New bounds for eigenvalues of strictly diagonally dominant tensors [1]

Yining Gu[2], Wei Wu[3]

School of Mathematical Sciences
Tianjin University, Tianjin 300350, China

Abstract

In this paper, we prove that the minimum eigenvalue of a strictly diagonally dominant Z-tensor with positive diagonal entries lies between the smallest and the largest row sums. The novelty comes from the upper bound. Moreover, we show that a similar upper bound does not hold for the minimum eigenvalue of a strictly diagonally dominant tensor with positive diagonal entries but with arbitrary off-diagonal entries. Furthermore, other new bounds for the minimum eigenvalue of nonsingular M-tensors are obtained.

Keywords: diagonal dominance, M-tensor, minimum eigenvalue.

AMS subject classifications: 15A18, 15A69, 65F10, 65F15.

1 Introduction

It is well known that the concepts of eigenvalues and eigenvectors of higher-order tensors were introduced by Qi [1] in 2005. Since then, the eigenvalue problems of high-order tensors have attracted attention of many mathematicians from different disciplines including applied mathematics branch and numerical multilinear algebra. Moreover, they also have a wide range of practical applications, see [1, 2, 3, 4]. Applications of eigenvalues of tensors include higher-order Markov chains [5], medical resonance imaging [4, 6], best-rank one approximation in data analysis [7], and positive definiteness of even-order multivariate forms in automatical control [8], etc.

Very recently, many contributions have been made on the bounds of the spectral radius of nonnegative tensor in [1, 5, 9, 10]. Furthermore, bounds for the Z-spectral

[1]This work was supported by the grant of NSF of China (11371276).
[2]Yining Gu(1992-), female, Shandong Province Binzhou Country, master, 449206085@qq.com.
[3]Corresponding author: Wei Wu(1969-), female, Jilin Province Jilin Country, associate professor, tensor, wuwei0423@126.com.

radius were given in [11] for the H-tensors. Also, in [12], He and Huang obtained the upper and lower bounds for the minimum H-eigenvalue of nonsingular M-tensors. It is important and meaningful to study the bounds for eigenvalues of a tensor. In this paper, we obtain the new bounds for the minimum eigenvalues of strictly diagonally dominant Z-tensors and nonsingular M-tensors.

The rest of the paper is organized as follows. First, we recall some definitions and theorems concerning our results. Then, the upper and lower bounds for the minimum eigenvalue of strictly diagonally dominant Z-tensors with positive diagonal entries are obtained. Moreover, we show that a similar upper bound does not hold for the minimum eigenvalue of strictly diagonally dominant tensors with positive diagonal entries but with arbitrary off-diagonal entries. Furthermore, we obtain the new bounds for the minimum eigenvalue of nonsingular M-tensors.

Denote the set of all nonnegative vectors in $\mathbb{R}^n$ by $\mathbb{R}^n_+$ and the set of all positive vectors in $\mathbb{R}^n$ by $\mathbb{R}^n_{++}$.

Now we recall some fundamental notions and properties on tensors.

A real mth-order n-dimensional square tensor $\mathcal{A}$ with n^m entries can be defined as follows:

$$\mathcal{A} = (a_{i_1 i_2 \ldots i_m}),\ a_{i_1 i_2 \ldots i_m} \in \mathbb{R},$$

where $i_j = 1, 2, \ldots, n$ for $j = 1, 2, \ldots, m$.

Furthermore, a real mth-order n-dimensional tensor $\mathcal{A}$ is called nonnegative if all the entries $a_{i_1 i_2 \ldots i_m}$ are nonnegative. The tensor $\mathcal{A}$ is called symmetric if its entries $a_{i_1 i_2 \ldots i_m}$ are invariant under any permutation of their indices $\{i_1, i_2, \ldots, i_m\}$. The mth-order n-dimensional identity tensor, denoted by $I = (\delta_{i_1 i_2 \ldots i_m})$, is the tensor with entries

$$\delta_{i_1 i_2 \ldots i_m} = \begin{cases} 1, \text{ if } i_1 = i_2 = \ldots = i_m, \\ 0, \text{ otherwise.} \end{cases}$$

Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be a real mth-order n-dimensional tensor, and $x \in \mathbb{C}^n$. Then $\mathcal{A}x^{m-1}$ is a vector in $\mathbb{C}^n$, with its ith component defined by

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, i_3, \ldots i_m = 1}^{n} a_{i i_2 \ldots i_m} x_{i_2} x_{i_3} \ldots x_{i_m}.$$

Let r be a positive integer. Then $x^{[r]} = (x_1^r, x_2^r, \ldots, x_n^r)^T$ is a vector in $\mathbb{C}^n$.

If there are a complex number λ and a nonzero complex vector x that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then we call λ an eigenvalue of $\mathcal{A}$ and x its corresponding eigenvector. In particular, if x is real, then λ is also real. In this case, we say that λ is an H-eigenvalue of $\mathcal{A}$ and x is its corresponding H-eigenvector. If $x \in \mathbb{R}^n_+$, then λ is called an H$^+$-eigenvalue of $\mathcal{A}$. If $x \in \mathbb{R}^n_{++}$, then λ is called an H^{++}-eigenvalue of $\mathcal{A}$. Moreover, we define $\sigma(\mathcal{A})$ as the set of all the eigenvalues of $\mathcal{A}$. When m is even and $\mathcal{A}$ is symmetric, the definition of eigenvalues of tensors was introduced by Qi [1]. When m is odd, Lim [2] used $(x_1^{m-1} sgn x_1, \ldots, x_n^{m-1} sgn x_n)$ on the right-hand side instead, and the notion has been generalized by Chang, Pearson, and Zhang [13], where $sgn(x)$ is defined by

$$sgn(x) = \begin{cases} 1, \text{ if } x > 0, \\ 0, \text{ if } x = 0, \\ -1, \text{ if } x < 0. \end{cases}$$

We now summarizes the Perron-Frobenius theorem for nonnegative tensors given in [13, 14, 15].

Let $\rho(\mathcal{A}) = \max\limits_{\lambda \in \sigma(\mathcal{A})} \{|\lambda|\}$, where $|\lambda|$ denotes the modulus of λ. We call $\rho(\mathcal{A})$ the spectral radius of tensor $\mathcal{A}$.

Theorem 1. *(The Perron-Frobenius theorem for nonnegative tensors)*

(a) If $\mathcal{A}$ is a nonnegative tensor of order m and dimension n, then $\rho(\mathcal{A})$ is an H^+-eigenvalue of $\mathcal{A}$.

(b) If moreover $\mathcal{A}$ is irreducible, then $\rho(\mathcal{A})$ is the unique H^{++}-eigenvalue of $\mathcal{A}$.

And the irreducibility of a tensor is defined as follows.

Definition 1. *A tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ of order m and dimension n is called reducible, if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that*

$$a_{i_1 i_2 \ldots i_m} = 0, \ \forall i_1 \in I, \ \forall i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

We now introduce some existing results on tensors. The following lemma was given by Qi [1].

Lemma 1. *Let $\mathcal{A}$ be an mth-order n-dimensional tensor. Suppose that $\mathcal{B} = a(\mathcal{A} + b\mathcal{I})$, where a and b are two real numbers. Then μ is an eigenvalue (H-eigenvalue) of $\mathcal{B}$ if and only if $\mu = a(\lambda + b)$ and λ is an eigenvalue (H-eigenvalue) of $\mathcal{A}$. In this case, they have the same eigenvectors (H-eigenvectors).*

Qi [1] introduced the Gerschogrin theorem for real symmetric tensors. Although this conclusion was proved in the case that $\mathcal{A}$ is a real symmetric tensor, it can be easily extended to a generic tensor of order m and dimension n.

Theorem 2. *(The Gerschogrin theorem for tensors) Let $\mathcal{A}$ be an mth-order n-dimensional tensor. The eigenvalues of $\mathcal{A}$ lie in the union of n disks in $\mathbb{C}$. These n disks have the diagonal elements of $\mathcal{A}$ as thier centers, and the sums of the absolute values of the off-diagonal elements as their radii.*

By [7], we now introduce a new tensor $\mathcal{B} = (b_{i_1 i_2 \ldots i_m})$. Let $\mathcal{A}$ be an mth-order n-dimensional tensor and $D = diag(d_1, d_2, \ldots, d_n)$ be a positive diagonal matrix. Then

$$\mathcal{B} = \mathcal{A} \cdot D^{-(m-1)} \cdot \overbrace{D \cdots D}^{m-1},$$

with

$$b_{i_1 i_2 \ldots i_m} = a_{i_1 i_2 \ldots i_m} d_{i_1}^{-(m-1)} d_{i_2} \cdots d_{i_m}.$$

Then we have the following lemma given in [14].

Lemma 2. *If λ is an eigenvalue of $\mathcal{A}$ with corresponding eigenvector x, then λ is also an eigenvalue of $\mathcal{B}$ with corresponding eigenvector $D^{-1}x$; if τ is an eigenvalue of $\mathcal{B}$ with corresponding eigenvector y, then τ is also an eigenvalue of $\mathcal{A}$ with corresponding eigenvector Dy.*

Zhang *et al* [16, 17, 18] extended the notion of M-matrices to higher-order tensors. Then they introduced the definitions of M-tensors and nonsingular M-tensors. Furthermore, they obtained some characterization theorems and basic properties for M-tensors and nonsingular M-tensors.

The mth-order n-dimensional tensor $\mathcal{A}$ is called a Z-tensor if all the off-diagonal entries are nonpositive.

Definition 2. *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be an mth-order n-dimensional tensor. $\mathcal{A}$ is called an M-tensor if there exist a nonnegative tensor $\mathcal{B}$ and a real number $c \geq \rho(\mathcal{B})$ such that*

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}.$$

If $c > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a nonsingular M-tensor.

Definition 3. *Let $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ be an mth-order n-dimensional tensor. $\mathcal{A}$ is diagonally dominant if for $i = 1, 2, \ldots, n$,*

$$\sum_{(i_2,\ldots,i_m) \neq (i,\ldots,i)} |a_{i i_2 \ldots i_m}| \leq |a_{ii\ldots i}|. \tag{1}$$

$\mathcal{A}$ is strictly diagonally dominant if the strict inequality holds in (1) for all i.

Definition 4. *Let $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ be an mth-order n-dimensional tensor. We call a tensor $\mathcal{M}(\mathcal{A}) = (m_{i_1 i_2 \dots i_m})$ the comparison tensor of $\mathcal{A}$ if*

$$m_{i_1 i_2 \dots i_m} = \begin{cases} +|a_{i_1 i_2 \dots i_m}|, & \text{if } (i_2, \dots, i_m) = (i_1, \dots, i_1), \\ -|a_{i_1 i_2 \dots i_m}|, & \text{if } (i_2, \dots, i_m) \neq (i_1, \dots, i_1). \end{cases}$$

Lemma 3. *Let $\mathcal{A}$ be a Z-tensor with nonnegative diagonal entries. If $\mathcal{A}$ is diagonally dominant, then $\mathcal{A}$ is an M-tensor. If $\mathcal{A}$ is strictly diagonally dominant, then $\mathcal{A}$ is a nonsingular M-tensor.*

Lemma 4. *Let $\mathcal{A}$ be a Z-tensor. Then $\mathcal{A}$ is a nonsingular M-tensor if and only if $\mathcal{A}$ has all positive diagonal entries and there exists a positive diagonal matrix $D = diag(d_1, d_2, \dots, d_n)$ such that $\mathcal{A}D^{m-1}$ is strictly diagonally dominant.*

We now define the ith row sum of a mth-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ as

$$R_i(\mathcal{A}) = \sum_{i_2, \dots, i_m = 1}^{n} a_{i i_2 \dots i_m}, \; i = 1, 2, \dots, n,$$

and denote the largest and the smallest row sums of $\mathcal{A}$ by

$$R_{\max}(\mathcal{A}) = \max_{i=1,2,\dots,n} R_i(\mathcal{A}), \; R_{\min}(\mathcal{A}) = \min_{i=1,2,\dots,n} R_i(\mathcal{A}),$$

respectively. Furthermore, we define

$$\widetilde{R}_i(\mathcal{A}) = |a_{ii\dots i}| - \sum_{(i_2, \dots, i_m) \neq (i, \dots, i)} |a_{i i_2 \dots i_m}|, \; i = 1, 2, \dots, n,$$
$$\widetilde{R}_{\max}(\mathcal{A}) = \max_{i=1,2,\dots,n} \widetilde{R}_i(\mathcal{A}), \; \widetilde{R}_{\min}(\mathcal{A}) = \min_{i=1,2,\dots,n} \widetilde{R}_i(\mathcal{A}).$$

Let $\lambda_{\min}(\mathcal{A}) = \min_{\lambda \in \sigma(\mathcal{A})} \{|\lambda|\}$, where $|\lambda|$ denotes the modulus of λ.

2 Main results

In this section, we obtain the upper and lower bounds for the minimum eigenvalue of strictly diagonally dominant Z-tensors with positive diagonal entries. Moreover, we show that a similar upper bound does not hold for the minimum eigenvalue of strictly diagonally dominant tensors with positive diagonal entries but with arbitrary off-diagonal entries. Furthermore, we obtain the new bounds for the minimum eigenvalue of nonsingular M-tensors.

Theorem 3. *Let $\mathcal{A}$ be a strictly diagonally dominant Z-tensor with positive diagonal entries. Then $\mathcal{A}$ has a positive H^+-eigenvalue $\lambda_{\min}(\mathcal{A})$ and satisfies*

$$0 < R_{\min}(\mathcal{A}) \leq \lambda_{\min}(\mathcal{A}) \leq R_{\max}(\mathcal{A}). \tag{2}$$

Proof. Since $\mathcal{A}$ is a strictly diagonally dominant Z-tensor with positive diagonal entries, by Lemma 3, $\mathcal{A}$ is a nonsingular M-tensor. Hence, there exist a nonnegative tensor $\mathcal{B}$ and a positive real number $c > \rho(\mathcal{B})$ such that

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}.$$

Denote $\tau(\mathcal{A}) = c - \rho(\mathcal{B})$, we have $\tau(\mathcal{A}) > 0$. By Theorem 1, $\rho(\mathcal{B})$ is an H$^+$-eigenvalue of $\mathcal{B}$. By Lemma 1, $\tau(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$. Moreover, $\tau(\mathcal{A})$ and $\rho(\mathcal{B})$ have the same eigenvectors. Hence, $\tau(\mathcal{A})$ is an H$^+$-eigenvalue of $\mathcal{A}$. Then

$$\tau(\mathcal{A}) \geq \lambda_{\min}(\mathcal{A}). \tag{3}$$

Let $\lambda \in \sigma(\mathcal{A})$. By Lemma 1, $c - \lambda$ is an eigenvalue of $\mathcal{B}$. Then

$$\rho(\mathcal{B}) = \max_{\lambda \in \rho(\mathcal{A})} \{|c - \lambda|\} \geq \max_{\lambda \in \rho(\mathcal{A})} \{c - |\lambda|\} = c - \min_{\lambda \in \rho(\mathcal{A})} \{|\lambda|\},$$

which, together with (3) and $\tau(\mathcal{A}) = c - \rho(\mathcal{B})$, yields

$$\tau(\mathcal{A}) = \lambda_{\min}(\mathcal{A}) > 0.$$

That is, $\lambda_{\min}(\mathcal{A})$ is a positive H$^+$-eigenvalue of $\mathcal{A}$. The first inequality of (2) is trivial and the second one is a consequence of the Gerschgorin theorem for tensors.

We now prove the last inequality of (2). Assume that $\mathcal{A}$ is irreducible. By Lemma 3, $\mathcal{A}$ is a nonsingular M-tensor. Hence, there exist a nonnegative tensor $\mathcal{B}$ and a positive real number $c > \rho(\mathcal{B})$ such that

$$\mathcal{A} = c\mathcal{I} - \mathcal{B}.$$

Since $\mathcal{A}$ is irreducible, $\mathcal{B}$ is also irreducible. By Theorem 1, $\rho(\mathcal{B})$ is the unique H^{++}-eigenvalue of $\mathcal{B}$ with positive eigenvector x. By Lemma 1, x is also an eigenvector of $\mathcal{A}$ associated with $\lambda_{\min}(\mathcal{A}) = c - \rho(\mathcal{B})$. Let us take $x = (x_1, x_2, \ldots, x_n)^T$ such that $x_i = \min\limits_{k=1,2,\ldots,n} \{x_k\} = 1$ and $x_j \geq 1$ for $j \neq i$. Since $a_{i_1 i_2 \ldots i_m} \leq 0$ when $(i_2, \ldots, i_m) \neq (i_1, \ldots, i_1)$, then

$$\begin{aligned}\lambda_{\min}(\mathcal{A}) = \lambda_{\min}(\mathcal{A})x_i^{m-1} = (\mathcal{A}x^{m-1})_i &= a_{ii\ldots i} + \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} a_{ii_2\ldots i_m} x_{i_2}x_{i_3}\ldots x_{i_m} \\ &\leq a_{ii\ldots i} + \sum_{(i_2,\ldots,i_m)\neq(i,\ldots,i)} a_{ii_2\ldots i_m} \\ &\leq R_{\max}(\mathcal{A}).\end{aligned}$$

The last inequality of (2) holds.

Now assume that $\mathcal{A}$ is reducible. For any $\varepsilon > 0$, we construct an mth-order n-dimensional tensor $\mathcal{A}^\varepsilon$ such that $\mathcal{A}^\varepsilon = \mathcal{A} - \varepsilon\mathcal{C}$, where $\mathcal{C}$ is an mth-order n-dimensional tensor with all of its elements being 1. Then we have

$$R_{\min}(\mathcal{A}^\varepsilon) = R_{\min}(\mathcal{A}) - n^{m-1}\varepsilon.$$

Hence, for any $0 < \varepsilon < (R_{\min}(\mathcal{A}^\varepsilon)/n^{m-1})$, $\mathcal{A}^\varepsilon$ is a strictly diagonally dominant Z-tensor with positive diagonal entries and $\mathcal{A}^\varepsilon$ is irreducible. Then $\mathcal{A}^\varepsilon$ has a positive H$^+$-eigenvalue $\lambda_{\min}(\mathcal{A}^\varepsilon)$ for any $0 < \varepsilon < (R_{\min}(\mathcal{A}^\varepsilon)/n^{m-1})$. Moreover, by the above conclusion, we have

$$\lambda_{\min}(\mathcal{A}^\varepsilon) \leq R_{\max}(\mathcal{A}^\varepsilon), \tag{4}$$

where $R_{\max}(\mathcal{A}^\varepsilon) = R_{\max}(\mathcal{A}) - n^{m-1}\varepsilon$.

Observe that $\mathcal{A}^\varepsilon \to \mathcal{A}$ and $R_{\max}(\mathcal{A}^\varepsilon) \to R_{\max}(\mathcal{A})$ as $\varepsilon \to 0$. By the continuity of the eigenvalues as functions of the elements of the tensor, we have $\lambda_{\min}(\mathcal{A}^\varepsilon) \to \lambda_{\min}(\mathcal{A})$. Hence, by considering (4) and letting $\varepsilon \to 0$, we have $\lambda_{\min}(\mathcal{A}) \leq R_{\max}(\mathcal{A})$.

Thus, we complete the proof. $\square$

The sharpness of the bounds of Theorem 3 depends on the dispersion of $R_i(\mathcal{A})$, $i = 1, 2, \ldots, n$. If $R_{\max}(\mathcal{A})$ and $R_{\min}(\mathcal{A})$ are very close, then we even have an accurate approach for $\lambda_{\min}(\mathcal{A})$.

Looking at the proof of Theorem 3, we can conclude that the lower bound of (2) holds for any strictly diagonally dominant tensor with positive diagonal entries but with arbitrary off-diagonal entries. By replacing $R_{\min}(\mathcal{A})$ by $\widetilde{R}_{\min}(\mathcal{A})$ and $\lambda_{\min}(\mathcal{A})$ by the minimal value of the real part of all eigenvalues of $\mathcal{A}$, we have

$$\widetilde{R}_{\min}(\mathcal{A}) \leq \min_{\lambda\in\sigma(\mathcal{A})} Re\lambda. \tag{5}$$

However, the upper bound of (2) cannot be extended for strictly diagonally dominant tensors whose off-diagonal entries have arbitrary sign. This fact can be illustrated by the following example.

Example 1. *Let $\mathcal{A}$ be a strictly diagonal dominant tensor of order 3 and dimension 2 with the entries defined as follows:*

$$a_{111} = 2,\ a_{122} = 1,\ a_{222} = 3,\ a_{211} = -2,\ \textit{and}\ a_{i_1 i_2 i_3} = 0,\ \textit{otherwise}.$$

Then the eigenvalue problem of tensor $\mathcal{A}$ becomes:

$$\begin{cases} 2x_1^2 + x_2^2 = \lambda x_1^2, \\ 3x_2^2 - 2x_1^2 = \lambda x_2^2. \end{cases} \tag{6}$$

Obviously, $\widetilde{R}_{\max}(\mathcal{A}) = 1$ and from (6), we have $\min\limits_{\lambda\in\sigma(\mathcal{A})} Re\lambda = \frac{5}{2}(> \widetilde{R}_{\max}(\mathcal{A}) = 1)$.

By the bounds (2) and (5), we have the following conclusion.

Proposition 1. *Let $\mathcal{A}$ be a strictly diagonally dominant tensor with positive diagonal entries such that $\widetilde{R}_{\max}(\mathcal{A}) = \widetilde{R}_{\min}(\mathcal{A})$. Then*

$$\widetilde{R}_i(\mathcal{A}) = \lambda_{\min}(\mathcal{M}(\mathcal{A})) \leq \min_{\lambda\in\sigma(\mathcal{A})} Re\lambda,$$

where $\mathcal{M}(\mathcal{A}) = (m_{i_1 i_2 \dots i_m})$ is the comparison tensor of $\mathcal{A}$.

Proof. Since $\widetilde{R}_{\max}(\mathcal{A})$ and $\widetilde{R}_{\min}(\mathcal{A})$ are the largest and the smallest row sums of $\mathcal{M}(\mathcal{A})$ respectively, by Theorem 3, we have $\lambda_{\min}(\mathcal{M}(\mathcal{A})) = \widetilde{R}_i(\mathcal{A})$. Now, the result follows from (5).

Thus, we complete the proof. □

According to Theorem 3, we obtain the new bounds for the minimum eigenvalue of nonsingular M-tensors.

Theorem 4. *Let $\mathcal{A}$ be a nonsingular M-tensor. Then $\mathcal{A}$ has a positive H^+-eigenvalue $\lambda_{min}(\mathcal{A})$ and satisfies*

$$0 < \frac{R_{\min}(\mathcal{A}D^{m-1})}{\max\limits_{i=1,2,\dots,n}\{d_i^{m-1}\}} \leq \lambda_{\min} \leq \frac{R_{\max}(\mathcal{A}D^{m-1})}{\min\limits_{i=1,2,\dots,n}\{d_i^{m-1}\}}, \tag{7}$$

where $D = diag(d_1, d_2, \dots, d_n)$ is a positive diagonal matrix.

Proof. By the proof of Theorem 3, $\mathcal{A}$ has a positive H$^+$-eigenvalue $\lambda_{\min}(\mathcal{A})$. We now prove (7). Since $\mathcal{A}$ is a nonsingular M-tensor, by Lemma 4, there exists a positive diagonal matrix $D = diag(d_1, d_2, \dots, d_n)$ such that $\mathcal{A}D^{m-1}$ is a strictly diagonally dominant Z-tensor with positive diagonal entries.

Denote $\mathcal{B} = \mathcal{A} \cdot D^{-(m-1)} \cdot \overbrace{D \cdots D}^{m-1}$. Then $\mathcal{B}$ is also a strictly diagonally dominant Z-tensor with positive diagonal entries. By Lemma 2, $\mathcal{A}$ and $\mathcal{B}$ have the same eigenvalues. Obviously, the vector of row sums of $\mathcal{B}$ has the form

$$(\frac{R_1(\mathcal{A}D^{m-1})}{d_1^{m-1}}, \ldots, \frac{R_n(\mathcal{A}D^{m-1})}{d_n^{m-1}})^T.$$

Hence, by Theorem 3, we have

$$\min_{i=1,2,\ldots,n}\{\frac{R_i(\mathcal{A}D^{m-1})}{d_i^{m-1}}\} \leq \lambda_{\min} \leq \max_{i=1,2,\ldots,n}\{\frac{R_i(\mathcal{A}D^{m-1})}{d_i^{m-1}}\}.$$

Then,

$$0 < \frac{R_{\min}(\mathcal{A}D^{m-1})}{\max\limits_{i=1,2,\ldots,n} d_i^{m-1}} \leq \lambda_{\min} \leq \frac{R_{\max}(\mathcal{A}D^{m-1})}{\min\limits_{i=1,2,\ldots,n} d_i^{m-1}}.$$

Thus, we complete the proof. $\square$

The sharpness of the bounds of Theorem 4 depends on the distance between $R_{\max}(\mathcal{A}D^{m-1})$ and $R_{\min}(\mathcal{A}D^{m-1})$, as well as on the distance between $\max\limits_{i=1,2,\ldots,n}\{d_i^{m-1}\}$ and $\min\limits_{i=1,2,\ldots,n}\{d_i^{m-1}\}$.

Acknowledgments

We are grateful to Professor Wenyu Sun for his instructive advice and useful suggestions, and this work was supported by the National Natural Science Foundation of China under the grant 11371276.

References

[1] L. Qi, Eigenvalues of a real supersymmetric tensor, *J. Symbolic Comput.*, **40** (2005), 1302–1324.

[2] L. H. Lim, Singular values and eigenvalues of tensors: A variational approach, *in CAMSAP'05: Proceeding of the IEEE International Workshop on Computational Advances in Multi-tensor Adaptive Processing*, 2005, 129–132.

[3] L. Qi, W. Sun and Y. Wang, Numerical multilinear algebra and its applications, *Front. Math. China*, **2** (2007), 501–526.

[4] L. Qi, Y. Wang and E. X. Wu, D-eigenvalues of diffusion kurtosis tensor, *Journal of Computational and Applied Mathematics*, **221** (2008), 150–157.

[5] M. Ng, L. Qi and G. Zhou, Finding the largest eigenvalue of a nonnegative tensor, *SLAM J. Matrix Anal. Appl.*, **31** (2009), 1090–1099.

[6] L. Bloy and R. Verma, On computing the underlying fiber directions from the diffusion orientation distribution function, *In: Medical Image Computing and Computer-Assisted Intervention-MICCAI 2008, Springer, Berlin/Heidelberg*, (2008), 1–8.

[7] L. De Lathauwer, B. D. Moor and J. Vandewalle, On the best rank-1 and rank-$(R_1, R_2, \ldots, R_N)$ approximation of higher-order tensors, *SIAM J. Matrix Anal. Appl.*, **21** (2000), 1324–1342.

[8] F. Wang, The tensor eigenvalue methods for the positive definiteness identification problem, *Hong Kong Polytechnic University*, (2006).

[9] K. C. Chang, K. Pearson, and T. Zhang, Some variational principles for Z-eigenvalues of nonnegative tensors, *Linear Algebra and Its Applications*, **438** (2013), 4166-4182.

[10] C. Li, Y. Li, and K. Xu, New eigenvalue inclusion sets for tensors, *Numerical Linear Algebra with Applications*, **21** (2014), 39-50.

[11] J. He and Z. Huang, Upper bound for the largest Z-eigenvalue of positive tensors, *Linear Algebra and Its Applications*, **38** (2014), 110-114.

[12] J. He and Z. Huang, Inequalities for M-tensors, *Journal of Inequalities and Applications*, (2014).

[13] K. C. Chang, K. Pearson and T. Zhang, Perron Frobenius Theorem for nonnegative tensors, *Commun. Math. Sci.*, **6** (2008), 507–520.

[14] Y. Yang and Q. Yang, Further Results for Perron-Frobenius Theorem for Nonnegative Tensors, *SIAM Journal on Matrix Analysis and Applications*, **31** (2011), 2517–2530.

[15] Q. Yang and Y. Yang , Further Results for Perron-Frobenius Theorem for Nonnegative Tensors II, *SIAM Journal on Matrix Analysis and Applications*, **32** (2011), 1236–1250.

[16] L. Zhang, L. Qi and G. Zhou, M-tensors and The Positive Definiteness of a Multivariate Form, *Mathematics*, (2012).

[17] W Ding, L. Qi and Y. Wei, M-Tensors and Nonsingular M-Tensors, *Linear Algebra and Its Applications*, **439** (2013), 3264–3278.

[18] L. Zhang, L. Qi and G. Zhou, M-tensors and some applications, *SIAM Journal on Matrix Analysis and Applications*, **35** (2014), 437–452.