$\begin{aligned} h(\varphi_\lambda^-, S_0) &= \Sigma^0 \quad (\lambda \in [-\varepsilon, 0)), \\ h(\varphi_\lambda^-, S_0) &= \Sigma^n \quad (\lambda \in (0, \varepsilon]). \end{aligned}$$

Since S_0 is a repeller of φ_λ for $\lambda \in [-\varepsilon, 0)$, it is an attractor of φ_λ^-. Repeating the argument above with φ_λ replaced by φ_λ^-, one immediately deduces that φ_λ has a nonempty maximal compact invariant set R_λ in $W_0 \setminus S_0$ for $\lambda \in [-\varepsilon, 0)$.

We show that R_λ is upper semicontinuous in λ. We only consider the case where $\lambda \in (0, \varepsilon]$. The argument for the case where $\lambda \in [-\varepsilon, 0)$ can be performed in the same manner by considering the inverse flow φ_λ^-, and so we omit the details.

Let $\mathcal{U}_\lambda = \mathcal{U}(S_0)$ be the attraction basin of S_0 in W with respect to φ_λ. For each fixed $\lambda > 0$, pick a number $r > 0$ such that $\bar{\mathrm{B}}_r \subset \mathcal{U}_\lambda$, where (and below) B_r denotes the ball in E^2 centered at 0 with radius r. Then, by the stability property of attraction basins (see, e.g., Li [13, Theorem 2.9]), there exists $\rho > 0$ such that $\bar{\mathrm{B}}_{r/2} \subset \mathcal{U}_{\lambda'}$ provided $|\lambda' - \lambda| \le \rho$. This implies that

$$R_{\lambda'} \cap \bar{\mathrm{B}}_{r/2} = \varnothing$$

for all $\lambda' \in (0, \varepsilon]$ with $|\lambda' - \lambda| \le \rho$. We check that $\lim_{\lambda' \to \lambda} d_{\mathrm{H}}(R_{\lambda'}, R_\lambda) = 0$, thus proving what we desired.

Suppose the contrary. There would then exist $\lambda_k \to \lambda$ and $\delta_0 > 0$ such that

$$d_{\mathrm{H}}(R_{\lambda_k}, R_\lambda) \ge \delta_0, \quad \forall\, k \ge 1.$$

We may assume $|\lambda_k - \lambda| \le \rho$, and hence $R_{\lambda_k} \subset W_0 \setminus \mathrm{B}_{r/2}$ for all k. Because of Lemma 2.2, it can be assumed that R_{λ_k} converges to a nonempty compact subset R'_λ of $W_0 \setminus \mathrm{B}_{r/2}$ in the sense of Hausdorff distance $\delta_{\mathrm{H}}(\,,\,)$. Then, $d_{\mathrm{H}}(R'_\lambda, R_\lambda) \ge \delta_0$. On the other hand, we can trivially verify that R'_λ is an invariant set of φ_λ. Thus, $R_\lambda \cup R'_\lambda$ is a compact invariant set of φ_λ in $W_0 \setminus S_0$. This contradicts the maximality of R_λ in $W_0 \setminus S_0$.

We are now ready to complete the proof of the theorem. Let U be the neighborhood of 0 given in Lemma 3.4. We may restrict U sufficiently small in advance so that $P^2 T_\lambda U \subset W_0$ for all $\lambda \in [-\varepsilon, \varepsilon]$, where $P^2 : E \to E^2$ is the projection, and T_λ is the operator in (H3). Let $K_\lambda = T_\lambda^{-1} \tilde{R}_\lambda$, where

$$\tilde{R}_\lambda = \{w + \xi_\lambda(w) \mid w \in R_\lambda\}.$$

Then, K_λ is upper semicontinuous in λ and is a compact invariant set of Φ_λ. By (4.9), we have $\lim_{\lambda \to 0} d_{\mathrm{H}}(R_\lambda, S_0) = 0$. It follows that $\lim_{\lambda \to 0} d_{\mathrm{H}}(K_\lambda, S_0) = 0$. Thus, we can assume ε is chosen sufficiently small so that $K_\lambda \subset U$ for $\lambda \in [-\varepsilon, \varepsilon]$.

We claim that K_λ is the maximal compact invariant set of Φ_λ in $U \setminus S_0$, which completes the proof of the theorem. Indeed, if this were false, then Φ_λ would have another compact invariant set $K'_\lambda \subset U \setminus S_0$ such that $K_\lambda \subsetneq K'_\lambda$. It follows that

$$R_\lambda = P^2 T_\lambda K_\lambda \subsetneq P^2 T_\lambda K'_\lambda := R'_\lambda.$$

By the invariance of K'_λ, it is easy to deduce that R'_λ is a compact invariant set of φ_λ in $W_0 \setminus S_0$. However, this contradicts the maximality of R_λ in $W_0 \setminus S_0$. □

4.3. Some remarks on static bifurcation. It is worth noticing that Theorem 4.3 may also give us information on the static bifurcation of the system in some cases. For example, if the stationary problem

$$Au = f_\lambda(u), \quad u \in E := X^\alpha \tag{4.12}$$

has a variational structure, then (1.1) is a gradient-like system, and each nonempty compact invariant set K of Φ_λ contains at least one equilibrium point, which is precisely a solution of (4.12). On the other hand, it is also easy to see that if K consists of at least two distinct points, then it contains at least two distinct equilibrium points of Φ_λ. Thus, under the hypotheses of Theorem 4.3, one immediately concludes that either there is a one-sided neighborhood I_1 of λ_0 such that (4.12) bifurcates two distinct nontrivial solutions for each $\lambda \in I_1 \setminus \{\lambda_0\}$, or there is a two-sided neighborhood I_2 of λ_0 such that (4.12) bifurcates at least one nontrivial solution for each $\lambda \in I_2 \setminus \{\lambda_0\}$.

We refer the interested reader to [4,32,33], [36], and so on for more detailed bifurcation results on such operator equations.

As another example, we consider the particular but important case where

$$n = \dim(X^2) = 1.$$

We first claim that each compact invariant set C_λ of Φ_λ close to 0 contains at least one equilibrium point which is a solution of (4.12). Indeed, each such invariant set C_λ is contained in the local invariant manifold $\mathcal{M}_\lambda^2$. Because $\mathcal{M}_\lambda^2$ is a C^1 curve, every connected component ℓ of C_λ is a segment of $\mathcal{M}_\lambda^2$. Since (1.1) reduces to a one-dimensional ODE on $\mathcal{M}_\lambda^2$ (hence, backward uniqueness holds on $\mathcal{M}_\lambda^2$), by invariance of ℓ it is trivial to deduce that the end points of ℓ are equilibria of Φ_λ.

Using the above basic fact, we can also easily verify that 0 is an isolated solution of (4.12) at λ_0 if and only if $S_0 = \{0\}$ is an isolated invariant set of Φ_{λ_0}. By Theorem 4.3, we immediately obtain the following bifurcation result, which generalizes Henry [10, Theorem 6.3.2].

Theorem 4.4. *Assume* (H1)–(H4) *are fulfilled with* $\dim(X^2) = 1$. *Then, one of the following alternatives occurs:*

(1) *There is a sequence u_k of nontrivial solutions of* (4.12) *at $\lambda = \lambda_0$ such that $u_k \to 0$ as $k \to \infty$.*
(2) *There is a one-sided neighborhood I_1 of λ_0 such that* (4.12) *bifurcates at least two nontrivial solutions for each $\lambda \in I_1 \setminus \{\lambda_0\}$.*
(3) *There is a two-sided neighborhood I_2 of λ_0 such that* (4.12) *bifurcates at least one nontrivial solution for each $\lambda \in I_2 \setminus \{\lambda_0\}$.*

Remark 4.5. When $\dim(X^2) = 1$, we can also use the classical Crandall-Rabinowitz Theorem (see [11, Theorem I.5.1] to derive more explicit static bifurcation results under some additional assumptions such as the *transversality condition*. (Some nice bifurcation results when the transversality condition mentioned above is violated can be found in [17], etc.) Other general bifurcation theorems such as the Krasnosel'skii Bifurcation Theorem (see [11, Theorem II.3.2]) also apply to dealing with this special case.

Remark 4.6. Whether the bifurcating invariant set K_λ contains equilibrium solutions is an interesting problem. In the case of attractor-bifurcation, this problem has already been addressed by Ma and Wang [22, p. 155, Theorem 6.1], where one can find an index formula on equilibrium solutions. For the general case treated here, results in this line will be reported in our forthcoming paper entitled "Equilibrium index of invariant sets and global static bifurcation for nonlinear evolution equations."

5. NONTRIVIALITY OF THE CONLEY INDICES OF THE BIFURCATING INVARIANT SETS

Our main goal in this section is to show that the bifurcating invariant set K_λ in Theorem 4.3 has nontrivial Conley index. This result will play a crucial role in establishing our global dynamic bifurcation theorem. However, it may also be of independent interest in its own right.

Let $m = \dim(X^1)$, $n = \dim(X^2)$ $(n \geq 1)$, and let K_λ be the bifurcating invariant set of Φ_λ in Theorem 4.3.

Theorem 5.1. *Suppose* (H1)–(H4) *are fulfilled* (*with* $\lambda_0 = 0$), *and* $S_0 = \{0\}$ *is an isolated invariant set of* Φ_0. *Then, there exists* $\varepsilon > 0$ *such that the following hold:*

(1) *If* $h(\Phi_0, S_0) \neq \Sigma^{m+n}$, *then*

$$h(\Phi_\lambda, K_\lambda) \neq \bar{0}, \quad \lambda \in [-\varepsilon, 0); \tag{5.1}$$

(2) *If* $h(\Phi_0, S_0) \neq \Sigma^m$, *then*

$$h(\Phi_\lambda, K_\lambda) \neq \bar{0}, \quad \lambda \in (0, \varepsilon].$$

Proof. Let U be the neighborhood of 0 given in Theorems 4.2 and 4.3. Since S_0 is an isolated invariant set of Φ_0, we can pick an $\varepsilon > 0$ sufficiently small such that U is an isolating neighborhood of the maximal compact invariant set S_λ of Φ_λ for all $\lambda \in [-\varepsilon, \varepsilon]$. We may also assume that U and ε are chosen sufficiently small so that Lemma 3.4 applies.

(1) Assume $h(\Phi_0, S_0) \neq \Sigma^{m+n}$. Let $\lambda \in [-\varepsilon, 0)$. Then, by (H1) and (H4),

$$h(\Phi_\lambda, S_0) = \Sigma^{m+n} \neq h(\Phi_0, S_0),$$

and the system bifurcates in $U \setminus S_0$ a maximal compact invariant set K_λ. By Lemma 3.4 one has $h(\Phi_\lambda, K_\lambda) = h(\Phi_\lambda^{12}, K_\lambda)$. Therefore, to prove (5.1), we need to check that $h(\Phi_\lambda^{12}, K_\lambda) \neq \bar{0}$.

Choose an isolating block $N = N_\lambda$ of S_λ in $\mathcal{M}_\lambda^{12}$. Since S_0 is a repeller of Φ_λ^{12} on $\mathcal{M}_\lambda^{12}$ (by (H4)), one can find an isolating block N_0 of S_0 in $\mathcal{M}_\lambda^{12}$ (depending upon λ) with $K_\lambda \cap N_0 = \emptyset$ such that $N_0^- = \partial N_0$, where ∂N_0 is the boundary of N_0 in $\mathcal{M}_\lambda^{12}$. Then, $M = N \setminus \operatorname{int} N_0$ is an isolating block of K_λ (see Figure 5.1 (1)).

As $h(\Phi_\lambda, S_0) = \Sigma^{m+n}$, one finds that

$$S^{m+n} \simeq N_0/\partial N_0 = N/M \cong (N/N^-)/\tilde{M}, \tag{5.2}$$

where $\tilde{M} = \pi_{N^-}(M)$, and $\pi_{N^-} : N \to N/N^-$ is the projection. Now, let us argue by contradiction, and suppose that $h(\Phi_\lambda^{12}, K_\lambda) = \bar{0}$. Noticing that we have

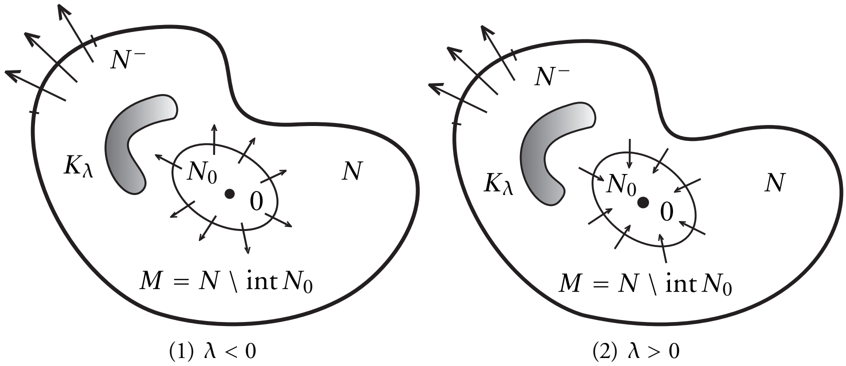


FIGURE 5.1.

$(M/N^-, [N^-]) \cong (\tilde{M}, [N^-])$ (here, we have used the same notation $[N^-]$ to denote both the base points in M/N^- and N/N^-), we deduce that

$$[(\tilde{M}, [N^-])] = \big[(M/N^-, [N^-])\big] = h(\Phi_\lambda^{12}, K_\lambda) = \bar{0},$$

where "$[\cdot]$" denotes homotopy type. This implies that $\tilde{M}$ is contractible.

By a standard argument, one can easily show that ∂N_0 is a strong deformation retract of $N_0 \setminus S_0$. Consequently, M is a strong deformation retract of $N \setminus S_0$. It then follows that $\tilde{M}$ is a strong deformation retract of $(N \setminus S_0)/N^-$. Hence, by [34, Chapter I, Proposition 3.6], we deduce that the pair $(N/N^-, \tilde{M})$ has the homotopy extension property. Further, by Lemma 2.5 and (5.2), it holds that

$$N/N^- \simeq (N/N^-)/\tilde{M} \simeq S^{m+n}. \tag{5.3}$$

On the other hand, by the continuation property of the index, we have

$$h(\Phi_\lambda, S_\lambda) = h(\Phi_0, S_0) \neq \Sigma^{m+n}, \quad \lambda \in [-\varepsilon, 0).$$

Since $h(\Phi_\lambda, S_\lambda) = h(\Phi_\lambda^{12}, S_\lambda)$, we find that

$$h(\Phi_\lambda^{12}, S_\lambda) = [(N/N^-, [N^-])] \neq \Sigma^{m+n}.$$

This implies that $N/N^- \not\simeq S^{m+n}$, which contradicts (5.3).

(2) Now, consider the case where $h(\Phi_0, S_0) \neq \Sigma^m$.

Let $\lambda \in (0, \varepsilon]$. Then, by (H1) and (H4), we have $h(\Phi_\lambda, S_0) = \Sigma^m$. Hence, $h(\Phi_\lambda, S_0) \neq h(\Phi_0, S_0)$, so the system bifurcates in $U \setminus S_0$ a maximal compact invariant set $K_\lambda \neq \emptyset$.

Let S_λ be the maximal compact invariant set of Φ_λ in U. Then, $S_\lambda \subset \mathcal{M}_\lambda^2$. By Lemma 3.4, we have

$$h(\Phi_\lambda, S_\lambda) = \Sigma^m \wedge h(\Phi_\lambda^2, S_\lambda). \tag{5.4}$$

On the other hand,

$$h(\Phi_\lambda, S_\lambda) = h(\Phi_0, S_0) \neq \Sigma^m. \tag{5.5}$$

Thus, by (5.4) and (5.5), we conclude that

$$h(\Phi_\lambda^2, S_\lambda) \neq \Sigma^0. \tag{5.6}$$

As Φ_λ^2 and the semiflow φ_λ generated by the ODE system (4.1) on W are conjugate, in the following argument we identify Φ_λ^2 with φ_λ, regardless of the conjugacy between them. By [7, Theorem 1.5], one can find a connected isolating block N of S_0 (with respect to φ_0) with smooth boundary ∂N. Further, by [7, Theorem 1.6], it can be assumed that ε is sufficiently small so that N is an isolating block of S_λ (with respect to φ_λ) for all $\lambda \in (0, \varepsilon]$ with $B_\lambda^- \equiv B_0^- := N^-$, where B_λ^- denotes the boundary exit set of N with respect to φ_λ. We claim that $N^- \neq \emptyset$. Indeed, if this were false, S_0 would be an attractor of φ_0 in N that attracts N. As S_0 is a singleton, it follows that N is contractible. Consequently,

$$[h(\varphi_\lambda, S_\lambda) = h(\varphi_0, S_0) = [(N/N^-, [N^-])] = [(N, \emptyset)] = \Sigma^0$$

for $\lambda \in (0, \varepsilon]$, which contradicts (5.6).

Because S_0 is an attractor of φ_λ in W for $\lambda \in (0, \varepsilon]$ (by (H4)), using an appropriate smooth Lyapunov function of S_0, we can find an arbitrarily small isolating block N_0 of S_0 (depending upon λ) with smooth boundary ∂N_0 such that $N_0^- = \emptyset$, where N_0^- is the boundary exit set of N_0 with respect to φ_λ. Note that $M := N \setminus \operatorname{int} N_0$ is then an isolating block of K_λ (with respect to φ_λ) with $M^- = N^- \cup \partial N_0$ (see Figure 5.1 (2)). We show that

$$CH_*(\varphi_\lambda, K_\lambda) = H_*(h(\varphi_\lambda, K_\lambda)) \neq 0, \tag{5.7}$$

where $CH_*(\varphi_\lambda, K_\lambda)$ is the homology Conley index of K_λ with respect to φ_λ.

First, we infer from [34] that the inclusion $M^- \subset M$ has the homotopy extension property. This implies that M^- is a strong deformation retract of one of its neighborhoods in M. As N^- and ∂N_0 are disjointed compact subsets of M, each of them is a strong deformation retract of a neighborhood of itself in M. We collapse N^- and ∂N_0 to two distinct points z and w (see Figure 5.1 (2)), respectively, and denote $\tilde{M}$ the corresponding quotient space. Let $\tilde{M}_0 = \{z, w\}$. Then,

$$h(\varphi_\lambda, K_\lambda) = [(M/M^-, [M^-])] = [(\tilde{M}/\tilde{M}_0, [\tilde{M}_0])]. \tag{5.8}$$

Consider the mapping cone C_f as depicted in Figure 5.2, where $f : \tilde{M}_0 \to \tilde{M}$ is the inclusion. Let

$$C\tilde{M}_0 = (\tilde{M}_0 \times I)/(\tilde{M}_0 \times \{1\}).$$

Then, $C\tilde{M}_0$ is homeomorphic to $I = [0,1]$. Hence, one can think of C_f as the space obtained by identifying the end points 0 and 1 of I with z and w, respectively, in the disjoint union of $\tilde{M}$ and I. We observe that $\tilde{M}_0$ is a strong deformation retract of an appropriate neighborhood in $\tilde{M}$. Consequently, $C\tilde{M}_0$ is a strong deformation retract of an appropriate neighborhood in C_f. Noticing that C_f is metrizable, by [34, Chapter I, Proposition 3.6], we deduce that the inclusion $C\tilde{M}_0 \subset C_f$ has the homotopy extension property. Since $C\tilde{M}_0$ is contractible, by the basic knowledge on homotopy equivalence (see, e.g., [9, Proposition 0.17]), we have

$$\tilde{M}/\tilde{M}_0 = C_f/C\tilde{M}_0 \simeq C_f. \tag{5.9}$$

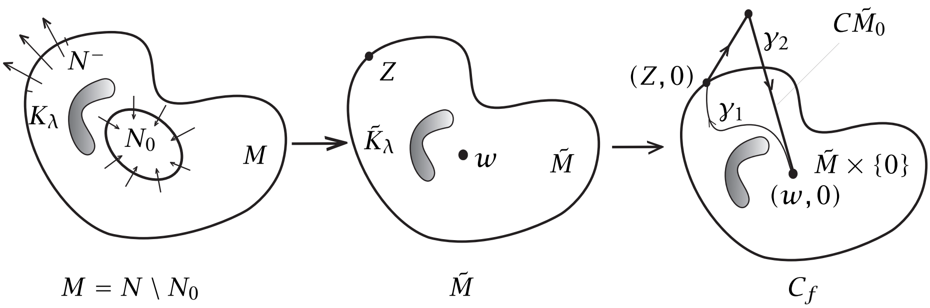


FIGURE 5.2. $M/M^- \simeq \tilde{M}/\tilde{M}_0 \simeq C_f/C\tilde{M}_0 \simeq C_f$

Because M is a domain in E^2 with smooth boundary, we deduce that M is path-connected. It then follows that $\tilde{M}$ is path-connected as well. Consequently C_f is a path-connected space. Let γ_1 be a path in $\tilde{M} \times \{0\}$ from $(w,0)$ to $(z,0)$ (see Figure 5.2), and γ_2 be a path in C_f from $(z,0)$ to $(w,0)$ along $C\tilde{M}_0$. Define a closed path γ in C_f from $(z,0)$ to $(z,0)$ to be the product $\gamma_1 * \gamma_2$ of γ_1 and γ_2. Then, by a simple continuity argument, it can be easily shown that γ is not homotopic to any constant path. Thus, the fundamental group $\pi_1(C_f) \neq 0$. Further, by some basic knowledge in the theory of algebraic topology, we know that $H_1(C_f) \neq 0$. In view of (5.8) and (5.9), we immediately conclude that $H_1(h(\varphi_\lambda, K_\lambda)) \neq 0$. This finishes the proof of (5.7).

Now, we verify that $h(\Phi_\lambda, K_\lambda) \neq \bar{0}$. By Lemma 3.4, it suffices to check that $h(\Phi_\lambda^{12}, K_\lambda) \neq \bar{0}$. Suppose the contrary. Then, we would have $CH_*(\Phi_\lambda^{12}, K_\lambda) = 0$. Invoking the Poincaré-Lefschetz duality theory on homology Conley index (see

McCord [18, Theorem 2.1]), it then holds that $CH^*((\Phi_\lambda^{12})^-, K_\lambda) = 0$, where $(\Phi_\lambda^{12})^-$ denotes the inverse flow of Φ_λ^{12}. On the other hand, for $(\Phi_\lambda^{12})^-$, we have

$$h((\Phi_\lambda^{12})^-, K_\lambda) = h((\Phi_\lambda^{2})^-, K_\lambda) = h(\varphi_\lambda^-, K_\lambda).$$

(Recall that we identify Φ_λ^2 with φ_λ, regardless of the conjugacy between them.) Hence, $CH^*(\varphi_\lambda^-, K_\lambda) = 0$. Again, by the Poincaré-Lefschetz duality theory, we find that $CH_*(\varphi_\lambda, K_\lambda) = 0$, which contradicts (5.7). □

6. GLOBAL DYNAMIC BIFURCATION

In this section, we establish a global dynamic bifurcation result.

6.1. Existence of a local bifurcation branch. We first prove an existence result for a local bifurcation branch.

Set $\mathcal{E} = E \times \mathbb{R}$, where $E = X^\alpha$. Here, $\mathcal{E}$ is equipped with the metric ρ defined as

$$\rho((u,\lambda),(v,\lambda')) = \|u - v\|_\alpha + |\lambda - \lambda'|, \quad \forall\, (u,\lambda),(v,\lambda') \in \mathcal{E}.$$

Let $\mathcal{Z} \subset \mathcal{E}$. For any $\lambda \in \mathbb{R}$, denote $\mathcal{Z}_\lambda$ the *λ-section* of $\mathcal{Z}$, $\mathcal{Z}_\lambda = \{u \mid (u,\lambda) \in \mathcal{Z}\}$.

Let $\tilde{\Phi}$ be the *skew-product flow* of the family Φ_λ ($\lambda \in \mathbb{R}$) on $\mathcal{E}$,

$$\tilde{\Phi}(t)(u,\lambda) = (\Phi_\lambda(t)u, \lambda), \quad \forall\, (u,\lambda) \in \mathcal{E}.$$

By the basic theory on abstract evolution equations (see, e.g., [10, Chaptet 3] or [34, Chapter 1, Theorem 4.4]), one can easily verify that $\tilde{\Phi}$ is *asymptotically compact*: that is, that $\tilde{\Phi}$ satisfies the hypothesis (AC) in Section 2.

For each $\lambda \in \mathbb{R}$, denote $\mathring{\mathcal{K}}_\lambda$ the *family of nonempty compact invariant sets K of Φ_λ with* $0 \notin K$. Given $\mathcal{U} \subset \mathcal{E}$, define

$$C(\mathcal{U}) = \overline{\bigcup \{K \times \{\lambda\} \subset \mathcal{U} \mid K \in \mathring{\mathcal{K}}_\lambda,\ \lambda \in \mathbb{R}\}}.$$

Definition 6.1 (Bifurcation branch). Let $(0,\lambda_0)$ be a bifurcation point, and $\mathcal{U} \subset \mathcal{E}$ be a closed neighborhood of $(0,\lambda_0)$. Then, the bifurcation branch in $\mathcal{U}$ from $(0,\lambda_0)$, denoted by $\Gamma_{\mathcal{U}}(0,\lambda_0)$, is defined to be the connected component of $C(\mathcal{U})$ which contains $(0,\lambda_0)$.

We now prove the following interesting result which ensures the existence of a local bifurcation branch.

Theorem 6.2. *Suppose the hypotheses* (H1)–(H4) *in Theorem 4.3 are fulfilled with $\lambda_0 = 0$, and that $S_0 = \{0\}$ is an isolated invariant set of Φ_0. Then, there exists $\varepsilon > 0$ such that $\Gamma \cap (U \times \{\pm\varepsilon\}) \neq \emptyset$, where $\Gamma = \Gamma_{\mathcal{U}}(0,0)$, and $\mathcal{U} = U \times [-\varepsilon, \varepsilon]$.*

Proof. Let U be the neighborhood of 0 given in Theorem 4.3, and let S_λ be the maximal compact invariant set of Φ_λ in U. Choose an $\varepsilon > 0$ such that the assertions in Theorem 5.1 hold. Let K_λ be the maximal compact invariant set of Φ_λ in $U \setminus S_0$. Since $\lim_{\lambda\to 0} d_{\mathrm{H}}(K_\lambda, 0) = 0$, we may also assume ε is sufficiently small so that there exists $r > 0$ such that

$$\mathrm{B}(K_\lambda, r) \subset U, \quad \forall\, \lambda \in [-\varepsilon, \varepsilon]. \tag{6.1}$$

We show that ε fulfills the requirement of the theorem.

For definiteness, by Theorem 5.1 it can be assumed that

$$h(\Phi_\lambda, K_\lambda) \neq \bar{0} \tag{6.2}$$

for $\lambda \in (0, \varepsilon]$. We check that $\Gamma \cap (U \times \{\varepsilon\}) \neq \emptyset$, thus completing the proof of the theorem.

We first prove that for any $0 < \mu < \varepsilon$, $C(\mathcal{U}_\mu)$ has a connected component $\mathcal{Z}$ such that

$$\mathcal{Z} \cap (U \times \{\mu\}) \neq \emptyset \neq \mathcal{Z} \cap (U \times \{\varepsilon\}), \tag{6.3}$$

where $\mathcal{U}_\mu = U \times [\mu, \varepsilon]$. For this purpose, let us first verify that

$$C(\mathcal{U}_\mu) = \bigcup_{\mu \le \lambda \le \varepsilon} K_\lambda \times \{\lambda\} := \mathcal{K}.$$

Indeed, we infer from the maximality of K_λ in $U \setminus S_0$ that $C(\mathcal{U}_\mu) = \bar{\mathcal{K}}$. On the other hand, it is clear that $\mathcal{K}$ is invariant under the skew-product flow $\bar{\Phi}$. Hence, by asymptotic compactness of $\bar{\Phi}$, we deduce that $\mathcal{K}$ is pre-compact. Further, by upper semicontinuity of K_λ in λ, one can easily verify that $\mathcal{K}$ is closed. Thus, $\mathcal{K}$ is compact. Consequently, $C(\mathcal{U}_\mu) = \bar{\mathcal{K}} = \mathcal{K}$.

The compactness of $\mathcal{K}$ also implies

$$d(0, K_\lambda) \ge 2\eta, \quad \forall\, \lambda \in [\mu, \varepsilon], \tag{6.4}$$

where $\eta > 0$ is a positive number independent of λ.

In what follows, we argue by contradiction and suppose that (6.3) fails to be true. Then, for any connected component $\mathcal{Z}$ of $C(\mathcal{U}_\mu)$, we have

$$\text{either } \mathcal{Z} \cap (U \times \{\mu\}) = \emptyset, \text{ or } \mathcal{Z} \cap (U \times \{\varepsilon\}) = \emptyset.$$

If there are only a finite number of components, then each component $\mathcal{Z}$ is isolated in $\mathcal{U}$. Because the λ-section $\mathcal{Z}_\lambda$ of $\mathcal{Z}$ is empty when λ is close to either μ or ε, by the continuation property of the Conley index, we see that $h(\Phi_\lambda, \mathcal{Z}_\lambda) \equiv \bar{0}$. Consequently, the "sum" of these indices equals $\bar{0}$. This contradicts (6.2) and justifies (6.3), as the union of $\mathcal{Z}_\lambda's$ is precisely K_λ. However, in general there is also the possibility that $C(\mathcal{U}_\mu)$ may contain infinitely many components. We will employ the separation lemma given in Section 2 to overcome this difficulty.

Set $\mathcal{O}_\mu = \mathcal{U}_\mu \setminus (\mathrm{B}(0,\eta) \times [\mu, \varepsilon])$. Then, clearly $C(\mathcal{O}_\mu) = C(\mathcal{U}_\mu)$. Denote $\mathcal{F}$ the family of connected components of $C(\mathcal{O}_\mu)$. By (6.1) and (6.4), we see that $\mathcal{O}_\mu$ is a neighborhood of $\mathcal{Z}$ in the space

$$\mathcal{H} = E \times [\mu, \varepsilon]$$

for each $\mathcal{Z} \in \mathcal{F}$. This allows us to pick for each $\mathcal{Z} \in \mathcal{F}$ a closed neighborhood $\Omega_{\mathcal{Z}}$ in $\mathcal{H}$ with $\Omega_{\mathcal{Z}} \subset \mathcal{O}_\mu$ such that if $\mathcal{Z} \cap (U \times \{\sigma\}) = \emptyset$ (where $\sigma = \mu$ or ε), then

$$\Omega_{\mathcal{Z}} \cap (U \times \{\sigma\}) = \emptyset \tag{6.5}$$

(see Figure 6.1).

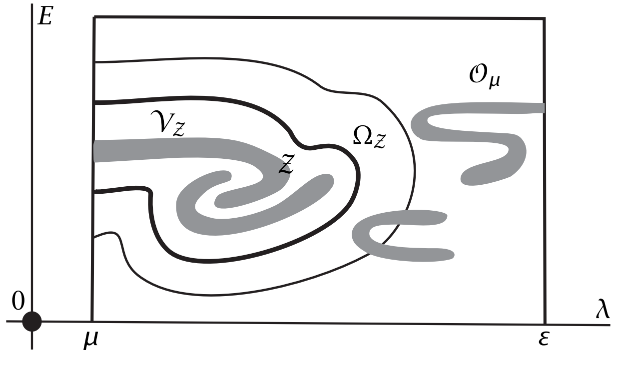


FIGURE 6.1. Separating neighborhoods of $\mathcal{Z}$ in $\mathcal{H}$

For any $\mathcal{O} \subset \mathcal{H}$, denote $\partial_{\mathcal{H}}\mathcal{O}$ the boundary of $\mathcal{O}$ in $\mathcal{H}$. Given $\mathcal{Z} \in \mathcal{F}$, set

$$\mathfrak{B} = \bigcup\{\mathcal{F} \in \mathcal{F} \mid \mathcal{F} \cap \partial_{\mathcal{H}}\Omega_{\mathcal{Z}} \neq \emptyset\},$$
$$\mathfrak{D} = \bigcup\{\mathcal{F} \in \mathcal{F} \mid \mathcal{F} \cap \Omega_{\mathcal{Z}} \neq \emptyset\}.$$

We claim that both $\mathfrak{B}$ and $\mathfrak{D}$ are closed. Indeed, if $b \in \bar{\mathfrak{B}}$, then there exists a sequence $b_k \in \mathfrak{B}$ such that $b_k \to b$. We may assume that $b_k \in \mathcal{F}_k$ for some $\mathcal{F}_k \in \mathcal{F}$ with $\mathcal{F}_k \cap \partial_{\mathcal{H}}\Omega_{\mathcal{Z}} \neq \emptyset$. By Lemma 2.2, we deduce that there exists a subsequence of $\mathcal{F}_k$, still denoted by $\mathcal{F}_k$, such that

$$\lim_{k\to\infty} \delta_{\mathrm{H}}(\mathcal{F}_k, \mathcal{F}_0) = 0.$$

One trivially checks that $\mathcal{F}_0$ is connected and contained in $C(\mathcal{O}_\mu)$; moreover, $\mathcal{F}_0 \cap \partial_{\mathcal{H}}\Omega_{\mathcal{Z}} \neq \emptyset$. Since $b \in \mathcal{F}_0$, we conclude that $b \in \mathfrak{B}$. Hence, $\mathfrak{B}$ is closed. Likewise, it can be shown that $\mathfrak{D}$ is closed.

Note that $\mathcal{Z} \cap \mathfrak{B} = \emptyset$. Since $\mathcal{Z}$ does not intersect any other connected component of $\mathfrak{D}$, by Lemma 2.1 there exist two disjoint closed subsets $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathfrak{D}$ such that $\mathfrak{D} = \mathcal{K}_1 \cup \mathcal{K}_2$, and $\mathcal{Z} \subset \mathcal{K}_1$, $\mathfrak{B} \subset \mathcal{K}_2$. It is clear that $\mathcal{K}_1$ is contained in the interior of $\Omega_{\mathcal{Z}}$ relative to $\mathcal{H}$.

Take a positive number $\delta_{\mathcal{Z}}$ with

$$\delta_{\mathcal{Z}} < \frac{1}{8}\min(d(\mathcal{K}_1, \mathcal{K}_2), d(\mathcal{K}_1, \partial_{\mathcal{H}}\Omega_{\mathcal{Z}})).$$

Let $\mathcal{V}_{\mathcal{Z}} = \mathrm{B}_{\mathcal{H}}(\mathcal{K}_1, 4\delta_{\mathcal{Z}})$ be the $4\delta_{\mathcal{Z}}$-neighborhood of $\mathcal{K}_1$ in $\mathcal{H}$. Then, $\mathcal{V}_{\mathcal{Z}} \subset \Omega_{\mathcal{Z}}$, and

$$\mathrm{B}_{\mathcal{H}}(\partial_{\mathcal{H}}\mathcal{V}_{\mathcal{Z}}, 2\delta_{\mathcal{Z}}) \cap C(\mathcal{O}_\mu) = \emptyset. \tag{6.6}$$

By the compactness of $C(\mathcal{O}_\mu)$, there exist a finite number of $\mathcal{Z} \in \mathcal{F}$, say, $\mathcal{Z}_1, \ldots, \mathcal{Z}_l$, such that $C(\mathcal{O}_\mu) \subset \bigcup_{1\le k\le l} \mathcal{V}_{\mathcal{Z}_k}$. Set

$$\mathcal{W}_k = \mathcal{V}_{\mathcal{Z}_k} \setminus (\bar{\mathcal{V}}_{\mathcal{Z}_1} \cup \cdots \cup \bar{\mathcal{V}}_{\mathcal{Z}_{k-1}}), \quad k = 1, 2, \ldots, l.$$

Then, $\mathcal{W}_k{}'s$ are disjoint open subsets of $\mathcal{H}$. We can easily check that

$$\partial_{\mathcal{H}}\mathcal{W}_k \subset \bigcup_{1\le i\le k} \partial_{\mathcal{H}}\mathcal{V}_{\mathcal{Z}_i}. \tag{6.7}$$

Thus, we deduce that $C(\mathcal{O}_\mu) \subset \bigcup_{1\le k\le l} \mathcal{W}_k$.

Let $S_k = C(\mathcal{O}_\mu) \cap \mathcal{W}_k$. We claim that

$$d(S_k, \partial_{\mathcal{H}}\mathcal{W}_k) > 0. \tag{6.8}$$

Indeed, if $w \in S_k$, then by (6.6) we have

$$d(w, \partial_{\mathcal{H}}\mathcal{V}_{\mathcal{Z}_i}) \ge 2\delta_{\mathcal{Z}_i} \ge 2\min_{1\le j\le l}\delta_{\mathcal{Z}_j} := \delta_0 > 0, \quad 1 \le i \le l,$$

and the conclusion follows from (6.7).

It follows by (6.8) that $S_k = C(\mathcal{O}_\mu) \cap \bar{\mathcal{W}}_k$. Hence, S_k is compact. It can be easily seen that S_k is the maximal compact invariant set of $\bar{\Phi}$ in $\bar{\mathcal{W}}_k$. Since $\bar{\mathcal{W}}_k$ is a neighborhood of S_k in $\mathcal{H}$, by Theorem 2.12, we have

$$h(\Phi_\lambda, S_{k,\lambda}) \equiv \text{const.}, \quad \lambda \in [\mu, \varepsilon], \tag{6.9}$$

where $S_{k,\lambda}$ is the λ-section of S_k. On the other hand, by (6.5) we have either $S_{k,\mu} = \emptyset$ or $S_{k,\varepsilon} = \emptyset$. Hence, by (6.9) it holds that

$$h(\Phi_\lambda, S_{k,\lambda}) \equiv \bar{0}, \quad \lambda \in [\mu, \varepsilon]. \tag{6.10}$$

Now, by (6.10) we conclude that

$$h(\Phi_\lambda, K_\lambda) = h(\Phi_\lambda, S_{1,\lambda}) \vee \cdots \vee h(\Phi_\lambda, S_{l,\lambda}) = \bar{0}.$$

This contradicts (6.2), and completes the proof of (6.3).

We are now ready to complete the proof of the theorem. Take a sequence of positive numbers $\mu_k \to 0$. For each μ_k, pick a connected component $\mathcal{Z}_k$ of $C(\mathcal{O}_{\mu_k})$ such that

$$\mathcal{Z}_k \cap (U \times \{\mu_k\}) \neq \emptyset \neq \mathcal{Z}_k \cap (U \times \{\varepsilon\}).$$

By Lemma 2.2, we may assume that

$$\lim_{k\to\infty} \delta_{\mathrm{H}}(\mathcal{Z}_k, \mathcal{Z}_0) = 0.$$

Then, $\mathcal{Z}_0$ is a continuum in $C(\mathcal{U})$ with $(0,0) \in \mathcal{Z}_0$ and $\mathcal{Z}_0 \cap (U \times \{\varepsilon\}) \neq \emptyset$. □

6.2. Global bifurcation. For the sake of convenience in statement, we make a convention that $\infty \in \partial\Omega$ if Ω is an unbounded subset of $\mathcal{E}$.

The main result in this subsection is the following theorem.

Theorem 6.3 (Global dynamic bifurcation). *Assume that the hypotheses in Theorem 4.3 are fulfilled. Let $\Omega \subset \mathcal{E}$ be a closed neighborhood of the bifurcation point $(0,0)$. Suppose that $S_0 = \{0\}$ is an isolated invariant set of Φ_0.*

Let $\Gamma = \Gamma_\Omega(0,0)$. Then, one of the following cases occurs:

(1) $\Gamma \cap \partial\Omega \neq \emptyset$ (*see Figure 6.2* (*1*)).

(2) $0 \in \overline{\Gamma_0 \setminus \{0\}}$, *where Γ_0 is the 0-section of Γ* (*see Figure 6.2* (*2*)).

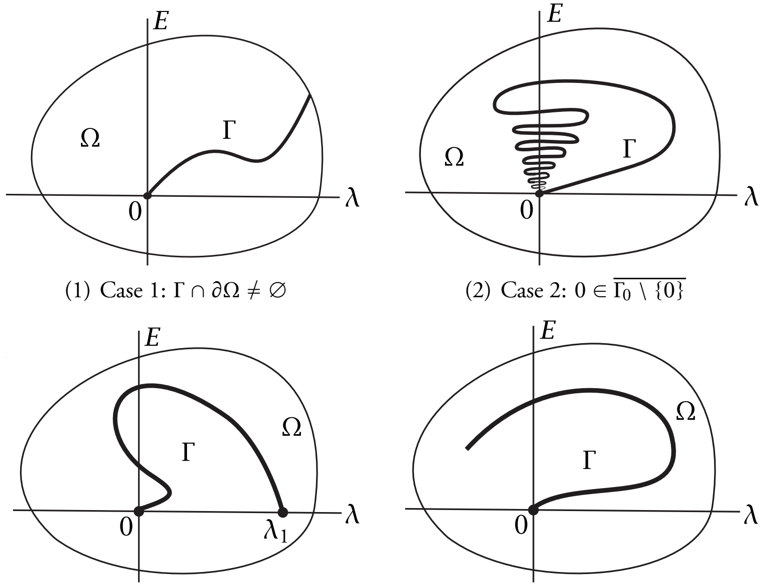
(3) *There exists $\lambda_1 \neq 0$ such that $(0, \lambda_1) \in \Gamma$* (*see Figure 6.2* (*3*)).

Proof. We argue by contradiction and suppose that none of the cases (1)–(3) occurs. Then, Γ is a bounded closed subset of $\mathcal{E}$ contained in the interior of Ω as depicted in Figure 6.2 (4). It is easy to see that Γ is invariant under the skew-product flow $\tilde{\Phi}$. Hence, by asymptotic compactness of $\tilde{\Phi}$, we deduce that Γ is compact.

Since $0 \notin \overline{\Gamma_0 \setminus S_0}$, we can write Γ_0 as $\Gamma_0 = S_0 \cup A_0$, where A_0 is a compact invariant set of Φ_0 with $A_0 \cap S_0 = \emptyset$. We only consider the case where $A_0 \neq \emptyset$. The argument for the case where $A_0 = \emptyset$ is a slight modification of that of the former one.

Let $U \subset E$ and $\varepsilon > 0$ be as in Theorem 6.2. Then, the system Φ_λ bifurcates for, say, each $0 < \lambda \leq \varepsilon$, a nonempty maximal compact invariant set K_λ in $U \setminus S_0$ with

$$\lim_{\lambda\to 0} d_{\mathrm{H}}(K_\lambda, S_0) = 0 \tag{6.11}$$



(1) Case 1: $\Gamma \cap \partial\Omega \neq \emptyset$ (2) Case 2: $0 \in \overline{\Gamma_0 \setminus \{0\}}$

(3) Case 3: $(0, \lambda_1) \in \Gamma$ (4) Case 4: This case never occurs

FIGURE 6.2.

and

$$h(\Phi_\lambda, K_\lambda) \neq \bar{0}, \quad \forall\, \lambda \in (0, \varepsilon]. \tag{6.12}$$

Pick a closed neighborhood V of 0 with $V \subset U$ and $d(A_0, V) := \sigma_0 > 0$. By (6.11) we can further restrict ε sufficiently small so that, for some $r_0 > 0$,

$$\mathrm{B}(K_\lambda, r_0) \subset V, \quad \forall\, \lambda \in (0, \varepsilon].$$

By the compactness of Γ it is easy to verify that the λ-section Γ_λ of Γ is upper semicontinuous in λ. Let $\mathcal{Z} = \Gamma \cap (V \times [0, \varepsilon])$. Then, $d_{\mathrm{H}}(\mathcal{Z}_\lambda, S_0) \to 0$ as $\lambda \to 0$. As $A_0 \cap S_0 = \emptyset$, it also holds that

$$\lim_{\lambda \to 0} d_{\mathrm{H}}(A_\lambda, A_0) \to 0,$$

where $A_\lambda = \Gamma_\lambda \setminus \mathcal{Z}_\lambda$ $(\lambda \in [0, \varepsilon])$. Thus, there exist $\eta_0 > 0$ and $0 < \varepsilon' \leq \varepsilon$ such that

$$\bar{\mathrm{B}}(\mathcal{Z}_\lambda, \eta_0) \subset V, \ \ \bar{\mathrm{B}}(A_\lambda, \eta_0) \cap V = \emptyset \tag{6.13}$$

for all $\lambda \in [0, \varepsilon']$. Note that both $\mathcal{Z}_\lambda$ and A_λ are compact invariant sets of Φ_λ.

Let $M_0 = \bigcup_{\lambda \geq \varepsilon'} \Gamma_\lambda$. It can be easily shown that M_0 is a compact subset of E. Clearly, $0 \notin M_0$; hence,

$$d(0, M_0) := \delta_0 > 0. \tag{6.14}$$

Fix a number $0 < r < \frac{1}{3}\min(\eta_0, \delta_0)$. Utilizing the separation lemma, by a similar argument as in the proof of Theorem 6.2, we can find a closed neighborhood $\mathcal{O}$ of Γ with $\mathcal{O} \subset \mathrm{B}_{\mathcal{E}}(\Gamma, r)$ such that

$$C(\Omega) \cap \partial\mathcal{O} = \emptyset. \tag{6.15}$$

Here, $\mathrm{B}_{\mathcal{E}}(\Gamma, r)$ denotes the r-neighborhood of Γ in $\mathcal{E}$. By the choice of r, it can be easily seen that if $\lambda \in (0, \varepsilon']$, then $\mathcal{O}_\lambda \subset \bar{\mathrm{B}}(\mathcal{Z}_\lambda, \eta_0) \cup \bar{\mathrm{B}}(A_\lambda, \eta_0)$ (see Figure 6.3 (1)). Set

$$G_\lambda = \mathcal{O}_\lambda \cap \bar{\mathrm{B}}(\mathcal{Z}_\lambda, \eta_0), \; H_\lambda = \mathcal{O}_\lambda \cap \bar{\mathrm{B}}(A_\lambda, \eta_0).$$

By (6.13), we have

$$\mathcal{O}_\lambda = G_\lambda \cup H_\lambda, \; G_\lambda \cap H_\lambda = \emptyset \tag{6.16}$$

for $\lambda \in (0, \varepsilon']$.

We claim there exists $\sigma > 0$ such that $\mathrm{B}_\sigma \subset G_\lambda$ for all λ sufficiently small, where (and below) B_R denotes the ball in E centered at 0 with radius R. Suppose the contrary. There would then exist sequences $\lambda_k \to 0$ and $x_k \in \partial G_{\lambda_k}$ such that $x_k \to 0$. Noticing that $(x_k, \lambda_k) \in \partial\mathcal{O}$, we conclude that $(0,0) \in \partial\mathcal{O}$, a contradiction!

By (6.11), we can find a number $0 < \mu \leq \varepsilon'/2$ such that

$$K_\lambda \subset \mathrm{B}_\sigma \subset G_\lambda, \quad \forall\, \lambda \in (0, 2\mu]. \tag{6.17}$$

Using the upper semicontinuity of K_λ in λ (see Theorems 4.3), we can easily show that $F = \bigcup_{\mu \leq \lambda \leq \varepsilon'} K_\lambda$ is closed in E. Because $\mathcal{F} = \bigcup_{\mu \leq \lambda \leq \varepsilon'} K_\lambda \times \{\lambda\}$ is invariant under the system $\tilde{\Phi}$, by asymptotic compactness of $\tilde{\Phi}$ we deduce that $\mathcal{F}$ is precompact in $\mathcal{E}$. It then follows that F is compact in E, and hence,

$$d(0, F) := d_0 > 0.$$

Take a $\Lambda > 0$ such that $\mathcal{O} \subset E \times (-\Lambda, \Lambda)$. Let ρ be a positive number with $\rho < \rho_0 := \frac{1}{2}\min(d_0, \delta_0)$, where δ_0 is the number given in (6.14). Set

$$\mathcal{V} = \mathcal{O} \cap \mathcal{H}, \; \mathcal{W} = \mathcal{V} \setminus (\mathrm{B}_\rho \times [\mu, \Lambda]),$$

where $\mathcal{H} = E \times [\mu, \Lambda]$ (see Figure 6.3 (2)). Clearly, $\mathcal{V}$ is closed in $\mathcal{H}$. Since $\mathrm{B}_\rho \times [\mu, \Lambda]$ is open in $\mathcal{H}$, we see that $\mathcal{W}$ is closed in $\mathcal{H}$ as well. We claim that

$$C(\mathcal{W}) = C(\mathcal{V}) := C, \quad \forall\, \rho < \rho_0. \tag{6.18}$$

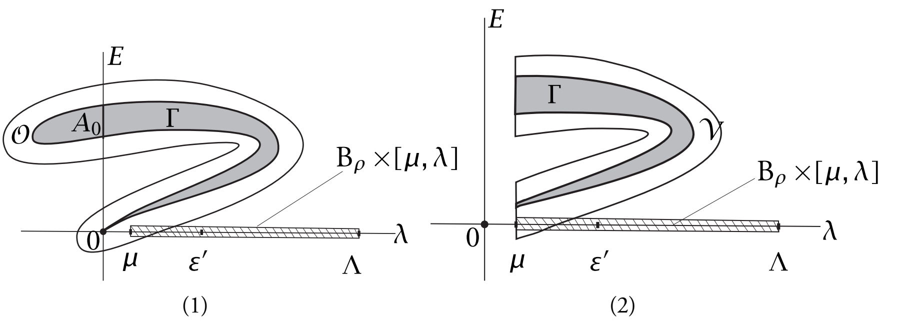


FIGURE 6.3.

To see this, by definition it suffices to show that if $\lambda \in [\mu, \Lambda]$, then any compact invariant set M of Φ_λ in $\mathcal{V}_\lambda \setminus S_0$ is necessarily contained in $\mathcal{W}_\lambda$.

We first consider $\lambda \in [\mu, \varepsilon']$. By (6.13) and the choice of r, we have

$$M \subset \mathcal{V}_\lambda \subset \bar{\mathrm{B}}(A_\lambda, \eta_0) \cup V.$$

Clearly, $M \cap \bar{\mathrm{B}}(A_\lambda, \eta_0) \subset \mathcal{W}_\lambda$. We observe that $M \cap V$ is a compact invariant set of Φ_λ in $V \setminus S_0$. Therefore, by the maximality of K_λ in $V \setminus S_0$, we have $M \cap V \subset K_\lambda$. Because $K_\lambda \cap \mathrm{B}_{2\rho} = \emptyset$ for $\mu \le \lambda \le \varepsilon'$, by the definition of $\mathcal{W}$ we see that $M \cap V \subset \mathcal{W}_\lambda$. Thus, $M \subset \mathcal{W}_\lambda$.

Now, assume that $\lambda > \varepsilon'$. Then, by the choice of ρ, we find that $\mathcal{O}_\lambda \cap \mathrm{B}_\rho = \emptyset$ (see Figure 6.3 (2)). It follows that $\mathcal{V}_\lambda = \mathcal{W}_\lambda$. This finishes the proof of what we desired. Hence, (6.18) holds true.

We show that $\mathcal{V}$ is a neighborhood of C in $\mathcal{H} := E \times [\mu, \Lambda]$. Suppose the contrary. Then, $C \cap \partial_{\mathcal{H}} \mathcal{V} \neq \emptyset$, where $\partial_{\mathcal{H}} \mathcal{V}$ denotes the boundary of $\mathcal{V}$ relative to $\mathcal{H}$. Noticing that

$$\partial_{\mathcal{H}} \mathcal{V} = \partial_{\mathcal{H}} (\mathcal{O} \cap \mathcal{H}) \subset \partial \mathcal{O} \cap \mathcal{H},$$

we have

$$C \cap \partial \mathcal{O} = C \cap (\partial \mathcal{O} \cap \mathcal{H}) \supset C \cap \partial_{\mathcal{H}} \mathcal{V} \neq \emptyset.$$

This contradicts (6.15).

By (6.18), we can fix a $\rho > 0$ sufficiently small so that $\mathcal{W} = \mathcal{V} \setminus (\mathrm{B}_\rho \times [\mu, \Lambda])$ is a neighborhood of C in $\mathcal{H}$. By the definitions of $C = C(\mathcal{W})$ and the skew-product flow, one can easily see that C is the maximal compact invariant set of $\tilde{\Phi}$ in $\mathcal{W}$. Hence, $\mathcal{W}$ is an isolating neighborhood of C in $\mathcal{H}$. It then follows by Theorem 2.12 that

$$h(\Phi_\lambda, C_\lambda) \equiv h(\Phi_\Lambda, C_\Lambda) = h(\Phi_\Lambda, \emptyset) = \bar{0}, \quad \lambda \in [\mu, \Lambda]. \tag{6.19}$$

On the other hand, if $\mu \le \lambda \le 2\mu$, then by (6.17) and the choice of ρ, we find that $\tilde{G}_\lambda := G_\lambda \setminus \mathrm{B}_\rho$ is a neighborhood of K_λ. Since K_λ is the maximal compact invariant of Φ_λ in $V \setminus S_0$ (and hence in $\tilde{G}_\lambda$), we infer from (6.16) that $C_\lambda \setminus K_\lambda$ is necessarily contained in H_λ (note that $\mathcal{W}_\lambda = \tilde{G}_\lambda \cup H_\lambda$). Thus,

$$h(\Phi_\lambda, C_\lambda) = h(\Phi_\lambda, K_\lambda) \vee h(\Phi_\lambda, C_\lambda \setminus K_\lambda).$$

Then, (6.19) implies that $h(\Phi_\lambda, K_\lambda) = \bar{0}$. This contradicts (6.12), which completes the proof of the theorem. □

7. AN EXAMPLE

In this section, we give an example to illustrate our theoretical results by considering the well-known Cahn-Hilliard equation describing the spinodal decomposition.

The nondimensional form of the equation reads (see [24])

$$\begin{cases} u_t + \Delta^2 u + \lambda \Delta u = \Delta(b_2 u^2 + b_3 u^3), & (x,t) \in \Omega \times R^+, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial(\Delta u)}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times R^+, \\ m(u) = 0, \end{cases} \tag{7.1}$$

where $\Omega \subset R^d$ ($d \le 3$) is a bounded domain with smooth boundary $\partial\Omega$, $b_3 > 0$, and

$$m(u) = \frac{1}{|\Omega|} \int_\Omega u \, dx.$$

The local attractor bifurcation and phase transition of the system have been extensively studied in Ma and Wang [24]. Other results related to bifurcation of the problem can be found in [2, 26], and so on. Here, by applying the theoretical results obtained above, we try to provide some new results about the dynamic bifurcation of the system, and demonstrate global features of the bifurcations.

7.1. Mathematical setting of the system. Denote by $(\ ,\ )$ and $|\ |$ the usual inner product and norm of $L^2(\Omega)$, respectively. For a mathematical setting, we introduce the Hilbert space H as follows:

$$H = \{u \in L^2(\Omega) \mid m(u) = 0\}.$$

Let $A_0 = -\Delta$ be the Laplacian in H associated with the homogeneous boundary condition

$$\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega.$$

Set $A = A_0^2$. Then, A is a positive-definite self-adjoint operator in H (and hence is a sectorial operator) with compact resolvent, and

$$D(A) = \left\{ u \in H^4(\Omega) \cap H \mid \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta u)}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

The spectral $\sigma(A_0)$ of A_0 consists of countably infinitely many eigenvalues:

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k \to +\infty.$$

Let $V := D(A_0) = D(A^{1/2})$. Denote $\| \ \|$ the norm in V.

Define

$$g_\lambda(u) = \Delta(b_2u^2 + b_3u^3), \quad u \in V.$$

Then, $g_\lambda : V \to H$ is locally Lipschitz, and the system (7.1) can be reformulated in an abstract form:

$$u_t + L_\lambda u = g_\lambda(u), \tag{7.2}$$

where $L_\lambda = A_0^2 - \lambda A_0$. We infer from Henry [10, Chapter 3] that, for each $u_0 \in V$, (7.2) has a unique global strong solution $u(t)$ in V with $u(0) = u_0$.

It is worth noticing that the problem has a natural Lyapunov function $J(u)$,

$$J(u) = \frac{1}{2}|\nabla u|^2 + \int_\Omega F_\lambda(u)\,\mathrm{d}x, \quad \text{where } F_\lambda(s) = -\frac{\lambda}{2}s^2 + \frac{b_2}{3}s^3 + \frac{b_3}{4}s^4.$$

7.2. Bifurcation from the trivial solution. It is obvious that each eigenvector w of A_0 corresponding to μ_k is also an eigenvector of L_λ corresponding to the eigenvalue

$$\beta_k(\lambda) := \mu_k^2 - \lambda\mu_k = \mu_k(\mu_k - \lambda).$$

Because H has a canonical basis consisting of eigenvectors of A_0, we deduce that $\beta_k(\lambda)$ $(k = 1, 2, \ldots)$ are precisely all the eigenvalues of L_λ.

Let Φ_λ be the semiflow generated by the system. We have the following result.

Theorem 7.1. *Assume $b_2 \neq 0$. Suppose A_0 has an eigenvector w corresponding to μ_j such that $\int_\Omega w^3\,\mathrm{d}x \neq 0$, and that 0 is an isolated equilibrium of Φ_{μ_j}.*

Then, there exist a closed neighborhood U of 0 in V and a two-sided neighborhood I_2 of μ_j such that Φ_λ has a nonempty maximal compact invariant set K_λ in $U \setminus \{0\}$ for each $\lambda \in I_2 \setminus \{\mu_j\}$. Consequently, for $\lambda \in I_2 \setminus \{\mu_j\}$, Φ_λ has at least one nontrivial equilibrium.

Proof. Since the system is a gradient-like one, by assumption it is easy to check that $S_0 = \{0\}$ is an isolated invariant set of Φ_{μ_j}. In what follows, we check that S_0 is neither an attractor nor a repeller of the restriction $\Phi^c_{\mu_j}$ of Φ_{μ_j} to $\mathcal{M}^c$, and hence the conclusion of the theorem immediately follows from Theorem 4.3.

Denote E_j the space spanned by the eigenvectors of A_0 corresponding to μ_j. Then, $H = E_j \oplus E_j^\perp$. Let $V_j^\perp = V \cap E_j^\perp$. Then, $V = E_j \oplus V_j^\perp$. We infer from [34, Chapter II, Theorem 2.1] that there is a small neighborhood W of 0 in E_j and a C^1 mapping $\xi : W \to V_j^\perp$ with

$$\xi(v) = 0(\|v\|^2) \quad (\text{as } \|v\| \to 0)$$

such that $\mathcal{M}^c = \{v + \xi(v) \mid v \in W\}$ is a local center manifold of Φ_{μ_j}.

For $u = v + \xi(v)$, where $v \in W$, simple computations show that

$$J(u) = \frac{1}{2}\int_\Omega |\xi'(v)\nabla v|^2 \,\mathrm{d}x + \frac{b_2}{3}\int_\Omega v^3 \,\mathrm{d}x + o(\|v\|^3).$$

Here, we have used the facts that $\xi(v), \Delta\xi(v) \in E_j^\perp$. Set $v = \tau w$, where w is the eigenvector of A_0 given in the theorem; then, since $\xi'(v) = 0(\|v\|)$, we have

$$J(\tau w) = \tau^3 \frac{b_2}{3}\int_\Omega w^3 \,\mathrm{d}x + o(|\tau|^3) \quad \text{as } \tau \to 0. \tag{7.3}$$

As $(b_2/3)\int_\Omega w^3 \,\mathrm{d}x \neq 0$, by (7.3) it is clear that 0 is neither a local maximum nor a minimum point of J, which completes the proof of what we desired. □

The following result demonstrates some global features of the dynamic bifurcation of the system.

Theorem 7.2. *Suppose* 0 *is an isolated equilibrium of* Φ_{μ_j}. *Let* Γ *be the bifurcation branch in* V *from the bifurcation point* $(0, \mu_j)$. *Set*

$$\Lambda_0 = \inf\{\lambda \mid \Gamma_\lambda \neq \varnothing\},$$
$$\Lambda_1 = \sup\{\lambda \mid \Gamma_\lambda \neq \varnothing\},$$

where $\Gamma_\lambda = \{u \mid (u, \lambda) \in \Gamma\}$ *is the* λ*-section of* Γ.

Then, $-\infty < \Lambda_0 < \Lambda_1 \le +\infty$, *and one of the following assertions holds:*

(1) $\Lambda_1 = +\infty$.
(2) $0 \in \overline{\Gamma_{\mu_j} \setminus \{0\}}$.
(3) *There exists* $\lambda_1 \neq \mu_j$ *such that* $(0, \lambda_1) \in \Gamma$. *Furthermore, one of the following holds:*
   (i) *Either there is a sequence* $(u_k, \nu_k) \in \Gamma$ *approaching* $(0, \lambda_1)$, *where* u_k *is a nontrivial equilibrium of* Φ_{ν_k} *for each* k;
   (ii) *Or* Γ_{λ_1} *contains at least two distinct complete trajectories* $\sigma^\pm$ *such that*

$$J(\alpha(\sigma^+)) \equiv \text{const.} > 0,\ \ \omega(\sigma^+) = \{0\},$$
$$J(\omega(\sigma^-)) \equiv \text{const.} < 0,\ \ \alpha(\sigma^-) = \{0\}.$$

Remark 7.3. It is worth noticing that both $\alpha(\sigma^+)$ and $\omega(\sigma^-)$ in (3) consist of nontrivial equilibrium points. Therefore, when (3) occurs, Φ_{λ_1} has at least two distinct nontrivial equilibria. When Γ_{λ_1} contains only a finite number of equilibria, each of the two limit sets $\alpha(\sigma^+)$ and $\omega(\sigma^-)$ consists of exactly one equilibrium. Consequently, $\sigma^\pm$ become heteroclinic orbits.

Proof of Theorem 7.2. It can be easily shown that if $\lambda < 0$ is large enough, then the trivial solution 0 is the global attractor of Φ_λ. Hence, we necessarily have $\Lambda_0 > -\infty$. The existence of a local bifurcation branch also implies $\Lambda_0 < \Lambda_1$.

Assume $\Lambda_1 < +\infty$ (otherwise, (1) holds true, and thus we are done). Then, $I = [\Lambda_0, \Lambda_1]$ is a compact interval. Therefore, we infer from the proof for the existence of a global attractor of the system in Temam [37] (see also [16], etc.) that the system is dissipative uniformly with respect to $\lambda \in I$. Specifically, there is a bounded set $B \subset V$ such that

$$\mathcal{A}_\lambda \subset B, \quad \forall \lambda \in I, \tag{7.4}$$

where $\mathcal{A}_\lambda$ is the global attractor of Φ_λ. Thus, the bifurcation branch Γ is bounded. Hence, by Theorem 6.3 we conclude that either (2) holds, or there is a $\lambda_1 \neq \mu_j$ such that $(0, \lambda_1) \in \Gamma$. To complete the proof of the theorem, it remains to check the alternatives in (3).

We therefore assume that $(0, \lambda_1) \in \Gamma$ for some $\lambda_1 \neq \mu_j$. Suppose the case (i) does not occur. Then, 0 is an isolated equilibrium of Φ_{λ_1}. Fix a $\delta_1 > 0$ such that Φ_{λ_1} has no equilibria other than the trivial one in the δ_1-neighborhood B_{δ_1} of 0 in V. By the definition of a bifurcation branch, we deduce there exists a sequence $\nu_k \to \lambda_1$ such that for each k, Φ_{ν_k} has a nonempty compact invariant set $M_k \subset \Gamma_{\nu_k}$ with $0 \notin M_k$ such that

$$\lim_{k\to\infty} d(0, M_k) = 0. \tag{7.5}$$

For convenience, denote $\mathcal{E}(\Phi_\lambda, M)$ the set of equilibria of Φ_λ in $M \subset V$. Let

$$\mathcal{E}_k := \mathcal{E}(\Phi_{\nu_k}, M_k).$$

Then, $\mathcal{E}_k$ is a nonempty compact subset of M_k. As we have assumed that (i) does not occur, it can be easily seen that there exists $0 < \delta < \delta_1$ such that

$$\liminf_{k\to\infty} d(0, \mathcal{E}_k) \geq 4\delta > 0. \tag{7.6}$$

By (7.5), for each k we can pick a $u_k \in M_k$ such that the sequence $u_k \to 0$ as $k \to \infty$. It can be assumed that

$$\|u_k\| < \delta \tag{7.7}$$

for all k (hence $d(u_k, \mathcal{E}_k) > 3\delta$). Let γ_k be a complete trajectory of Φ_{ν_k} contained in M_k with $\gamma_k(0) = u_k$. We have

$$\min_{t\leq 0} J(\gamma_k(t)) = J(\gamma_k(0)) = J(u_k) \to 0, \quad \text{as } k \to \infty.$$

Set

$$t_k = \min\{s < 0 \mid \max_{t\in[s,0]} \|\gamma_k(t) - u_k\| \leq 2\delta\}.$$

Noticing that $\alpha(\gamma_k) \subset \mathcal{E}_k$, we deduce by (7.6) and (7.7) that $t_k > -\infty$, and hence $\|\gamma_k(t_k) - u_k\| = 2\delta$. Therefore,

$$\delta \leq \|\gamma_k(t_k)\| \leq 3\delta, \quad k \geq 1. \tag{7.8}$$

Define a sequence of complete trajectories σ_k as

$$\sigma_k(t) = \gamma_k(t_k + t), \quad t \in \mathbb{R}.$$

Since all these trajectories are contained in the bounded set B in (7.4), by a very standard argument it can be shown that σ_k has a subsequence (still denoted by σ_k) converging uniformly on any compact interval to a complete trajectory σ^+. It is trivial to check that σ^+ is contained in Γ_{λ_1}. Observing that $\sigma^+(0) = \lim_{k\to\infty} \gamma_k(t_k)$, by (7.8) we deduce that

$$\delta \leq \|\sigma^+(0)\| \leq 3\delta. \tag{7.9}$$

Because

$$J(\sigma_k(0)) \geq J(\sigma_k(-t_k)) = J(\gamma_k(0)) = J(u_k) \to 0 \quad \text{as } k \to \infty,$$

we also have $J(\sigma^+(0)) \geq 0$.

On the other hand, as Φ_{λ_1} has no equilibrium in $\mathrm{B}_{\delta_1} \setminus \{0\}$, by (7.9) we see that $\sigma^+(0)$ is not an equilibrium of Φ_{λ_1}. Hence, there is a small open interval $I_\varepsilon = (-\varepsilon, \varepsilon)$ such that $J(\sigma^+(t))$ is strictly decreasing in t on I_ε. Consequently,

$$J(\alpha(\sigma^+)) \equiv \text{const.} > J(\sigma^+(0)) \geq 0.$$

In what follows, we show that $\omega(\sigma^+) = \{0\}$. If t_k has a bounded subsequence (still denoted by t_k) with $t_k \to -\tau \leq 0$, then

$$\sigma^+(\tau) = \lim_{k\to\infty} \sigma_k(-t_k) = \lim_{k\to\infty} \gamma_k(0) = \lim_{k\to\infty} u_k = 0.$$

Hence, $\sigma^+(t) \equiv 0$ for $t \geq \tau$, which contradicts (7.9). Thus, we know that $t_k \to -\infty$. Since $\|\gamma_k(t) - u_k\| \leq 2\delta$ for $t \in [t_k, 0]$, we have

$$\|\gamma_k(t)\| \leq \|u_k\| + 2\delta \leq 3\delta, \quad t \in [t_k, 0],$$

and therefore,

$$\|\sigma_k(t)\| \leq 3\delta, \quad t \in [0, -t_k],$$

from which it follows that $\|\sigma^+(t)\| \leq 3\delta$ for all $t \geq 0$. As 0 is the unique equilibrium of Φ_{λ_1} in B_{δ_1} and $3\delta < \delta_1$, we immediately conclude $\omega(\sigma^+) = \{0\}$.

Likewise, we can prove there is a complete trajectory σ^- in Γ_{λ_1} such that

$$J(\omega(\sigma^-)) \equiv \text{const.} < 0, \quad \alpha(\sigma^-) = \{0\}.$$

The proof of the theorem is finished. □

Remark 7.4. We have assumed in Theorems 7.1 and 7.2 that the trivial solution 0 of the system is an isolated equilibrium of Φ_{μ_j}. In general, it seems to be difficult to verify this condition because of the degeneracy. However, in some particular but important cases one can really do so. For instance, if $b_2 = 0$ then it can be shown that the equilibrium 0 is isolated with respect to Φ_{μ_j} (see the proof of Theorem 9.4 in Ma and Wang [23]).

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DESHENG LI:
Department of Mathematics
Tianjin University
Tianjin 300072, P. R. China
E-MAIL: lidsmath@tju.edu.cn

ZHI-QIANG WANG:
Center for Applied Mathematics
Tianjin University
Tianjin 300072, P. R. China

AND

Department of Mathematics and Statistics
Utah State University
Animal Science, 3900 Old Main Hill
Logan, UT 84322, USA
E-MAIL: zhi-qiang.wang@usu.edu

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