Existence of Periodic Solutions for Second Order Hamiltonian Systems with Asymptotically Linear Conditions

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Abstract This paper considers a class of asymptotically linear nonautonomous second order Hamiltonian Systems. Using Saddle Point Theorem, the existence result is obtained, which extends some previously known results.

Keywords existence, periodic solution, second order Hamiltonian systems, Saddle Point Theorem

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1 Introduction and main results

Let us consider the following second order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) + \nabla_u F(t, u(t)) = 0, \ \forall t \in \mathbf{R}, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \quad T > 0, \end{cases} \tag{1}$$

where $F(t, u) = -K(t, u) + W(t, u)$ and $K, W \in C^2(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ with conditions that $K(t+T, u) = K(t, u)$ and $W(t+T, u) = W(t, u)$ hold for all t and u.

In recent decades, the existence results for system (1) are obtained via minimax methods in critical point theory, such as papers [3]-[10],[12]-[20] and their references therein. For example, under the assumption that $K(t, x) \equiv 0$, papers [5] and [18] considered the case that $W(t, x)$ satisfies subquadratic potential condition. Under the assumption that $K(t, x) = \frac{1}{2}(B(t)x, x)$, where $B(t)$ is a $n \times n$ symmetric matrix function, continuous and T-periodic, papers [13] and [19] considered the case that $W(t, x)$ satisfies superquadratic potential condition. Different from above papers [5, 13, 18, 19], papers [14] and [20] considered the case that $\nabla W(t, x)$ satisfies an asymptotically linear condition. And the multiplicity of periodic solutions for system (1) with symmetric assumption for $W(t, x)$ was proved in paper [6]. If $K(t, x)$ is not a quadratic form, there are also some results, such as papers [3], [14], [20], etc. Paper [20]

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obtained an existence result, if $K(t,x)$ satisfies the "pinching" condition, that is, $q_1|x|^2 \leqslant K(t,x) \leqslant q_2|x|^2$, where constants $q_1, q_2 > 0$. In the sequence, paper [14] generalized the result in paper [20] replacing the "pinching" condition by (K1) and (K2), that is,

(K1) there exist constants $d_1 > 0$ and $\gamma \in (1,2]$ such that

$$K(t,\mathbf{0}) = 0 \text{ and } \ K(t,x) \geqslant d_1|x|^\gamma, \ (t,x) \in [0,T] \times \mathbf{R}^n,$$

(K2) $(\nabla K(t,x), x) \leqslant 2K(t,x)$, $(t,x) \in [0,T] \times \mathbf{R}^n$.

In this paper, we continue to discuss the case that $\nabla W(t,x)$ satisfies an asymptotically linear condition. Different from paper [14], we replace conditions (K1) and (K2) by

(K1*) there exist a constant $d > 0$ and a function $f_1 \in L^1([0,T],\mathbf{R})$ such that

$$K(t,x) \geqslant -d|x|^2 + f_1(t), \ (t,x) \in [0,T] \times \mathbf{R}^n,$$

(K2*) there exists a constant $L_1 > 0$ such that

$$(\nabla K(t,x), x) \leqslant 2K(t,x), \ t \in [0,T] \text{ and } |x| \geqslant L_1.$$

Then we obtain the following existence result.

Theorem 1. *Suppose that function K satisfies* (K1*), (K2*) *and function W satisfies*

(W1) *there exist a constant $0 < a < \frac{6-dT^2}{T^2}$ and a function $f_2 \in L^1([0,T],\mathbf{R})$ such that*

$$W(t,x) \leqslant a|x|^2 + f_2(t), \ \forall x \in \mathbf{R}^n \text{ and } t \in [0,T],$$

(W2) $(\nabla W(t,x), x) - 2W(t,x) \to +\infty$ *uniformly for $t \in [0,T]$ as $|x| \to +\infty$,*

in addition, functions K and W also satisfy the following conditions (F1) *and* (F2) *,*

(F1) *there exists a constant $L_3 > 0$, for every $c \geqslant L_3$,*

$$\max_{|x|=c} K(t,x) < \min_{|x|=c} W(t,x) \text{ for all } t \in [0,T],$$

(F2) *there exists a constant $L_4 > 0$ such that $\nabla F(t,x) \not\equiv \mathbf{0}$ for all $t \in [0,T]$ and $|x| \leqslant L_4$ and*

$$\int_0^T F(t,x)\mathrm{d}t > \int_0^T [f_2(t) - f_1(t)]\mathrm{d}t \text{ for all } t \in [0,T] \text{ and } |x| > L_4.$$

Then system (1) possesses a nontrivial T-periodic solution.

Here, we state three aspects which illustrate that Theorem 1 is different from [[14], Theorem 1.1]. Firstly, paper [14] used Mountain Pass Lemma, however, we use Saddle Point Theorem. Secondly, conditions (K1*) and (K2*) generalize the conditions (K1) and (K2) respectively. For example, set $K(t,x) = \frac{1}{2}(B(t)x,x)$, where $B(t)$ is a $n \times n$ symmetric matrix function, continuous and T-periodic, then $K(t,x)$ satisfies (K1*) and (K2*), however, does not always satisfy (K1), unless $B(t)$ is positive definite for all t. Thirdly, paper [14] supposed that $\limsup_{|x|\to 0} \frac{W(t,x)}{|x|^2} < d_1$ for all $t \in [0,T]$. However, the limit condition at origin of $W(t,x)$ has been got rid of in our Theorem 1.1.

Functions satisfying Theorem 1 do really exist, but may not be covered by [[14], Theorem 1.1] (see Example 3.1 in Section 3).

2 Proof of Theorem 1

Set $H_T^1 = \Big\{ u : [0,T] \to \mathbf{R}^n | u$ is absolutely continuous, $u(0) = u(T)$ and $\dot{u} \in L^2([0,T], \mathbf{R}^n) \Big\}$, then H_T^1 is a Hilbert space with the norm defined by

$$\|u\| = \left[\int_0^T (|\dot{u}(t)|^2 + |u(t)|^2) \mathrm{d}t \right]^{\frac{1}{2}}. \tag{2}$$

Define a functional

$$\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \mathrm{d}t + \int_0^T K(t,u(t)) \mathrm{d}t - \int_0^T W(t,u(t)) \mathrm{d}t, \ \forall u \in H_T^1. \tag{3}$$

The Book [10] tells us that K and $W \in C^1(\mathbf{R} \times \mathbf{R}^n, \mathbf{R})$ implies that the functional φ is continuously differentiable in H_T^1. Moreover, for every u, v in H_T^1, one has

$$\langle \varphi'(u), v \rangle = \int_0^T (\dot{u}(t), \dot{v}(t)) \mathrm{d}t + \int_0^T (\nabla K(t,u(t)), v(t)) \mathrm{d}t - \int_0^T (\nabla W(t,u(t)), v(t)) \mathrm{d}t, \tag{4}$$

if $u \in H_T^1$ is a solution of the corresponding Euler-Lagrange equation $\varphi'(u) = \mathbf{0}$, then $u(t)$ satisfies the system (1).

Lemma 1. *Suppose that $W(t,x)$ satisfies* (W2) *and $K(t,x)$ satisfies* (K2*), *then there exists a constant $M > 0$ large enough such that*

$$W(t,x) \geqslant \frac{|x|^2}{M^2} \cdot \min_{|x|=M} W(t,x), \quad \text{if } |x| \geqslant M \text{ and } t \in [0,T], \tag{5}$$

$$K(t,x) \leqslant \frac{|x|^2}{M^2} \cdot \max_{|x|=M} K(t,x), \ \text{if } |x| \geqslant M \text{ and } t \in [0,T]. \tag{6}$$

Proof. For every fixed $x \in \mathbf{R}^n \backslash \{\mathbf{0}\}$ and $t \in [0,T]$, set functions $f(s) = W(t,sx)$ and $g(s) = f'(s)s - 2f(s)$. By (W2), there exists a constant $M > 0$ such that

$$g(s) \geqslant 0 \text{ as } s \geqslant \frac{M}{|x|} \text{ and all } t \in [0,T].$$

Solving the ordinary differential equation $f'(s)s - 2f(s) - g(s) = 0$, we obtain

$$
\begin{aligned}
f(s) &= \exp\Big(\int_{\frac{M}{|x|}}^{s} \frac{2}{t}\mathrm{d}t\Big) \cdot \Big[\int_{\frac{M}{|x|}}^{s} \frac{g(t)}{t} \exp\Big(-\int_{\frac{M}{|x|}}^{t} \frac{2}{r}\mathrm{d}r\Big)\mathrm{d}t + f(\frac{M}{|x|})\Big] \\
&= \frac{s^2|x|^2}{M^2} f(\frac{M}{|x|}) + s^2 \int_{\frac{M}{|x|}}^{s} \frac{g(t)}{t^3}\mathrm{d}t \\
&\geqslant \frac{s^2|x|^2}{M^2} f(\frac{M}{|x|}).
\end{aligned}
$$

So

$$
W(t, sx) \geqslant \frac{s^2|x|^2}{M^2} \cdot W(t, \frac{Mx}{|x|}) \geqslant \frac{s^2|x|^2}{M^2} \cdot \min_{|x|=M} W(t, x), \ \forall s \geqslant \frac{M}{|x|} \text{ and } t \in [0, T],
$$

which implies that (5) holds. Similar to the above process for $K(t, x)$, we have that (6) holds. $\square$

Recall the (C) condition (see definition in paper [2]), that is, a sequence $\{u_m\} \subset H_T^1$ has a convergent sequence, if $\{\varphi(u_m)\}$ is bounded and $\|\varphi'(u_m)\|(1+\|u_m\|) \to 0$, as $m \to +\infty$.

Lemma 2. *(see paper [9]) Suppose that E is a Lebesgue measurale subset of $\mathbf{R}$ with meas$(E) < +\infty$ ("meas" denotes the Lebesgue measure) and $f_n(t)$ is a sequence of Lebesgue measurable functions such that $f_n(t) \to +\infty$ as $n \to +\infty$ for a.e. $t \in E$. Then there exists, for every $\delta > 0$, a subset E_δ with meas$(E \backslash E_\delta)< \delta$ such that $f_n(t) \to +\infty$ as $n \to +\infty$ uniformly for all $t \in E_\delta$.*

Lemma 3. *If the function K satisfies* (K1*) *and* (K2*)*, the function W satisfies* (W1) *and* (W2)*, then the functional φ satisfies the (C) condition.*

Proof. Let $\{u_m\}$ be a (C)-sequence in H_T^1, that is,

$$
\sup_{m \in \mathbf{N}^*} \{|\varphi(u_m)|\} < +\infty \text{ and } (1 + \|u_m\|)\|\varphi'(u_m)\| \to 0, \text{ as } m \to +\infty.
$$

Then, there exists a constant $M_0 > 0$ such that

$$
|\varphi(u_m)| \leqslant M_0, \quad (1 + \|u_m\|)\|\varphi'(u_m)\| \leqslant M_0 \text{ for all } m \in \mathbf{N}^*.
$$

Firstly, we will show that $\{u_m\}$ is bounded.

Arguing in an indirect way, we may suppose that $\|u_{m_k}\| \to +\infty$, as $k \to +\infty$, we still denote $\{u_{m_k}\}$ by $\{u_m\}$.

Set $z_m = \frac{u_m}{\|u_m\|}$, then $\|z_m\| = 1$, so there exists a $z \in H_T^1$ such that $z_m \rightharpoonup z$ in H_T^1, then $\|z\| \leqslant 1$. By Sobolev's Imbedding Theorem, we have $z_m \to z$ in $C([0, T], \mathbf{R}^n)$ as $m \to +\infty$.

The following discussion is divided into two cases.

Case 1. $z \not\equiv \mathbf{0}$. Set $E := \{t \in [0,T] : |z(t)| > 0\}$, then $meas(E) > 0$ ($meas$ denotes the Lebesgue measure). By Lemma 2 and $\|u_m\| \to +\infty$, there exists a set $E_\delta \subset E$ with $meas(E_\delta) > 0$ such that

$$|u_m(t)| = \|u_m\| \cdot |z_m(t)| \to +\infty \text{ uniformly for all } \ t \in E_\delta \text{ as } n \to +\infty. \quad (7)$$

For every fixed $m \in \mathbf{N}^*$ and $\lambda > \max\{L_1, M\}$, from (K2*), we have

$$\begin{aligned} &\int_0^T [2K(t,u_m(t)) - (\nabla K(t,u_m(t)), u_m(t))]\mathrm{d}t \\ =& \int_{\{t\in[0,T]:|u_m(t)|>\lambda\}} [2K(t,u_m(t)) - (\nabla K(t,u_m(t)), u_m(t))]\mathrm{d}t \\ &+ \int_{\{t\in[0,T]:|u_m(t)|\leqslant\lambda\}} [2K(t,u_m(t)) - (\nabla K(t,u_m(t)), u_m(t))]\mathrm{d}t \\ \geqslant& \int_{\{t\in[0,T]:|u_m(t)|\leqslant\lambda\}} [2K(t,u_m(t)) - (\nabla K(t,u_m(t)), u_m(t))]\mathrm{d}t \\ \geqslant& - M_1, \end{aligned} \quad (8)$$

where

$$M_1 = T\cdot \max_{t\in[0,T]} \left\{ \max_{|x|\leqslant\lambda} \{2|K(t,x)| + |\nabla K(t,x)| \cdot |x|, 2|W(t,x)| + |\nabla W(t,x)| \cdot |x|\} \right\}.$$

Set $E_\delta^c = [0,T]\backslash E_\delta$. Similar to (8), by (W2), we have

$$\begin{aligned} &\int_{E_\delta^c} [(\nabla W(t,u_m(t)), u_m(t)) - 2W(t,u_m(t))]\mathrm{d}t \\ \geqslant& \int_{E_\delta^c \cap \{t\in[0,T]:|u_m(t)|\leqslant\lambda\}} [(\nabla W(t,u_m(t)), u_m(t)) - 2W(t,u_m(t))]\mathrm{d}t \geqslant -M_1. \end{aligned} \quad (9)$$

By (W2) and (7), we have

$$\int_{E_\delta} [(\nabla W(t,u_m(t)), u_m(t)) - 2W(t,u_m(t))]\mathrm{d}t \to +\infty, \text{as } m \to +\infty. \quad (10)$$

By (3), (4), (8), (9) and (10), one has

$$
\begin{aligned}
3M_0 \geqslant & 2\varphi(u_m) - \langle \varphi'(u_m), u_m \rangle \\
= & \int_0^T [(\nabla W(t, u_m(t)), u_m(t)) - 2W(t, u_m(t))]\mathrm{d}t \\
& + \int_0^T [2K(t, u_m(t)) - (\nabla K(t, u_m(t)), u_m(t))]\mathrm{d}t \\
= & \int_{E_\delta} [(\nabla W(t, u_m(t)), u_m(t)) - 2W(t, u_m(t))]\mathrm{d}t \\
& + \int_{E_\delta^c} [(\nabla W(t, u_m(t)), u_m(t)) - 2W(t, u_m(t))]\mathrm{d}t \\
& + \int_0^T [2K(t, u_m(t)) - (\nabla K(t, u_m(t)), u_m(t))]\mathrm{d}t \\
\geqslant & \int_{E_\delta} [(\nabla W(t, u_m(t)), u_m(t)) - 2W(t, u_m(t))]\mathrm{d}t - 2M_1 \\
\to & +\infty, \text{ as } m \to +\infty,
\end{aligned}
$$

which yields a contradiction.

Case 2. $z \equiv \mathbf{0}$. By (2) and (3), we have

$$
\int_0^T W(t, u_m(t))\mathrm{d}t - \int_0^T K(t, u_m(t))\mathrm{d}t = \frac{1}{2}\|u_m\|^2 - \frac{1}{2}\int_0^T |u_m(t)|^2\mathrm{d}t - \varphi(u_m).
$$

Divided by $\|u_m\|^2$ on both sides, then we have

$$
\int_0^T \frac{W(t, u_m(t)) - K(t, u_m(t))}{\|u_m\|^2}\mathrm{d}t \to \frac{1}{2} \text{ as } m \to +\infty. \tag{11}
$$

By (W1), (K1*), one has

$$
\begin{aligned}
\int_0^T \frac{W(t, u_m(t)) - K(t, u_m(t))}{\|u_m\|^2}\mathrm{d}t \leqslant & \int_0^T \frac{(a+d)|u_m(t)|^2}{\|u_m(t)\|^2}\mathrm{d}t + \frac{\int_0^T (f_2(t) - f_1(t))\mathrm{d}t}{\|u_m\|^2} \\
\leqslant & (a+d)T\|z_m\|_\infty^2 + \frac{M_2}{\|u_m\|^2} \to 0, \text{ as } m \to +\infty,
\end{aligned}
$$

which contradicts to (11). Hence, $\{u_m\}$ is bounded in H_T^1.

In a similar way to Proposition 4.3 in book [10], there exists $u \in H_T^1$ such that $u_m \rightharpoonup u$ in H_T^1. One has

$$\begin{aligned}\int_0^T |\dot{u}_m(t)-\dot{u}(t)|^2 \mathrm{d}t =& \langle \varphi'(u_m)-\varphi'(u), u_m-u\rangle \\ &- \int_0^T (\nabla K(t,u_m)-\nabla K(t,u), u_m-u)\mathrm{d}t \\ &+ \int_0^T (\nabla W(t,u_m)-\nabla W(t,u), u_m-u)\mathrm{d}t \\ &\to 0, \text{ as } m\to+\infty,\end{aligned}$$

which implies that $\|\dot{u}_m-\dot{u}\|_{L^2}\to 0$. So we have $u_m\to u$ in H_T^1. Hence φ satisfies (C) condition. □

Lemma 4. *(Saddle Point Theorem, see book [11]) Let $\mathscr{H}$ be a real Banach space with $\mathscr{H}=V\bigoplus X$, where $V\neq\{\mathbf{0}\}$ is finite dimensional. Suppose that $\varphi\in C^1(\mathscr{H},\mathbf{R})$ satisfies (PS) condition and*
(i) there is a constant α and a bounded neighborhood D of $\mathbf{0}$ in V such that $\varphi|_{\partial D}\leqslant\alpha$ and
(ii) there is a constant $\beta>\alpha$ such that $\varphi|_X\geqslant\beta$.
Then φ possesses a critical value $c\geqslant\beta$ which can be characterized as

$$c=\inf_{h\in\tau}\max_{u\in\bar{D}}\varphi(h(u)),\ \textit{where } \tau=\{h\in C(\overline{D},\mathscr{H})|h=\mathrm{id} \text{ on } \partial D\}.$$

Remark 1. *As shown in paper [1], a deformation lemma can be proved with condition (C) replacing the usual (PS) condition, and it turns out that Lemma 4 holds under condition (C).*

Set $\tilde{u}(t)=u(t)-\bar{u}$ with $\overline{u}=\frac{1}{T}\int_0^T u(t)\mathrm{d}t$, then book [10] tells us that $H_T^1=\tilde{H}_T^1\bigoplus\mathbf{R}^n$, where $\tilde{H}_T^1:=\{u\in H_T^1|\overline{u}=\mathbf{0}\}$. Page 9 of book [10] tells us

$$\|\tilde{u}\|_\infty^2\leqslant\frac{T}{12}\int_0^T|\dot{u}(t)|^2\mathrm{d}t,\ \forall u\in\tilde{H}_T^1.\quad \text{(Sobolev's inequality)} \tag{12}$$

Proof of Theorem 1. Lemma 3 tells us that φ satisfies (C) condition. So, it needs only to check (i) and (ii) in Lemma 4.

Step 1. Set $V=\mathbf{R}^n$. We claim that (i) in Lemma 4 holds. In fact, by Lemma 1 and (F1), for fixed $x_0\in\mathbf{R}^n$ with $|x_0|=1$, if $s\geqslant\max\{M,L_3\}$, then we have

$$\begin{aligned}\varphi(sx_0)&=\int_0^T K(t,sx_0)\mathrm{d}t-\int_0^T W(t,sx_0)\mathrm{d}t \\ &\leqslant\frac{s^2}{M^2}\int_0^T\left[\max_{|x|=M}K(t,x)-\min_{|x|=M}W(t,x)\right]\mathrm{d}t \\ &\leqslant\frac{s^2T}{M^2}\max_{t\in[0,T]}\left\{\max_{|x|=M}K(t,x)-\min_{|x|=M}W(t,x)\right\} \\ &\to-\infty,\ \text{as } s\to+\infty,\end{aligned}$$

which implies that there exist constant $r > 0$ large enough and constant $\alpha := \int_0^T [f_1(t) - f_2(t)]\mathrm{d}t - 1$ such that $\varphi|_{\partial B_{r(\mathbf{0})} \cap V} \leqslant \alpha$.

Step 2. Set $X = \tilde{H}_T^1$. We claim that (ii) in Lemma 4 holds. In fact, by (K1*), (W1) and (12), for $u \in \tilde{H}_T^1$, we obtain

$$\begin{aligned}
\varphi(u) =& \frac{1}{2}\int_0^T |\dot{u}(t)|^2 \mathrm{d}t + \int_0^T K(t,u)\mathrm{d}t - \int_0^T W(t,u)\mathrm{d}t \\
\geqslant& \frac{1}{2}\int_0^T |\dot{u}(t)|^2 \mathrm{d}t - (d+a)\int_0^T |u(t)|^2 \mathrm{d}t + \int_0^T (f_1(t) - f_2(t))\mathrm{d}t \\
=& \left(\frac{1}{2} - \frac{(d+a)T^2}{12}\right)\int_0^T |\dot{u}(t)|^2 \mathrm{d}t + \int_0^T (f_1(t) - f_2(t))\mathrm{d}t \\
\geqslant& \int_0^T (f_1(t) - f_2(t))\mathrm{d}t, \quad \forall u \in \tilde{H}_T^1,
\end{aligned}$$

which implies that there exists a constant $\beta := \int_0^T (f_1(t) - f_2(t))\mathrm{d}t$ such that $\varphi\,|_X \geqslant \beta$.

So Lemma 4 tells us that φ possesses a critical value $c \geqslant \beta$, which can be characterized as $c = \inf\limits_{h\in\tau}\max\limits_{u\in\bar{D}} \varphi(h(u))$, where $\tau = \{h \in C(\overline{D}, \mathscr{H})|h = \mathrm{id} \text{ on } \partial D\}$. We suppose that $\varphi(u) = c$ and $\varphi'(u) = \mathbf{0}$, then we know that u satisfies $\int_0^T (\dot{u}\cdot \dot{h} - \nabla F(t,u)\cdot h)\mathrm{d}t = 0$ for $\forall h \in H_T^1$.

Similarly to the proof in page 96 of book [8], under the assumption of $K, W \in C^2(\mathbf{R}\times\mathbf{R}^n, \mathbf{R})$, the weak solution of system (1) is classical solution.

Step 3. By (F2), the above u is a nontrivial solution.

This completes the proof. $\square$

3 Example

Now, we give an example to illustrate an application of the Theorem 1 and the difference between the Theorem 1 and [[14], Theorem 1.1].

Example 3.1 Set $T = 1$, define $K, W : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ with $K(t,x) = \frac{\sin^2(\pi t)}{5}(B(t)x, x)$, where $B(t) \equiv \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & (-1)^{n+1} \end{pmatrix}_{n\times n}$ and

$$W(t,x) = \frac{1+\sin^2(\pi t)}{4}|x|^2\left[1 - \frac{1}{\ln(10^{10} + |x|^2)}\right]$$

for all $t \in [0,1]$ and $x \in \mathbf{R}^n$. Then functions $K, W \in C^2(\mathbf{R}\times\mathbf{R}^n, \mathbf{R})$ hold and are 1-periodic with respect to the variable t.

Obviously, $K(t,x)$ satisfies (K1*) with $d \equiv \frac{1}{5}$, $f_1(t) \equiv 0$ and (K2*). For $W(t,x)$, set $a = \frac{1}{2}$, $f_2(t) \equiv 0$, then $a + d < 6$, so (W1) holds. In addition,

$(\nabla W(t,x),x)-2W(t,x)=\frac{(1+\sin^2(\pi t))|x|^4}{2\ln^2(10^{10}+|x|^2)(10^{10}+|x|^2)}\rightrightarrows+\infty$, as $|x|\to+\infty$, so (W2) holds. Lastly, for any $c\in\mathbf{R}_+$ large enough, we have

$$\max_{|x|=c}K(t,x)\leqslant\frac{1}{5}c^2<\min_{|x|=c}W(t,x),$$

and $\int_0^T F(t,x)\mathrm{d}t>\int_0^T[f_2(t)-f_1(t)]\mathrm{d}t$ for all $t\in[0,T]$. Hence, (F1) and (F2) hold. Hence, system (1) possesses a nontrivial 1-periodic solution for above functions K and W.

However, above $K(t,x)$ and $W(t,x)$ can't be covered by [[14], Theorem 1.1], because $K(t,x)$ does not satisfy (K1) and $W(t,x)$ does not satisfy the condition that $\limsup\limits_{|x|\to 0}\frac{W(t,x)}{|x|^2}<d_1$ (where d_1 appears in (K1)).

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