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# Differences of integral-type operators from weighted Bergman spaces to $\beta$-Bloch–Orlicz spaces

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**Abstract.** We found several characterizations for the boundedness of the differences of integral-type operators from weighted Bergman spaces to $\beta$-Bloch–Orlicz spaces on the unit disk. In particular, their descriptions in terms of the $n$-th power of the induced analytic self-maps were also found. After that we estimated their essential norms, which can provide new compactness criteria. Finally, we completed this paper with analogous results for the differences of relevant integral-type operators acting from weighted Bergman spaces to $\beta$-Bloch–Orlicz spaces, which extend and strengthen several existing results in the literature.

## 1. Introduction

Let $H(\mathbb{D})$ be the space of all holomorphic functions on the open unit disk $\mathbb{D}$, and $S(\mathbb{D})$ the collection of all holomorphic self-maps on $\mathbb{D}$, where $\mathbb{D}$ is the unit disk in the complex plane $\mathbb{C}$. Given a continuous linear operator $T$ on a Banach space $X$, its essential norm is the distance from the operator $T$ to compact operators on $X$, that is, $\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}$. It is trivial that $\|T\|_e = 0$ if and only if $T$ is compact, see, e.g., [3]–[5], [10]–[11], [14], and their references therein.

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For $a \in \mathbb{D}$, let $\varphi_a$ be the automorphism of $\mathbb{D}$ exchanging 0 for $a$, that is, $\varphi_a(z) = (a-z)/(1-\bar{a}z)$. For $z, w \in \mathbb{D}$, the pseudo-hyperbolic distance between $z$ and $w$ is given by

$$\rho(z,w) = |\varphi_w(z)| = \left|\frac{z-w}{1-\bar{w}z}\right|.$$

In what follows, we will denote $\rho(z) = \rho(\phi(z), \psi(z))$ for $\phi, \psi \in S(\mathbb{D})$.

For an analytic self-map $\phi : \mathbb{D} \to \mathbb{D}$, the composition operator $C_\phi : H(\mathbb{D}) \to H(\mathbb{D})$ is defined by

$$C_\phi f = f \circ \phi, \quad f \in H(\mathbb{D}).$$

The study of composition operators is a fairly active field. For general motivations on the theory of composition operators, see the excellent books [1] by Cowen and MacCluer, and [16] by Shapiro, for more information. Based on our previous work, we generalized the results in [8] to some extent. We concentrate our attention on the boundedness, compactness and essential norm estimations of the differences of integral-type operators acting from weighted Bergman spaces to $\beta$-Bloch–Orlicz spaces, and then we list several similar characterizations for other relevant integral-type operators. It is popular for the investigations on the operator theoretic properties of integral-type operators expressed in terms of function theoretic conditions on symbols, which have been a subject of high interest. Devoted readers can refer to the very recent papers [6]–[8], [11]–[12], [17], and their reference therein. To begin with, we provided four integral-type operators, which have close relationships.

(a) Given $g \in H(\mathbb{D})$, the operator $T^g$ is defined by

$$T^g f(z) = \int_0^z f(t)g(t)dt, \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

(b) Given $g \in H(\mathbb{D})$, the operator $T_g$ is defined by

$$T_g f(z) = \int_0^z f(t)g'(t)dt, \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

(c) Let $\phi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, then the operator $P_\phi^g$ is defined by

$$P_\phi^g f(z) = \int_0^z f(\phi(t))g(t)dt, \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

(d) Let $\phi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, then the operator $T_g C_\phi$ is defined by

$$T_g C_\phi f(z) = \int_0^z f(\phi(t))g'(t)dt, \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

Indeed, these integral-type operators are closely related. On the one hand, let $\phi = id$ the identity map in $P_\phi^g$ and $T_g C_\phi$, then $P_{id}^g = T^g$ and $T_g C_{id} = T_g$. That is, the operators $T^g$ and $T_g$ are special cases of $P_\phi^g$ and $T_g C_\phi$, respectively. On the other hand, if we let $g = k' \in H(\mathbb{D})$ in $P_\phi^g$, then $P_\phi^{k'} = T_g C_\phi$. Motivated by the above observations, we will first collect some interesting consequences about $P_\phi^g - P_\psi^h$ acting from weighted Bergman space to $\beta$-Bloch–Orlicz space for $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$. And then the analogous results for the differences of another three integral-type operators will apparently follow. We refer the readers to [6] and its references therein for some descriptions about integral-type operators acting on several holomorphic function spaces. In what follows, the definitions for holomorphic spaces we investigated were exhibited in details.

Let $\mu$ be a weight; that is, $\mu$ is a positive continuous function on $\mathbb{D}$. We recall that the $\mu$-Bloch space $\mathcal{B}_\mu = \mathcal{B}_\mu(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \sup_{z \in \mathbb{D}} \mu(z)|f'(z)| < \infty.$$

It is a well-known fact that the $\mu$-Bloch space $\mathcal{B}_\mu$ is a Banach space under the norm $\|f\|_{\mathcal{B}_\mu}$. In particular, if $\mu(z) = (1 - |z|^2)^\alpha$, it follows that $\mathcal{B}_\mu = \mathcal{B}^\alpha$ (see [14] and [21]). For $\alpha = 1$, $\mathcal{B}^\alpha = \mathcal{B}$, the classical Bloch space (see, e.g. [8]); if $0 < \alpha < 1$, we have $\mathcal{B}^\alpha = \mathrm{Lip}_{1-\alpha}$ (see, e.g. [23]), the analytic Lipschitz space, which consists of all $f \in H(\mathbb{D})$ satisfying

$$|f(z) - f(w)| \leq C|z - w|^{1-\alpha},$$

for some constant $C > 0$ and all $z, w \in \mathbb{D}$; when $\alpha > 1$, $\mathcal{B}^\alpha = H_{\alpha-1}^\infty$, the $\alpha - 1$-weighted-type space of analytic functions that contains all $f \in H(\mathbb{D})$ satisfying

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha-1}|f(z)| < \infty.$$

More generally, let $v$ be a strictly positive continuous and bounded function (weight) on $\mathbb{D}$. The weighted-type space $H_v^\infty$ is defined to be the collection of all functions $f \in H(\mathbb{D})$ that satisfy

$$\|f\|_v = \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty,$$

provided we identify that differ by a constant, and then $H_v^\infty$ is a Banach space endowed with the norm $\|.\|_v$, see, e.g. [2], [5], [22], and their references therein. In particular, let $v(z) = (1 - |z|^2)^\alpha$, then $H_{(1-|z|^2)^\alpha}^\infty$ was denoted as $H_\alpha^\infty$, which was called an $\alpha$-weighted-type space endowed with the norm

$$\|f\|_{H_\alpha^\infty} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)|.$$

Recently, Ramos-Fernández employed Young's functions to define the Bloch–Orlicz space $\mathcal{B}^{\varphi}$ [15], which is a generalization of the classical Bloch space $\mathcal{B}$. More precisely, let $\varphi : [0, +\infty) \to [0, +\infty)$ be an $\mathcal{N}$-function, that is, $\varphi$ is a strictly increasing convex function such that $\varphi(0) = 0$, which implies that $\lim_{t\to\infty} \varphi(t) = +\infty$. The Bloch–Orlicz space linked with the function $\varphi$, denoted by $\mathcal{B}^{\varphi} = \mathcal{B}^{\varphi}(\mathbb{D})$, is the collection of all $f \in H(\mathbb{D})$ fulfilling

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)\varphi(\lambda|f'(z)|) < \infty,$$

for some $\lambda > 0$ depending on $f$. We can further suppose that $\varphi^{-1}$ is continuously differentiable. If $\varphi^{-1}$ is not differentiable everywhere, we can define the function

$$\psi(t) = \int_0^t \frac{\varphi(x)}{x} dx, \quad t \geq 0,$$

then $\psi$ is differentiable, whence $\psi^{-1}$ is differentiable everywhere on $[0, \infty)$. Since $\varphi$ is a strictly increasing, convex function satisfying $\varphi(0) = 0$, therefore the function $\varphi(t)/t,\ t > 0$, is increasing and

$$\varphi(t) \geq \psi(t) \geq \int_{t/2}^t \frac{\varphi(x)}{x} dx \geq \varphi\left(\frac{t}{2}\right) \quad \text{for all } t \geq 0.$$

As a consequence, $\mathcal{B}^{\varphi} = \mathcal{B}^{\psi}$. Because of the convexity of $\varphi$, we can show that the Minkowski functional

$$\|f\|_{\varphi} = \inf\left\{k > 0 : S_{\varphi}\left(\frac{f'}{k}\right) \leq 1\right\}$$

defines a seminorm for $\mathcal{B}^{\varphi}$, where

$$S_{\varphi}(f) = \sup_{z\in\mathbb{D}}(1 - |z|^2)\varphi(|f(z)|).$$

Furthermore, we can verify that $\mathcal{B}^{\varphi}$ is a Banach space under the norm

$$\|f\|_{\mathcal{B}^{\varphi}} = |f(0)| + \|f\|_{\varphi}.$$

Observing that

$$S_{\varphi}\left(\frac{f'}{\|f\|_{\mathcal{B}^{\varphi}}}\right) \leq 1,$$

we get the following lemma.

**Lemma 1.1.** *The Bloch–Orlicz space is isometrically equal to a $\mu_1^\varphi$-Bloch space, where*

$$\mu_1^\varphi(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{1-|z|^2}\right)}, \quad z \in \mathbb{D}.$$

*Whence for any $f \in \mathcal{B}^\varphi$,*

$$\|f\|_{\mathcal{B}^\varphi} = |f(0)| + \sup_{z\in\mathbb{D}} \mu_1^\varphi(z)|f'(z)|.$$

As far as we all know, the readers can consult, e.g., [15], [19] and the references therein, for the Bloch–Orlicz spaces. It is evident that Bloch–Orlicz spaces generalize some other spaces. For example, if $\varphi(t) = t^p$ with $p > 0$, then $\mathcal{B}^\varphi$ coincides with an $\alpha$-Bloch space $\mathcal{B}^\alpha$, where $\alpha = 1/p$; if $\varphi(t) = t\log(1+t)$, then $\mathcal{B}^\varphi$ coincides with the log-Bloch space (see, e.g. [20]). By a parallel generalization of a $\beta$-Bloch space $\mathcal{B}^\beta$ for $\beta > 0$, we define the $\beta$-Bloch–Orlicz space $\mathcal{B}^\varphi_\beta = \mathcal{B}^\varphi_\beta(\mathbb{D})$ [6], which is the class of all $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \varphi(\lambda|f'(z)|) < \infty,$$

for some $\lambda > 0$ depending on $f$. Besides, the $\beta$-Bloch–Orlicz space $\mathcal{B}^\varphi_\beta$ is also a Banach space under the norm

$$\|f\|_{\mathcal{B}^\varphi_\beta} = |f(0)| + \|f\|_{\varphi,\beta},$$

where

$$\|f\|_{\varphi,\beta} = \inf\left\{k > 0 : S_{\varphi,\beta}\left(\frac{f'}{k}\right) \le 1\right\},$$

and

$$S_{\varphi,\beta}(f) = \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta \varphi(|f(z)|).$$

It turns out that $\mathcal{B}^\varphi_\beta = \mathcal{B}^\varphi$ when $\beta = 1$. Furthermore, a standard fact is that

$$S_{\varphi,\beta}\left(\frac{f'}{\|f\|_{\mathcal{B}^\varphi_\beta}}\right) \le 1,$$

which yields a lemma linking with Lemma 1.1.

**Lemma 1.2.** *The $\beta$-Bloch–Orlicz space is isometrically equal to a $\mu_\beta^\varphi$-Bloch space, where*

$$\mu_\beta^\varphi(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{(1-|z|^2)^\beta}\right)}, \quad z \in \mathbb{D}.$$

*Whence for any $f \in \mathcal{B}^\varphi_\beta$,*

$$\|f\|_{\mathcal{B}^\varphi_\beta} = |f(0)| + \sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z)|f'(z)|. \tag{1.1}$$

This norm allows us to define the little $\beta$-Bloch–Orlicz space, denoted by $\mathcal{B}^{\varphi}_{\beta,0}$, which consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1} \mu^{\beta}_{\varphi}(z)|f'(z)| = 0.$$

Clearly, $\mathcal{B}^{\varphi}_{\beta,0}$ is a closed subspace of $\mathcal{B}^{\varphi}_{\beta}$.

*Remark 1.3.* For the sake of convenience in our writing, we will always use $\mu^{\varphi}_{\beta}(z)$ to stand for $1/\varphi^{-1}\left(\frac{1}{(1-|z|^2)^{\beta}}\right)$. In the sequel, we will employ the norm given in Lemma 1.2 to show our main results concerning differences of all integral-type operators.

We recall that $dA(z) = (1/\pi)drd\theta$ is the normalized Lebesgue measure on $\mathbb{D}$, and let $dA_{\alpha}(z) = (\alpha+1)(1-|z|^2)^{\alpha}dA(z)$ be the weighted Lebesgue measure for $-1 < \alpha < \infty$ such that $A_{\alpha}(\mathbb{D}) = 1$. The surface measure on $\partial\mathbb{D}$ (the boundary of the unit disk) will be denoted by $d\sigma$ satisfying $\sigma(\partial\mathbb{D}) = 1$. Interestingly, [24, Lemma 1.8] showed that the measures $dA_{\alpha}$ and $d\sigma$ are related by the the polar coordinates formula

$$\int_{\mathbb{D}} f(z)dA(z) = 2\int_0^1 rdr \int_{\partial\mathbb{D}} f(r\zeta)d\sigma(\zeta). \tag{1.2}$$

We recall that the weighted Bergman space $A^p_{\alpha} = A^p_{\alpha}(\mathbb{D})$ consisting of those functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|^p_{A^p_{\alpha}} = \int_{\mathbb{D}} |f(z)|^p dA_{\alpha}(z) < \infty.$$

It is easy to check that if $p \le q$, then $A^q_{\alpha} \subset A^p_{\alpha}$ and $A^p_{\alpha} \subset A^p_{\alpha+1}$. Finally, $H^{\infty}$ is the space of bounded analytic functions on $\mathbb{D}$, with $\|f\|_{H^{\infty}} = \sup\{|f(z)| : \ z \in \mathbb{D}\}$. Then $H^{\infty} \subset A^p_{\alpha}$, and this inclusion is proper if $p < \infty$. Based on the polar transformation and (1.2), we calculate that

$$\begin{aligned}
\|z^n\|^p_{A^p_{\alpha}} &= \int_{\mathbb{D}} |z^n|^p dA_{\alpha}(z) = \int_{\mathbb{D}} |z|^{np}(\alpha+1)(1-|z|^2)^{\alpha}dA(z) \\
&= 2(\alpha+1)\int_0^1 rdr \int_{\partial\mathbb{D}} r^{np}(1-r^2)^{\alpha}d\sigma(\zeta) \\
&= (\alpha+1)\int_0^1 (r^2)^{1+np/2-1}(1-r^2)^{\alpha+1-1}d(r^2) \\
&= (\alpha+1)\int_0^1 t^{1+np/2-1}(1-t)^{\alpha+1-1}dt = B(1+np/2, \alpha+1) \\
&= (\alpha+1)\frac{\Gamma(1+np/2)\Gamma(\alpha+1)}{\Gamma(2+np/2+\alpha)} \approx (\alpha+1)n^{-\alpha-1}, \quad \text{as } n \to \infty,
\end{aligned}$$

where the last equation is due to the Stirling's formula. That is to say that

$$\|z^n\|_{A^p_\alpha} \approx n^{-(\alpha+1)/p}, \quad \text{as } n \to \infty. \tag{1.3}$$

In 2009, a great interest was paid to describing some properties of the composition operator $C_\phi$ on Bloch-type spaces in terms of the $n$-th power of the analytic self-map $\phi$ of the open unit disk $\mathbb{D}$. For Bloch-type spaces, Zhao [21] obtained that $\|C_\phi\|_{e,\mathcal{B}^\alpha \to \mathcal{B}^\beta} \approx \limsup_{n\to\infty} n^{\alpha-1}\|\phi^n\|_\beta$ for $0 < \alpha, \beta < \infty$. Since then, many mathematicians have contributed to the development of this new characterizations for some classical operators, interested readers can refer to [3]–[4], [9]–[11] and [18]. As far as we all know, there has been no such new descriptions for differences of classical linear operators, especially for differences of integral-type operators acting from weighted Bergman spaces to $\beta$-Bloch–Orlicz spaces. Hence these problems are in desired need for response, and we will start with these investigations. By constructing some more suitable test functions, we resolved this problem partially. Of particular interest is that the Orlicz-type spaces $\mathcal{B}^\varphi_\beta$ are not quite often used in the literature. The outline of the paper is organized as follows: the properties of $P^g_\phi - P^h_\psi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$ were exhibited in Section 2, and then the similar properties of $P^g_\phi - P^h_\psi : A^p_\alpha \to \mathcal{B}^\varphi_{\beta,0}$ were investigated in Section 3. Finally, some corollaries were presented in Section 4. In summary, this paper provides a systematic exposition of equivalent conditions for the differences of integral-type operators from weighted Bergman spaces to $\beta$-Bloch–Orlicz spaces.

We want to finish this introduction by mentioning that in what follows, for two positive quantities $A$ and $B$, the notations $A \approx B$, $A \preceq B$, $A \succeq B$ mean that there may be different positive constants $C$ such that $B/C \le A \le CB$, $A \le CB$, $CB \le A$. Throughout this paper, constants are denoted by $C$, they are positive and may differ from one occurrence to the other. Besides, we denote by $\mathbb{N}_0$ the set of all nonnegative integers and denote $\gamma = (2+\alpha)/p$ in order to simplify our writings.

## 2. The properties of $P^g_\phi - P^h_\psi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$

**2.1. The boundedness of $P^g_\phi - P^h_\psi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$.** In this section, we will give several characterizations for the boundedness of $P^g_\phi - P^h_\psi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$. For $a \in \mathbb{D}$, define two families test functions,

$$f_a(z) = \frac{(1-|a|^2)^\gamma}{(1-\bar{a}z)^{2\gamma}}, \qquad \hat{f}_a(z) = \frac{(1-|a|^2)^\gamma}{(1-\bar{a}z)^{2\gamma}} \cdot \frac{a-z}{1-\bar{a}z}. \tag{2.1}$$

By using a standard procedure, it turns out that

$$\|\hat{f}_a\|^p_{A^p_\alpha} \preceq \|f_a\|^p_{A^p_\alpha} = \int_{\mathbb{D}} \frac{(1-|a|^2)^{p\gamma}}{|1-\bar{a}z|^{2p\gamma}} dA_\alpha(z) = \int_{\mathbb{D}} (\alpha+1) \frac{(1-|z|^2)^\alpha (1-|a|^2)^{2+\alpha}}{|1-\bar{a}z|^{2(2+\alpha)}} dA(z) \preceq 1,$$

which is due to [24, Theorem 1.12]. That is to say, $\sup_{a\in\mathbb{D}} \|f_a\|_{A^p_\alpha} \preceq 1$ and $\sup_{a\in\mathbb{D}} \|\hat{f}_a\|_{A^p_\alpha} \preceq 1$. For our further use, we denote two notations as below:

$$\mathcal{T}^\beta_\gamma(g\phi)(z) = \frac{\mu^\varphi_\beta(z)g(z)}{(1-|\phi(z)|^2)^\gamma}, \qquad \mathcal{T}^\beta_\gamma(h\psi)(z) = \frac{\mu^\varphi_\beta(z)h(z)}{(1-|\psi(z)|^2)^\gamma}. \tag{2.2}$$

**Lemma 2.1** ([24, Theorem 2.1]). *Suppose $0 < p < \infty$ and $\alpha > -1$. Then*

$$|f(z)| \leq \frac{\|f\|_{A^p_\alpha}}{(1-|z|^2)^\gamma},$$

*for all $f \in A^p_\alpha$ and $z \in \mathbb{D}$.*

**Lemma 2.2.** *Let $1 < p < \infty$ and $f \in A^p_\alpha$, then it holds that*

$$|(1-|z|^2)^\gamma f(z) - (1-|w|^2)^\gamma f(w)| \leq C\|f\|_{A^p_\alpha}\rho(z,w),$$

*for all $z, w \in \mathbb{D}$.*

Proof. By Lemma 2.1, it follows that if $f \in A^p_\alpha$, then $f \in H^\infty_\gamma$ and $\|f\|_{H^\infty_\gamma} \leq \|f\|_{A^p_\alpha}$. Then by [2, Lemma 3.2], there is a constant $C > 0$ such that

$$|(1-|z|^2)^\gamma f(z) - (1-|w|^2)^\gamma f(w)| \leq C\|f\|_{H^\infty_\gamma}\rho(z,w) \leq C\|f\|_{A^p_\alpha}\rho(z,w),$$

for each $f \in A^p_\alpha$ and $z, w \in \mathbb{D}$. This completes the proof. □

In the following lemma, we employ the test functions defined in (2.1) and the notations given in (2.2).

**Lemma 2.3.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$, and $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$, then the following three inequalities hold:*

(i) $$\sup_{z\in\mathbb{D}} \left|\mathcal{T}^\beta_\gamma(g\phi)(z)\right| \rho(z) \leq \sup_{a\in\mathbb{D}} \|(P^g_\phi - P^h_\psi)f_a\|_{\mathcal{B}^\varphi_\beta} + \sup_{a\in\mathbb{D}} \|(P^g_\phi - P^h_\psi)\hat{f}_a\|_{\mathcal{B}^\varphi_\beta}; \tag{2.3}$$

(ii) $$\sup_{z\in\mathbb{D}} \left|\mathcal{T}^\beta_\gamma(h\psi)(z)\right| \rho(z) \leq \sup_{a\in\mathbb{D}} \|(P^g_\phi - P^h_\psi)f_a\|_{\mathcal{B}^\varphi_\beta} + \sup_{a\in\mathbb{D}} \|(P^g_\phi - P^h_\psi)\hat{f}_a\|_{\mathcal{B}^\varphi_\beta}; \tag{2.4}$$

(iii) $$\sup_{z\in\mathbb{D}} \left|\mathcal{T}^\beta_\gamma(g\phi)(z) - \mathcal{T}^\beta_\gamma(h\psi)(z)\right| \leq \sup_{a\in\mathbb{D}} \|(P^g_\phi - P^h_\psi)f_a\|_{\mathcal{B}^\varphi_\beta} + \sup_{w\in\mathbb{D}} \|(P^g_\phi - P^h_\psi)\hat{f}_a\|_{\mathcal{B}^\varphi_\beta}. \tag{2.5}$$

Proof. Firstly, we employ (1.1) and $((P^g_\phi - P^h_\psi)f)' = f(\phi(z))g(z) - f(\psi(z))h(z)$ to express the norm $\|(P^g_\phi - P^h_\psi)f_{\phi(a)}\|_{\mathcal{B}^\varphi_\beta}$ as below:

$$
\begin{aligned}
&\|(P_\phi^g - P_\psi^h) f_{\phi(a)}\|_{\mathcal{B}_\beta^\varphi} \\
&\quad = \sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z) |f_{\phi(a)}(\phi(z)) g(z) - f_{\phi(a)}(\psi(z)) h(z)| \\
&\quad \geq \mu_\beta^\varphi(a) |f_{\phi(a)}(\phi(a)) g(a) - f_{\phi(a)}(\psi(a)) h(a)| \qquad (2.6)\\
&\quad \geq \frac{\mu_\beta^\varphi(a)|g(a)|}{(1-|\phi(a)|^2)^\gamma} - \frac{(1-|\phi(a)|^2)^\gamma (1-|\psi(a)|^2)^\gamma}{|1-\overline{\phi(a)}\psi(a)|^{2\gamma}} \frac{\mu_\beta^\varphi(a)|h(a)|}{(1-|\psi(a)|^2)^\gamma} \\
&\quad = |\mathcal{T}_\gamma^\beta(g\phi)(a)| - \frac{(1-|\phi(a)|^2)^\gamma (1-|\psi(a)|^2)^\gamma}{|1-\overline{\phi(a)}\psi(a)|^{2\gamma}} |\mathcal{T}_\gamma^\beta(h\psi)(a)|. \qquad (2.7)
\end{aligned}
$$

Similarly, it turns out that

$$
\begin{aligned}
\|(P_\phi^g - P_\psi^h) \hat{f}_{\phi(a)}\|_{\mathcal{B}_\beta^\varphi} &\geq \mu_\beta^\varphi(a) |\hat{f}_{\phi(a)}(\phi(a)) g(a) - \hat{f}_{\phi(a)}(\psi(a)) h(a)| \\
&= \mu_\beta^\varphi(a)|h(a)| \frac{(1-|\phi(a)|^2)^\gamma}{|1-\overline{\phi(a)}\psi(a)|^{2\gamma}} \rho(a) \\
&= \frac{(1-|\phi(a)|^2)^\gamma (1-|\psi(a)|^2)^\gamma}{|1-\overline{\phi(a)}\psi(a)|^{2\gamma}} \frac{\mu_\beta^\varphi(a)|h(a)|}{(1-|\psi(a)|^2)^\gamma} \rho(a) \\
&= \frac{(1-|\phi(a)|^2)^\gamma (1-|\psi(a)|^2)^\gamma}{|1-\overline{\phi(a)}\psi(a)|^{2\gamma}} |\mathcal{T}_\gamma^\beta(h\psi)(a)| \rho(a). \qquad (2.8)
\end{aligned}
$$

Putting (2.8) into (2.7), we deduce that

$$
\begin{aligned}
\left|\mathcal{T}_\gamma^\beta(g\phi)(a)\right| \rho(a) &\leq \|(P_\phi^g - P_\psi^h) f_{\phi(a)}\|_{\mathcal{B}_\beta^\varphi} \rho(a) \\
&\quad + \frac{(1-|\phi(a)|^2)^\gamma (1-|\psi(a)|^2)^\gamma}{|1-\overline{\phi(a)}\psi(a)|^{2\gamma}} |\mathcal{T}_\gamma^\beta(h\psi)(a)| \rho(a) \\
&\leq \|(P_\phi^g - P_\psi^h) f_{\phi(a)}\|_{\mathcal{B}_\beta^\varphi} + \|(P_\phi^g - P_\psi^h) \hat{f}_{\phi(a)}\|_{\mathcal{B}_\beta^\varphi}, \qquad (2.9)
\end{aligned}
$$

where the last inequality is due to the fact $\rho(a) \leq 1$. Analogously, we deduce that

$$
\left|\mathcal{T}_\gamma^\beta(h\psi)(a)\right| \rho(a) \leq \|(P_\phi^g - P_\psi^h) f_{\psi(a)}\|_{\mathcal{B}_\beta^\varphi} + \|(P_\phi^g - P_\psi^h) \hat{f}_{\psi(a)}\|_{\mathcal{B}_\beta^\varphi}. \qquad (2.10)
$$

Taking the supremum about $a \in \mathbb{D}$ in (2.9) and (2.10), we arrive at

$$
\begin{aligned}
\text{(i)} \quad \sup_{a\in\mathbb{D}} \left|\mathcal{T}_\gamma^\beta(g\phi)(a)\right| \rho(a) &\leq \sup_{a\in\mathbb{D}} \Big( \|(P_\phi^g - P_\psi^h) f_{\phi(a)}\|_{\mathcal{B}_\beta^\varphi} + \|(P_\phi^g - P_\psi^h) \hat{f}_{\phi(a)}\|_{\mathcal{B}_\beta^\varphi} \Big) \\
&\leq \sup_{a\in\mathbb{D}} \Big( \|(P_\phi^g - P_\psi^h) f_a\|_{\mathcal{B}_\beta^\varphi} + \|(P_\phi^g - P_\psi^h) \hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \Big); \qquad (2.11)
\end{aligned}
$$

$$\text{(ii)} \quad \sup_{a\in\mathbb{D}} \left|\mathcal{T}_\gamma^\beta(h\psi)(a)\right| \rho(a) \le \sup_{a\in\mathbb{D}} \Big( \|(P_\phi^g - P_\psi^h) f_a\|_{\mathcal{B}_\beta^\varphi} + \|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \Big). \tag{2.12}$$

On the other hand, we change (2.6) into

$$\begin{aligned}
\|(P_\phi^g - P_\psi^h) f_{\phi(a)}\|_{\mathcal{B}_\beta^\varphi} &\ge \mu_\beta^\varphi(a) \left| \frac{g(a)}{(1-|\phi(a)|^2)^\gamma} - \frac{h(a)(1-|\phi(a)|^2)^\gamma}{(1-\overline{\phi(a)}\psi(a))^{2\gamma}} \right| \\
&\ge \left|\mathcal{T}_\gamma^\beta(g\phi)(a) - \mathcal{T}_\gamma^\beta(h\psi)(a)\right| - \frac{\mu_\beta^\varphi(a)|h(a)|}{(1-|\psi(a)|^2)^\gamma} \\
&\quad \times \left|(1-|\phi(a)|^2)^\gamma f_{\phi(a)}(\phi(a)) - (1-|\psi(a)|^2)^\gamma f_{\phi(a)}(\psi(a))\right| \\
&= \left|\mathcal{T}_\gamma^\beta(g\phi)(a) - \mathcal{T}_\gamma^\beta(h\psi)(a)\right| - \left|\mathcal{T}_\gamma^\beta(h\psi)(a)\right| \\
&\quad \times \left|(1-|\phi(a)|^2)^\gamma f_{\phi(a)}(\phi(a)) - (1-|\psi(a)|^2)^\gamma f_{\phi(a)}(\psi(a))\right| \\
&\succeq \left|\mathcal{T}_\gamma^\beta(g\phi)(a) - \mathcal{T}_\gamma^\beta(h\psi)(a)\right| - \left|\mathcal{T}_\gamma^\beta(h\psi)(a)\right| \rho(a). \qquad (2.13)
\end{aligned}$$

The last inequality in (2.13) is due to Lemma 2.2. The above inequalities imply

$$\begin{aligned}
\text{(iii)} \quad & \sup_{a\in\mathbb{D}} \left|\mathcal{T}_\gamma^\beta(g\phi)(a) - \mathcal{T}_\gamma^\beta(h\psi)(a)\right| \\
&\preceq \sup_{a\in\mathbb{D}} \Big( \|(P_\phi^g - P_\psi^h) f_{\phi(a)}\|_{\mathcal{B}_\beta^\varphi} + \left|\mathcal{T}_\gamma^\beta(h\psi)(a)\right| \rho(a) \Big) \\
&\preceq \sup_{a\in\mathbb{D}} \Big( \|(P_\phi^g - P_\psi^h) f_a\|_{\mathcal{B}_\beta^\varphi} + \|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \Big). \qquad (2.14)
\end{aligned}$$

(2.11), (2.12) together with (2.14) imply the statements (2.3)–(2.4). This completes the proof. □

Very interestingly, the next lemma includes the $n$-th power of the induced analytic self-maps on $\mathbb{D}$.

**Lemma 2.4.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$, and $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$, then the following statements hold:*

$$\text{(i)} \quad \sup_{a\in\mathbb{D}} \|(P_\phi^g - P_\psi^h) f_a\|_{\mathcal{B}_\beta^\varphi} \preceq \sup_{n\in\mathbb{N}_0} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi};$$

$$\text{(ii)} \quad \sup_{a\in\mathbb{D}} \|(P_\phi^g - P_\psi^h) \hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \preceq \sup_{n\in\mathbb{N}_0} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.$$

Proof. For $\gamma = (\alpha+2)/p > 0$, we recall that

$$\frac{1}{(1-\bar{a}z)^{2\gamma}} = \sum_{k=0}^\infty \frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!} (\bar{a}z)^k.$$

Hence we can express the Maclaurin expansion of $f_a$ as

$$f_a(z) = \frac{(1-|a|^2)^\gamma}{(1-\bar{a}z)^{2\gamma}} = (1-|a|^2)^\gamma \sum_{k=0}^{\infty} \frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}(\bar{a}z)^k. \tag{2.15}$$

It further yields that

$$\begin{aligned}
\hat{f}_a(z) &= \frac{(1-|a|^2)^\gamma}{(1-\bar{a}z)^{2\gamma}} \cdot \frac{a-z}{1-\bar{a}z} \\
&= (1-|a|^2)^\gamma \left(\sum_{k=0}^{\infty} \frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}\bar{a}^k z^k\right)\left(\frac{a(1-\bar{a}z)+|a|^2 z - z}{1-\bar{a}z}\right) \\
&= (1-|a|^2)^\gamma \left(\sum_{k=0}^{\infty} \frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}\bar{a}^k z^k\right)\left(a-(1-|a|^2)\frac{z}{1-\bar{a}z}\right) \\
&= (1-|a|^2)^\gamma \left(\sum_{k=0}^{\infty} \frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}\bar{a}^k z^k\right)\left(a-(1-|a|^2)\sum_{k=0}^{\infty}\bar{a}^k z^{k+1}\right) \\
&= a(1-|a|^2)^\gamma \sum_{k=0}^{\infty} \frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}\bar{a}^k z^k \\
&\quad - (1-|a|^2)^{\gamma+1}\left(\sum_{k=0}^{\infty} \frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}\bar{a}^k z^k\right)\left(\sum_{k=0}^{\infty}\bar{a}^k z^{k+1}\right) \\
&= a(1-|a|^2)^\gamma \sum_{k=0}^{\infty} \frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}\bar{a}^k z^k - (1-|a|^2)^{\gamma+1}\sum_{k=1}^{\infty}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+2\gamma)}{\Gamma(2\gamma)l!}\right)\bar{a}^{k-1}z^k \\
&= af_a(z) - (1-|a|^2)^{\gamma+1}\sum_{k=1}^{\infty}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+2\gamma)}{\Gamma(2\gamma)l!}\right)\bar{a}^{k-1}z^k. 
\end{aligned} \tag{2.16}$$

On the account of the Maclaurin expansion in (2.15), we verify

$$\begin{aligned}
&\|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} \\
&\quad \le (1-|a|^2)^\gamma \sum_{k=0}^{\infty} \frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}|\bar{a}|^k \|(P_\phi^g - P_\psi^h)z^k\|_{\mathcal{B}_\beta^\varphi} \\
&\quad = (1-|a|^2)^\gamma \sum_{k=0}^{\infty} \frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}|\bar{a}|^k \sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z)|g(z)\phi^k(z) - h(z)\psi^k(z)| \\
&\quad = (1-|a|^2)^\gamma \sum_{k=0}^{\infty} \frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}|\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi}
\end{aligned} \tag{2.17}$$

$$\preceq (1-|a|^2)^{\gamma}\sum_{k=0}^{\infty}\frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}|\bar{a}|^k\left(\frac{2k}{k+1}\right)^{\gamma}\|g\phi^k-h\psi^k\|_{\mu_\beta^\varphi}$$
$$\preceq (1-|a|^2)^{\gamma}\sum_{k=0}^{\infty}\frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}|\bar{a}|^k(k+1)^{-\gamma}\cdot\sup_{n\in\mathbb{N}_0}n^{\gamma}\|g\phi^n-h\psi^n\|_{\mu_\beta^\varphi}. \tag{2.18}$$

By Stirling's formula, it follows that

$$\frac{\Gamma(k+\gamma)}{k!\Gamma(\gamma)}\approx(k+1)^{\gamma-1},\quad \text{as } k\to\infty;$$
$$\frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}(k+1)^{-\gamma}\approx(k+1)^{\gamma-1},\quad \text{as } k\to\infty.$$

Therefore we conclude that

$$\frac{1}{(1-|a|)^{\gamma}}=\sum_{k=0}^{\infty}\frac{\Gamma(k+\gamma)}{k!\Gamma(\gamma)}|a|^k$$
$$\approx\sum_{k=0}^{\infty}(k+1)^{\gamma-1}|a|^k\approx\sum_{k=0}^{\infty}\frac{\Gamma(k+2\gamma)}{\Gamma(2\gamma)k!}|\bar{a}|^k(k+1)^{-\gamma}. \tag{2.19}$$

Putting (2.19) into (2.18), we deduce that

$$\|(P_\phi^g-P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi}\le(1-|a|^2)^{\gamma}\frac{1}{(1-|a|)^{\gamma}}\cdot\sup_{n\in\mathbb{N}_0}n^{\gamma}\|g\phi^n-h\psi^n\|_{\mu_\beta^\varphi}$$
$$\preceq\sup_{n\in\mathbb{N}_0}n^{\gamma}\|g\phi^n-h\psi^n\|_{\mu_\beta^\varphi}. \tag{2.20}$$

Using the Maclaurin expansion in (2.16), it turns out that

$$\|(P_\phi^g-P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi}$$
$$\preceq\|(P_\phi^g-P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi}+(1-|a|^2)^{\gamma+1}\sum_{k=1}^{\infty}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+2\gamma)}{\Gamma(2\gamma)l!}\right)|\bar{a}|^{k-1}\|(P_\phi^g-P_\psi^h)z^k\|_{\mathcal{B}_\beta^\varphi}$$
$$=\|(P_\phi^g-P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi}+(1-|a|^2)^{\gamma+1}\sum_{k=1}^{\infty}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+2\gamma)}{\Gamma(2\gamma)l!}\right)|\bar{a}|^{k-1}$$
$$\times\sup_{z\in\mathbb{D}}\left[\mu_\beta^\varphi(z)|g(z)\phi^k(z)-h(z)\psi^k(z)|\right]$$
$$=\|(P_\phi^g-P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi}+(1-|a|^2)^{\gamma+1}\sum_{k=1}^{\infty}\left(\sum_{l=0}^{k-1}\frac{\Gamma(l+2\gamma)}{\Gamma(2\gamma)l!}\right)|\bar{a}|^{k-1}\cdot\|g\phi^k-h\psi^k\|_{\mu_\beta^\varphi}. \tag{2.21}$$

Furthermore by Stirling's formula again, we obtain

$$\sum_{l=0}^{k-1} \frac{\Gamma(l+2\gamma)}{\Gamma(2\gamma)l!} \approx \sum_{l=0}^{k-1}(l+1)^{2\gamma-1} \approx k^{2\gamma}, \quad \text{as } k \to \infty.$$

The second equivalent display is due to the fact below. For simplicity, we denote $a_k = \sum_{l=0}^{k-1}(l+1)^{2\gamma-1}$, and employ the Binomial theorem to obtain

$$\begin{aligned} k^{2\gamma} - (k-1)^{2\gamma} &= (k-1+1)^{2\gamma} - (k-1)^{2\gamma} \\ &= (k-1)^{2\gamma} + (2\gamma)(k-1)^{2\gamma-1} + \cdots + 1 - (k-1)^{2\gamma} \\ &= (2\gamma)(k-1)^{2\gamma-1} + \cdots + 1, \end{aligned}$$

here we may observe that the sum on the right-hand is not necessarily finite if $\gamma$ is not an integer, which does not essentially affect the following estimates. And then we deduce from the Stole–Cesáro formula that

$$\begin{aligned} \lim_{k\to\infty} \frac{a_k}{k^{2\gamma}} &= \lim_{k\to\infty} \frac{a_k - a_{k-1}}{k^{2\gamma} - (k-1)^{2\gamma}} = \lim_{k\to\infty} \frac{k^{2\gamma-1}}{k^{2\gamma} - (k-1)^{2\gamma}} \\ &= \lim_{k\to\infty} \frac{k^{2\gamma-1}}{(2\gamma)(k-1)^{2\gamma-1} + \cdots + 1} = \frac{1}{2\gamma}. \end{aligned}$$

The above facts entail (2.21) into

$$\begin{aligned} &\|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \\ &\preceq \|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + (1-|a|^2)^{\gamma+1} \sum_{k=1}^{\infty} k^{2\gamma}|\bar{a}|^{k-1}\|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \qquad (2.22) \\ &\le \|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + (1-|a|^2)^{\gamma+1} \sum_{k=1}^{\infty} k^{2\gamma}|\bar{a}|^{k-1}k^{-\gamma} \cdot \sup_{n\in\mathbb{N}_0} n^\gamma\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\ &\preceq \|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + (1-|a|^2)^{\gamma+1} \sum_{k=1}^{\infty} k^{\gamma}|\bar{a}|^{k-1} \cdot \sup_{n\in\mathbb{N}_0} n^\gamma\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\ &\preceq \|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + (1-|a|^2)^{\gamma+1} \frac{1}{(1-|a|)^{\gamma+1}} \cdot \sup_{n\in\mathbb{N}_0} n^\gamma\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\ &\preceq \sup_{n\in\mathbb{N}_0} n^\gamma\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}, \qquad (2.23) \end{aligned}$$

here the second line from the bottom is due to

$$\frac{1}{(1-|a|)^{\gamma+1}} \approx \sum_{k=1}^{\infty} k^\gamma|\bar{a}|^{k-1}$$

by (2.19). After that, we take the supremum about $a \in \mathbb{D}$ in (2.20) and (2.23), which completes the proof. □

Following is one of our main results, which combines the previous lemmas.

**Theorem 2.5.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$ and $\varphi : [0, \infty) \to [0, \infty)$ be an $\mathcal{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$, then the following statements are equivalent:*

(i) $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ *is bounded*;

(ii) $\sup\limits_{z\in\mathbb{D}} |\mathcal{T}_\gamma^\beta(g\phi)(z)|\rho(z) + \sup\limits_{z\in\mathbb{D}} \left|\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)\right| < \infty$,
$\sup\limits_{z\in\mathbb{D}} |\mathcal{T}_\gamma^\beta(h\psi)(z)|\rho(z) + \sup\limits_{z\in\mathbb{D}} \left|\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)\right| < \infty$;

(iii) $\sup\limits_{a\in\mathbb{D}} \|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + \sup\limits_{a\in\mathbb{D}} \|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} < \infty$.

*Moreover, if* (iv) $\sup\limits_{n\in\mathbb{N}_0} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} < \infty$ *holds, then* (i) *is true. Besides, if* (i) *holds, it yields the statement* (v) $\sup\limits_{n\in\mathbb{N}_0} \mathrm{n}^{\gamma-1/\mathrm{p}} \|\mathrm{g}\phi^\mathrm{n} - \mathrm{h}\psi^\mathrm{n}\|_{\mu_\beta^\varphi} < \infty$. *That is, the following relationships hold:* (iv) $\Rightarrow$ (iii) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (i) $\Rightarrow$ (v).

Proof. The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) follow from Lemma 2.4 and Lemma 2.3, respectively. It is easy to check that the implication (i) $\Rightarrow$ (iii) can be deduced from the facts $\sup\limits_{a\in\mathbb{D}} \|f_a\|_{A_\alpha^p} \preceq 1$, $\sup\limits_{a\in\mathbb{D}} \|\hat{f}_a\|_{A_\alpha^p} \preceq 1$, and the boundedness of $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$. To be specific, it turns out that

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + \sup_{a\in\mathbb{D}} \|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \\
&\quad \preceq \|(P_\phi^g - P_\psi^h)\|_{A_\alpha^p \to \mathcal{B}_\beta^\varphi} \sup_{a\in\mathbb{D}} (\|f_a\|_{A_\alpha^p} + \|\hat{f}_a\|_{A_\alpha^p}) < +\infty.
\end{aligned}$$

In what follows, we will prove (i) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (v). Suppose that $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ is bounded. As we all know, the monomial function $z^n \in A_\alpha^p$ and $\|z^n\|_{A_\alpha^p} \approx n^{-(\alpha+1)/p} = n^{-\gamma+1/p}$, as $n \to \infty$, from (1.3), we conclude that

$$\begin{aligned}
\infty > \|P_\phi^g - P_\psi^h\|_{A_\alpha^p \to \mathcal{B}_\beta^\varphi} &\succeq \left\|(P_\phi^g - P_\psi^h)\frac{z^n}{\|z^n\|_{A_\alpha^p}}\right\|_{\mathcal{B}_\beta^\varphi} \\
&\succeq n^{\gamma-1/p} \sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z)|g(z)\phi^n(z) - h(z)\psi^n(z)| = n^{\gamma-1/p}\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

The above formulas imply

$$\sup_{n\in\mathbb{N}_0} n^{\gamma-1/p}\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \preceq \|P_\phi^g - P_\psi^h\|_{A_\alpha^p \to \mathcal{B}_\beta^\varphi} < \infty,$$

which shows the statement (i) $\Rightarrow$ (v).

(ii) $\Rightarrow$ (i). For any $f \in A^p_\alpha$, we employ Lemma 2.2 to show that

$$\begin{aligned}
&\|(P^g_\phi - P^h_\psi)f\|_{\mathcal{B}^\varphi_\beta} \\
&\quad = \sup_{z\in\mathbb{D}} \mu^\varphi_\beta(z)|g(z)f(\phi(z)) - h(z)f(\psi(z))| \\
&\quad \preceq \sup_{z\in\mathbb{D}} \left|\mathcal{T}^\beta_\gamma(g\phi)(z)\right| \left|(1-|\phi(z)|^2)^\gamma f(\phi(z)) - (1-|\psi(z)|^2)^\gamma f(\psi(z))\right| \\
&\qquad + \sup_{z\in\mathbb{D}} (1-|\psi(z)|^2)^\gamma |f(\psi(z))| \left|\mathcal{T}^\beta_\gamma(g\phi)(z) - \mathcal{T}^\beta_\gamma(h\psi)(z)\right| \\
&\quad \preceq \sup_{z\in\mathbb{D}} |\mathcal{T}^\beta_\gamma(g\phi)(z)|\rho(z) + \sup_{z\in\mathbb{D}} \left|\mathcal{T}^\beta_\gamma(g\phi)(z) - \mathcal{T}^\beta_\gamma(h\psi)(z)\right| < \infty. \qquad (2.24)
\end{aligned}$$

Analogously to (2.24), we can also obtain that

$$\begin{aligned}
&\|(P^g_\phi - P^h_\psi)f\|_{\mathcal{B}^\varphi_\beta} \\
&\quad \preceq \sup_{z\in\mathbb{D}} |\mathcal{T}^\beta_\gamma(h\psi)(z)|\rho(z) + \sup_{z\in\mathbb{D}} \left|\mathcal{T}^\beta_\gamma(g\phi)(z) - \mathcal{T}^\beta_\gamma(h\psi)(z)\right| < \infty. \qquad (2.25)
\end{aligned}$$

The two inequalities (2.24) and (2.25) imply that each one of conditions (ii) can ensure the boundedness of $P^g_\phi - P^h_\psi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$. This finishes the proof. □

**2.2. The essential norm of $P^g_\phi - P^h_\psi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$.** In this section, we deduce several estimations for the essential norm of $P^g_\phi - P^h_\psi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$. Firstly, we collect some parallel results from Lemma 2.3 as follows.

**Lemma 2.6.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$, and $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$, then the following inequalities hold:*

$$\begin{aligned}
\text{(i)}\quad & \lim_{r\to 1} \sup_{|\phi(z)|>r} \left|\mathcal{T}^\beta_\gamma(g\phi)(z)\right| \rho(z) \\
&\quad \preceq \limsup_{|a|\to 1} \|(P^g_\phi - P^h_\psi)f_a\|_{\mathcal{B}^\varphi_\beta} + \limsup_{|a|\to 1} \|(P^g_\phi - P^h_\psi)\hat{f}_a\|_{\mathcal{B}^\varphi_\beta}; \\
\text{(ii)}\quad & \lim_{r\to 1} \sup_{|\psi(z)|>r} \left|\mathcal{T}^\beta_\gamma(h\psi)(z)\right| \rho(z) \\
&\quad \preceq \limsup_{|a|\to 1} \|(P^g_\phi - P^h_\psi)f_a\|_{\mathcal{B}^\varphi_\beta} + \limsup_{|a|\to 1} \|(P^g_\phi - P^h_\psi)\hat{f}_a\|_{\mathcal{B}^\varphi_\beta}; \\
\text{(iii)}\quad & \lim_{r\to 1} \sup_{\min\{|\phi(z)|,|\psi(z)|\}>r} \left|\mathcal{T}^\beta_\gamma(g\phi)(z) - \mathcal{T}^\beta_\gamma(h\psi)(z)\right| \\
&\quad \preceq \limsup_{|a|\to 1} \|(P^g_\phi - P^h_\psi)f_a\|_{\mathcal{B}^\varphi_\beta} + \limsup_{|a|\to 1} \|(P^g_\phi - P^h_\psi)\hat{f}_a\|_{\mathcal{B}^\varphi_\beta}.
\end{aligned}$$

Proof. These results can be deduced from the inequalities (2.9), (2.10) and (2.13) in Lemma 2.3. □

**Lemma 2.7.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$, and $\varphi : [0, \infty) \to [0, \infty)$ be an $\mathcal{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$ such that the operator $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ is bounded, then the following statements hold:*

$$\text{(i)}\quad \limsup_{|a|\to 1} \|(P_\phi^g - P_\psi^h) f_a\|_{\mathcal{B}_\beta^\varphi} \preceq \limsup_{n\to\infty} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}; \tag{2.26}$$

$$\text{(ii)}\quad \limsup_{|a|\to 1} \|(P_\phi^g - P_\psi^h) \hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \preceq \limsup_{n\to\infty} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}. \tag{2.27}$$

Proof. For any $a \in \mathbb{D}$ and each positive integer $N$, employing (2.17) we obtain

$$\begin{aligned}
\|(P_\phi^g - P_\psi^h) f_a\|_{\mathcal{B}_\beta^\varphi} &\le (1 - |a|^2)^\gamma \sum_{k=0}^{\infty} \frac{\Gamma(k + 2\gamma)}{\Gamma(2\gamma) k!} |\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\le (1 - |a|^2)^\gamma \sum_{k=0}^{N} \frac{\Gamma(k + 2\gamma)}{\Gamma(2\gamma) k!} |\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\quad + (1 - |a|^2)^\gamma \sum_{k=N+1}^{\infty} \frac{\Gamma(k + 2\gamma)}{\Gamma(2\gamma) k!} |\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi}.
\end{aligned} \tag{2.28}$$

We denote

$$\begin{aligned}
J_1 &:= (1 - |a|^2)^\gamma \sum_{k=0}^{N} \frac{\Gamma(k + 2\gamma)}{\Gamma(2\gamma) k!} |\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi}, \\
J_2 &:= (1 - |a|^2)^\gamma \sum_{k=N+1}^{\infty} \frac{\Gamma(k + 2\gamma)}{\Gamma(2\gamma) k!} |\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi}.
\end{aligned}$$

On the one hand, for $k \in \{0, \cdots, N\}$, we can choose $z^k \in A_\alpha^p$. Using the boundedness of $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$, it turns out that $\|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} < \infty$ for $k = 0, 1, \ldots, N$. Hence

$$\limsup_{|a|\to 1} J_1 = 0. \tag{2.29}$$

On the other hand, it follows from (2.19) that

$$\begin{aligned}
J_2 &= (1 - |a|^2)^\gamma \sum_{k=N+1}^{\infty} \frac{\Gamma(k + 2\gamma)}{\Gamma(2\gamma) k!} |\bar{a}|^k \|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\preceq (1 - |a|^2)^\gamma \sum_{k=N+1}^{\infty} \frac{\Gamma(k + 2\gamma)}{\Gamma(2\gamma) k!} |\bar{a}|^k (k+1)^{-\gamma} \cdot \sup_{n \ge N+1} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\
&\preceq \frac{(1 - |a|^2)^\gamma}{(1 - |a|)^\gamma} \cdot \sup_{n \ge N+1} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \preceq \sup_{n \ge N+1} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

Furthermore, letting $|a| \to 1$ in the above inequality, it leads to

$$\limsup_{|a|\to 1} J_2 \preceq \sup_{n\geq N+1} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}. \tag{2.30}$$

Putting (2.29) and (2.30) into (2.28) and letting $n \to \infty$, we arrive at (2.26). Similarly, by (2.22), we conclude that

$$\begin{aligned}
&\|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \\
&\preceq \|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + (1-|a|^2)^{\gamma+1}\sum_{k=1}^{N} k^{2\gamma}|\bar{a}|^{k-1}\|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\quad + (1-|a|^2)^{\gamma+1}\sum_{k=N+1}^{\infty} k^{2\gamma}|\bar{a}|^{k-1}\|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\leq \|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + (1-|a|^2)^{\gamma+1}\sum_{k=1}^{N} k^{2\gamma}|\bar{a}|^{k-1}\|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
&\quad + (1-|a|^2)^{\gamma+1}\sum_{k=N+1}^{\infty} k^{2\gamma}|\bar{a}|^{k-1}k^{-\gamma}\cdot \sup_{n\geq N+1} n^\gamma\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

Taking (2.19) into consideration, we deduce that

$$\begin{aligned}
\|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \preceq \|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + (1-|a|^2)^{\gamma+1}\sum_{k=1}^{N} k^{2\gamma}|\bar{a}|^{k-1}\|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} \\
+ \sup_{n\geq N+1} n^\gamma\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

In view of the condition $\|g\phi^k - h\psi^k\|_{\mu_\beta^\varphi} < \infty$ for $k = 0, 1, \ldots, N$, and letting $|a| \to 1$ in the above formulas, we get that

$$\limsup_{|a|\to 1}\|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \preceq \limsup_{|a|\to 1}\|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + \sup_{n\geq N+1} n^\gamma\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.$$

We firstly let $n \to \infty$ in the above inequality and then combine with (2.26), which can verify (2.27). This ends the proof. $\square$

The proof of the lemma below can be shown by an argument similar to that of [1, Proposition 3.11], consequently, we omit the details.

**Lemma 2.8.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$, and $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$. Then $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ is compact if and only if $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $A_\alpha^p$ with $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, and then $\|(P_\phi^g - P_\psi^h)f_k\|_{\mathcal{B}_\beta^\varphi} \to 0$ as $k \to \infty$.*

Suppose the operators $P_\phi^g : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ and $P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ are bounded, then

$$\sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z)|g(z)| < \infty \quad \text{and} \quad \sup_{z\in\mathbb{D}} \mu_\beta^\varphi(z)|h(z)| < \infty.$$

It is trivial that if $\|\phi\|_\infty < 1$ (or $\|\psi\|_\infty < 1$), then $P_\phi^g : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ (or $P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$) is a compact operator. Indeed, for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in $A_\alpha^p$ converging to zero on the compact subset of $\mathbb{D}$, we arrive at $\lim\limits_{k\to\infty} \|P_\phi^g f_k\|_{\mathcal{B}_\beta^\varphi} = \lim\limits_{k\to\infty} \mu_\beta^\varphi(z)|g(z)||f_k(\phi(z))| = 0$. Thus Lemma 2.8 tells the operator $P_\phi^g : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ is compact. A similar case holds for the operator $P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$. That is, the difference operator $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ is compact under the case $\|\phi\|_\infty < 1$ and $\|\psi\|_\infty < 1$. Hence we are interested in the case when $\max\{\|\phi\|_\infty,\ \|\psi\|_\infty\} = 1$. The following is our main theorem in this section.

**Theorem 2.9.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$ and $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function. Suppose $g, h \in H(\mathbb{D})$ and $\phi, \psi \in S(\mathbb{D})$ satisfying $\max\{\|\phi\|_\infty,\ \|\psi\|_\infty\} = 1$. If the operators $P_\phi^g,\ P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ are bounded, then the following displays hold:*

$$\begin{aligned}
&\limsup_{n\to\infty} n^{\gamma-1/p}\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\
&\quad \preceq \|P_\phi^g - P_\psi^h\|_{e, A_\alpha^p \to \mathcal{B}_\beta^\varphi} \approx \lim_{r\to 1} \sup_{|\phi(z)|>r} \left|\mathcal{T}_\gamma^\beta(g\phi)(z)\right| \rho(z) \\
&\qquad + \lim_{r\to 1} \sup_{|\psi(z)|>r} \left|\mathcal{T}_\gamma^\beta(h\psi)(z)\right| \rho(z) \\
&\qquad + \lim_{r\to 1} \sup_{\min\{|\phi(z)|,|\psi(z)|\}>r} \left|\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)\right| \\
&\quad \approx \limsup_{|a|\to 1} \|(P_\phi^g - P_\psi^h) f_a\|_{\mathcal{B}_\beta^\varphi} + \limsup_{|a|\to 1} \|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \\
&\quad \preceq \limsup_{n\to\infty} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

Proof. Firstly, the boundedness of $P_\phi^g : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ and $P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ imply that $M_g = \sup\limits_{z\in\mathbb{D}} \mu_\beta^\varphi(z)|g(z)| < \infty$ and $M_h = \sup\limits_{z\in\mathbb{D}} \mu_\beta^\varphi(z)|h(z)| < \infty$. Lemma 2.6 together with Lemma 2.7 verify that

$$\begin{aligned}
&\lim_{r\to 1} \sup_{|\phi(z)|>r} \left|\mathcal{T}_\gamma^\beta(g\phi)(z)\right| \rho(z) + \lim_{r\to 1} \sup_{|\psi(z)|>r} \left|\mathcal{T}_\gamma^\beta(h\psi)(z)\right| \rho(z) \\
&\qquad + \lim_{r\to 1} \sup_{\min\{|\phi(z)|,|\psi(z)|\}>r} \left|\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)\right| \\
&\quad \preceq \limsup_{|a|\to 1} \|(P_\phi^g - P_\psi^h) f_a\|_{\mathcal{B}_\beta^\varphi} + \limsup_{|a|\to 1} \|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} \\
&\quad \preceq \limsup_{n\to\infty} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi}.
\end{aligned}$$

In conclusion, we need to show the following inequalities:

$$\begin{aligned}
&\|P_\phi^g - P_\psi^h\|_{e,A_\alpha^p\to\mathcal{B}_\beta^\varphi} \\
&\quad \succeq \limsup_{|a|\to1} \|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + \limsup_{|a|\to1} \|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi}; \qquad (2.31)
\end{aligned}$$

$$\begin{aligned}
&\limsup_{n\to\infty} n^{\gamma-1/p}\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} \\
&\quad \preceq \|P_\phi^g - P_\psi^h\|_{e,A_\alpha^p\to\mathcal{B}_\beta^\varphi} \preceq \lim_{r\to1} \sup_{|\phi(z)|>r} \left|\mathcal{T}_\gamma^\beta(g\phi)(z)\right| \rho(z) \\
&\qquad + \lim_{r\to1} \sup_{|\psi(z)|>r} \left|\mathcal{T}_\gamma^\beta(h\psi)(z)\right| \rho(z) \\
&\qquad + \lim_{r\to1} \sup_{\min\{|\phi(z)|,|\psi(z)|\}>r} \left|\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)\right|. \qquad (2.32)
\end{aligned}$$

On the one hand, $\{f_a\}_{a\in\mathbb{D}}$ and $\{\hat{f}_a\}_{a\in\mathbb{D}}$ are bounded sequences in $A_\alpha^p$ and converge to 0 uniformly on compact subsets of $\mathbb{D}$ as $|a|\to1$, and it yields $\lim\limits_{|a|\to1}\|Kf_a\|_{\mathcal{B}_\beta^\varphi} = \lim\limits_{|a|\to1}\|K\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} = 0$ for any compact operator $K : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ by Lemma 2.8. Therefore, we deduce that

$$\begin{aligned}
\|P_\phi^g - P_\psi^h\|_{e,A_\alpha^p\to\mathcal{B}_\beta^\varphi} &\succeq \limsup_{|a|\to1} \inf_K \|(P_\phi^g - P_\psi^h - K)f_a\|_{\mathcal{B}_\beta^\varphi} \\
&\geq \limsup_{|a|\to1} \inf_K \Big(\|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} - \|Kf_a\|_{\mathcal{B}_\beta^\varphi}\Big) \\
&\geq \limsup_{|a|\to1} \|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi}, \qquad (2.33)
\end{aligned}$$

which also holds for the function sequence $\{\hat{f}_a\}$. Hence (2.31) is true.

On the other hand, the first inequality in (2.32) is due to the bounded sequence $\{f_n(z) = z^n/\|z^n\|_{A_\alpha^p}\}$ in $A_\alpha^p$ converging to 0 uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. Hence replacing $f_a$ by $f_n$ in (2.33), we conclude that

$$\begin{aligned}
\|P_\phi^g - P_\psi^h\|_{e,A_\alpha^p\to\mathcal{B}_\beta^\varphi} &\succeq \limsup_{n\to\infty} \|(P_\phi^g - P_\psi^h)f_n\|_{\mathcal{B}_\beta^\varphi} \\
&= \limsup_{n\to\infty} \sup_{z\in\mathbb{D}} \frac{\mu_\beta^\varphi(z)|g(z)\phi^n(z) - h(z)\psi^n(z)|}{\|z^n\|_{A_\alpha^p}} \\
&\succeq \limsup_{n\to\infty} n^{\gamma-1/p}\|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi},
\end{aligned}$$

the last line follows from $\|z^n\|_{A_\alpha^p} \approx n^{-\gamma+1/p}$ as $n\to\infty$. Next, we prove the second inequality in (2.32). Consider the operators on $H(\mathbb{D})$ defined by $P_k(f)(z) =$

$f(\frac{k}{k+1}z)$, $k\in\mathbb{N}$. Denote $\Phi_k(z)=kz/(k+1)$. Since $\Phi_k\in S(\mathbb{D})$ with $\|\Phi_k\|_\infty<1$, the composition operator $C_{\Phi_k}(\equiv P_k)$ can be proved compact on $A^p_\alpha$ by the analogous ideas in Lemma 2.8. Furthermore, these operators are continuous on compact open topology and $P_k(f)\to f$ on compact subsets of $\mathbb{D}$ as $k\to\infty$. Since the integral means $M_p(f,r)=\Big(\int_{\partial\mathbb{D}}|f(r\zeta)|^p d\sigma(\zeta)\Big)^p$ are nondecreasing in $r$, and by the polar coordinates formula (1.2), it follows that

$$\begin{aligned}\|P_k(f)\|^p_{A^p_\alpha}&=\int_{\mathbb{D}}|P_kf(z)|^p dA_\alpha(z)=\int_{\mathbb{D}}(\alpha+1)\left|f\left(\frac{k}{k+1}z\right)\right|^p(1-|z|^2)^\alpha dA(z)\\&=2(\alpha+1)\int_0^1 r(1-r^2)^\alpha dr\int_{\partial\mathbb{D}}\left|f\left(r\frac{k}{k+1}\zeta\right)\right|^p d\sigma(\zeta)\\&\le 2(\alpha+1)\int_0^1 r(1-r^2)^\alpha dr\int_{\partial\mathbb{D}}|f(r\zeta)|^p d\sigma(\zeta)=\|f\|^p_{A^p_\alpha},\end{aligned}$$

which is equivalent to saying that $\|P_k(f)\|_{A^p_\alpha}\le\|f\|_{A^p_\alpha}$ for all $k\in\mathbb{N}$. Consequently,

$$\sup_{k\in\mathbb{N}}\|P_k\|_{A^p_\alpha\to A^p_\alpha}\le 1.$$

Given a function $f\in A^p_\alpha$ with $\|f\|_{A^p_\alpha}\le 1$, we set the notation $G_k:=(I-P_k)f, k\in\mathbb{N}$, and it is trivial to show that $G_k\in A^p_\alpha$ for all $k\in\mathbb{N}$ with $\sup\limits_{k\in\mathbb{N}}\|G_k\|_{A^p_\alpha}\le 2$. Since the operator $P^g_\phi-P^h_\psi:A^p_\alpha\to\mathcal{B}^\varphi_\beta$ is bounded, and $P_k:A^p_\alpha\to A^p_\alpha$ is compact, the operator $(P^g_\phi-P^h_\psi)P_k:A^p_\alpha\to\mathcal{B}^\varphi_\beta$ is also compact. As a consequence, by the definition of essential norm, we arrive at

$$\begin{aligned}&\|P^g_\phi-P^h_\psi\|_{e,A^p_\alpha\to\mathcal{B}^\varphi_\beta}\\&\le\limsup_{k\to\infty}\|(P^g_\phi-P^h_\psi)-(P^g_\phi-P^h_\psi)P_k\|_{A^p_\alpha\to\mathcal{B}^\varphi_\beta}=\limsup_{k\to\infty}\|(P^g_\phi-P^h_\psi)(I-P_k)\|_{A^p_\alpha\to\mathcal{B}^\varphi_\beta}\\&=\limsup_{k\to\infty}\sup_{\|f\|_{A^p_\alpha}\le 1}\|(P^g_\phi-P^h_\psi)(I-P_k)f\|_{\mathcal{B}^\varphi_\beta}=\limsup_{k\to\infty}\sup_{\|f\|_{A^p_\alpha}\le 1}\|(P^g_\phi-P^h_\psi)G_k\|_{\mathcal{B}^\varphi_\beta}\\&=\limsup_{k\to\infty}\sup_{\|f\|_{A^p_\alpha}\le 1}\sup_{z\in\mathbb{D}}\mu^\varphi_\beta(z)\left|g(z)G_k(\phi(z))-h(z)G_k(\psi(z))\right|.\end{aligned}\tag{2.34}$$

For an arbitrary $r\in(0,1)$, we denote

$$\begin{aligned}&\mathbb{D}_1=\{z\in\mathbb{D}:|\phi(z)|\le r,|\psi(z)|\le r\},\quad\mathbb{D}_2=\{z\in\mathbb{D}:|\phi(z)|\le r,|\psi(z)|>r\},\\&\mathbb{D}_3=\{z\in\mathbb{D}:|\phi(z)|>r,|\psi(z)|\le r\},\quad\mathbb{D}_4=\{z\in\mathbb{D}:|\phi(z)|>r,|\psi(z)|>r\};\\&I_i:=\sup_{z\in\mathbb{D}_i}\mu^\varphi_\beta(z)\left|g(z)G_k(\phi(z))-h(z)G_k(\psi(z))\right|,\quad\text{for } i=1,2,3,4.\end{aligned}$$

Employing the estimate $|f(z)| \preceq \|f\|_{A^p_\alpha}/(1-|z|)^\gamma$ (Lemma 2.1) and the fact $(I-P_k)f$ converges to zero uniformly on compact subsets of $(H(\mathbb{D}), co)$, it turns out that

$$\lim_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} \sup_{|\zeta|\le r} |(I-P_k)(f)(\zeta)| = \lim_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} \sup_{|\zeta|\le r} |G_k(\zeta)| = 0. \tag{2.35}$$

By (2.35), it yields that

$$\begin{aligned}\limsup_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} I_1 &\le \limsup_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} \sup_{|\phi(z)|\le r} (\mu^\varphi_\beta(z)|g(z)|)|G_k(\phi(z))| \\ &\quad + \limsup_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} \sup_{|\psi(z)|\le r} (\mu^\varphi_\beta(z)|h(z)|)|G_k(\psi(z))| \\ &\le M_g \limsup_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} \sup_{|\phi(z)|\le r} |G_k(\phi(z))| \\ &\quad + M_h \limsup_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} \sup_{|\psi(z)|\le r} |G_k(\psi(z))| = 0. \end{aligned}\tag{2.36}$$

On the other hand, we formulate that

$$\begin{aligned}&\mu^\varphi_\beta(z)\,|g(z)G_k(\phi(z)) - h(z)G_k(\psi(z))| \\ &\preceq \frac{\mu^\varphi_\beta(z)|g(z)|}{(1-|\phi(z)|^2)^\gamma}|(1-|\phi(z)|^2)^\gamma G_k(\phi(z)) - (1-|\psi(z)|^2)^\gamma G_k(\psi(z))| \\ &\quad + (1-|\psi(z)|^2)^\gamma\,|G_k(\psi(z))| \left|\frac{\mu^\varphi_\beta(z)g(z)}{(1-|\phi(z)|^2)^\gamma} - \frac{\mu^\varphi_\beta(z)h(z)}{(1-|\psi(z)|^2)^\gamma}\right| \\ &\preceq \left|\mathcal{T}^\beta_\gamma(g\phi)(z)\right|\rho(z) + (1-|\psi(z)|^2)^\gamma|G_k(\psi(z))|\left|\mathcal{T}^\beta_\gamma(g\phi)(z) - \mathcal{T}^\beta_\gamma(h\psi)(z)\right|. \end{aligned}\tag{2.37}$$

Analogously, we can transform the above formula into

$$\begin{aligned}&\mu^\varphi_\beta(z)\,|g(z)G_k(\phi(z)) - h(z)G_k(\psi(z))| \\ &\preceq |\mathcal{T}^\beta_\gamma(h\psi)(z)|\rho(z) + (1-|\phi(z)|^2)^\gamma\,|G_k(\phi(z))|\left|\mathcal{T}^\beta_\gamma(g\phi)(z) - \mathcal{T}^\beta_\gamma(h\psi)(z)\right|. \end{aligned}\tag{2.38}$$

Since the operators $P^g_\phi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$ and $P^h_\psi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$ are bounded, hence $P^g_\phi - P^h_\psi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$ is bounded. It turns out that $\left|\mathcal{T}^\beta_\gamma(g\phi)(z) - \mathcal{T}^\beta_\gamma(h\psi)(z)\right| < \infty$ by Theorem 2.5. Employing (2.35), we can show that

$$\begin{aligned}&\limsup_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} I_2 \\ &\le \limsup_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} \sup_{|\psi(z)|> r} |\mathcal{T}^\beta_\gamma(h\psi)(z)|\rho(z) \\ &\quad + \limsup_{k\to\infty} \sup_{\|f\|_{A^p_\alpha}\le 1} \sup_{|\phi(z)|\le r} (1-|\phi(z)|^2)^\gamma\,|G_k(\phi(z))|\left|\mathcal{T}^\beta_\gamma(g\phi)(z) - \mathcal{T}^\beta_\gamma(h\psi)(z)\right| \end{aligned}$$

$$
\begin{aligned}
&\preceq \sup_{|\psi(z)|>r} |\mathcal{T}_\gamma^\beta(h\psi)(z)|\rho(z) + \limsup_{k\to\infty} \sup_{\|f\|_{A_\alpha^p}\le 1} \sup_{|\phi(z)|\le r} |G_k(\phi(z))| \\
&= \sup_{|\psi(z)|>r} |\mathcal{T}_\gamma^\beta(h\psi)(z)|\rho(z). \qquad (2.39)
\end{aligned}
$$

Similarly, utilizing (2.35) and (2.37), we can prove that

$$\limsup_{k\to\infty} \sup_{\|f\|_{A_\alpha^p}\le 1} I_3 \preceq \sup_{|\phi(z)|>r} |\mathcal{T}_\gamma^\beta(g\phi)(z)|\rho(z). \qquad (2.40)$$

Finally, we deduce from (2.37) that

$$
\begin{aligned}
&\limsup_{k\to\infty} \sup_{\|f\|_{A_\alpha^p}\le 1} I_4 \\
&\preceq \sup_{|\phi(z)|>r} |\mathcal{T}_\gamma^\beta(g\phi)(z)|\rho(z) + \|G_k\|_{A_\alpha^p} \sup_{\min\{|\phi(z)|,|\psi(z)|\}>r} \left|\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)\right| \\
&\preceq \sup_{|\phi(z)|>r} |\mathcal{T}_\gamma^\beta(g\phi)(z)|\rho(z) + \sup_{\min\{|\phi(z)|,|\psi(z)|\}>r} \left|\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)\right|. \qquad (2.41)
\end{aligned}
$$

Similarly, (2.38) entails that

$$
\begin{aligned}
\limsup_{n\to\infty} \sup_{\|f\|_{A_\alpha^p}\le 1} I_4 \preceq& \sup_{|\psi(z)|>r} |\mathcal{T}_\gamma^\beta(h\psi)(z)|\rho(z) \\
&+ \sup_{\min\{|\phi(z)|,|\psi(z)|\}>r} \left|\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)\right|. \qquad (2.42)
\end{aligned}
$$

Consequently, we put (2.36), (2.39), (2.40) and (2.41), (2.42) into (2.34), and let $r \to 1$ on both sides to derive that

$$
\begin{aligned}
\|P_\phi^g - P_\psi^h\|_{e, A_\alpha^p \to \mathcal{B}_\beta^\varphi} \preceq& \lim_{r\to 1} \sup_{|\phi(z)|>r} \left|\mathcal{T}_\gamma^\beta(g\phi)(z)\right| \rho(z) + \lim_{r\to 1} \sup_{|\psi(z)|>r} \left|\mathcal{T}_\gamma^\beta(h\psi)(z)\right| \rho(z) \\
&+ \lim_{r\to 1} \sup_{\min\{|\phi(z)|,|\psi(z)|\}>r} \left|\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)\right|.
\end{aligned}
$$

This ends all the proof for essential norm estimation. □

Subsequently, Theorem 2.9 indicates several equivalent characterizations for the compactness of $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$.

**Theorem 2.10.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$ and $\varphi : [0, \infty) \to [0, \infty)$ be an $\mathcal{N}$-function. Suppose $g, h \in H(\mathbb{D})$ and $\phi, \psi \in S(\mathbb{D})$ satisfying $\max\{\|\phi\|_\infty, \ \|\psi\|_\infty\} = 1$. If the operators $P_\phi^g$, $P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ are bounded,* *then $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ is compact if and only if one of the following statements hold:*

$$\text{(i)}\quad \lim_{r\to 1}\sup_{|\phi(z)|>r}\left|\mathcal{T}_\gamma^\beta(g\phi)(z)\right|\rho(z) + \lim_{r\to 1}\sup_{|\psi(z)|>r}\left|\mathcal{T}_\gamma^\beta(h\psi)(z)\right|\rho(z) + \lim_{r\to 1}\sup_{\min\{|\phi(z)|,|\psi(z)|\}>r}\left|\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)\right| = 0;$$

$$\text{(ii)}\quad \limsup_{|a|\to 1}\|(P_\phi^g - P_\psi^h)f_a\|_{\mathcal{B}_\beta^\varphi} + \limsup_{|a|\to 1}\|(P_\phi^g - P_\psi^h)\hat{f}_a\|_{\mathcal{B}_\beta^\varphi} = 0.$$

*Moreover, if the condition $\limsup_{n\to\infty} n^\gamma \|g\phi^n - h\psi^n\|_{\mu_\beta^\varphi} = 0$ holds, then $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ is also compact.*

## 3. The properties of $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_{\beta,0}^\varphi$

**3.1. The boundedness of $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_{\beta,0}^\varphi$.** In this section, we will use the following three conditions:

$$\lim_{|z|\to 1}\mu_\beta^\varphi(z)|g(z) - h(z)| = 0; \tag{3.1}$$

$$\lim_{|z|\to 1}\mu_\beta^\varphi(z)|\phi(z) - \psi(z)||g(z)| = 0; \tag{3.2}$$

$$\lim_{|z|\to 1}\mu_\beta^\varphi(z)|\phi(z) - \psi(z)||h(z)| = 0. \tag{3.3}$$

**Theorem 3.1.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$ and $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$, then the following statements are equivalent:*

(i) *$P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_{\beta,0}^\varphi$ is bounded;*

(ii) *$P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ is bounded,* (3.1) *and* (3.2) *hold;*

(iii) *$P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ is bounded,* (3.1) *and* (3.3) *hold.*

Proof. (i) $\Rightarrow$ (ii). Suppose that $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_{\beta,0}^\varphi$ is bounded. It is obvious that $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_\beta^\varphi$ is bounded. Taking the function $f(z) = 1 \in A_\alpha^p$, we get

$$\begin{aligned}\lim_{|z|\to 1}\mu_\beta^\varphi(z)|((P_\varphi^g - P_\psi^h)f)'(z)| &= \lim_{|z|\to 1}\mu_\beta^\varphi(z)|f(\varphi(z))g(z) - f(\psi(z))h(z)|\\ &= \lim_{|z|\to 1}\mu_\beta^\varphi(z)|g(z) - h(z)| = 0,\end{aligned}$$

which yields (3.1). Similarly, taking $f(z)=z\in A^p_\alpha$, we verify that

$$\begin{aligned}\lim_{|z|\to1}\mu^\varphi_\beta(z)|((P^g_\phi-P^h_\psi)f)'(z)| &= \lim_{|z|\to1}\mu^\varphi_\beta(z)|f(\phi(z))g(z)-f(\psi(z))h(z)|\\ &= \lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi(z)g(z)-\psi(z)h(z)|=0. \qquad (3.4)\end{aligned}$$

The displays (3.1) together with (3.4) imply that

$$\begin{aligned}&\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi(z)-\psi(z)||g(z)|\\ &\quad=\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi(z)g(z)-\psi(z)h(z)+\psi(z)h(z)-\psi(z)g(z)|\\ &\quad\le\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi(z)g(z)-\psi(z)h(z)|+\lim_{|z|\to1}\mu^\varphi_\beta(z)|\psi(z)||h(z)-g(z)|=0.\end{aligned}$$

Thus equation (3.2) holds.

(ii) $\Rightarrow$ (iii). We need to show (3.3) holds. By (3.1) and (3.2), it yields that

$$\begin{aligned}&\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi(z)-\psi(z)||h(z)|\\ &\quad=\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi(z)h(z)-\phi(z)g(z)+\phi(z)g(z)-\psi(z)g(z)+\psi(z)g(z)-\psi(z)h(z)|\\ &\quad\le\lim_{|z|\to1}\mu^\varphi_\beta(z)\left(|\phi(z)||g(z)-h(z)|+|\phi(z)-\psi(z)||g(z)|+|\psi(z)||g(z)-h(z)|\right)\\ &\quad\le\lim_{|z|\to1}2\mu^\varphi_\beta(z)|g(z)-h(z)|+\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi(z)-\psi(z)||g(z)|=0.\end{aligned}$$

The above inequalities imply (3.3), and then the statement (iii) holds.

(iii) $\Rightarrow$ (i). Suppose that $P^g_\varphi-P^h_\psi: A^p_\alpha\to\mathcal{B}^\varphi_\beta$ is bounded, (3.1) and (3.3) are true. Then we choose a monomial $f_m=z^m\in A^p_\alpha$ for $m\in\mathbb{N}$, we conclude that

$$\begin{aligned}&\lim_{|z|\to1}\mu^\varphi_\beta(z)|((P^g_\phi-P^h_\psi)f_m)'(z)|\\ &=\lim_{|z|\to1}\mu^\varphi_\beta(z)|f_m(\phi(z))g(z)-f_m(\psi(z))h(z)|=\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi^m(z)g(z)-\psi^m(z)h(z)|\\ &\le\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi^m(z)-\psi^m(z)||h(z)|+\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi(z)|^m|g(z)-h(z)|\\ &\le\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi(z)-\psi(z)||h(z)||\phi^{m-1}(z)+\phi^{m-2}(z)\psi(z)+\cdots+\psi^{m-1}(z)|\\ &\quad+\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi(z)|^m|g(z)-h(z)|\\ &\le m\lim_{|z|\to1}\mu^\varphi_\beta(z)|\phi(z)-\psi(z)||h(z)|+\lim_{|z|\to1}\mu^\varphi_\beta(z)|g(z)-h(z)|=0,\end{aligned}$$

where the last equation follows from the formulas in (iii). That is equivalent to saying that $P^g_\phi-P^h_\psi$ maps all monomials $z^m$ into $\mathcal{B}^\varphi_{\beta,0}$ for $m\in\mathbb{N}$. Thus

$(P_{\varphi}^g - P_{\psi}^h)p \in \mathcal{B}_{\beta,0}^{\varphi}$ for any polynomial $p$. Since the set of all polynomials is dense in $A_{\alpha}^p$, for any $f \in A_{\alpha}^p$, there is a sequence of polynomials $\{q_k\}_{k\in\mathbb{N}}$ satisfying $\|f - q_k\|_{A_{\alpha}^p} \to 0$ as $k \to \infty$. Thus by the boundedness of $P_{\phi}^g - P_{\psi}^h : A_{\alpha}^p \to \mathcal{B}_{\beta}^{\varphi}$, it turns out that

$$\|(P_{\phi}^g - P_{\psi}^h)(f - q_k)\|_{\mathcal{B}_{\beta,0}^{\varphi}} \le \|P_{\phi}^g - P_{\psi}^h\|_{A_{\alpha}^p \to \mathcal{B}_{\beta,0}^{\varphi}} \|f - q_k\|_{A_{\alpha}^p} \to 0, \quad k \to 0.$$

Due to the boundedness of $P_{\phi}^g - P_{\psi}^h : A_{\alpha}^p \to \mathcal{B}_{\beta}^{\varphi}$ and that $\mathcal{B}_{\beta,0}^{\varphi}$ is a closed subset of $\mathcal{B}_{\beta}^{\varphi}$, we validate that $(P_{\phi}^g - P_{\psi}^h)(A_{\alpha}^p) \subset \mathcal{B}_{\beta,0}^{\varphi}$. That is, $P_{\phi}^g - P_{\psi}^h : A_{\alpha}^p \to \mathcal{B}_{\beta,0}^{\varphi}$ is bounded. This completes the proof. $\square$

The following lemma can be proved analogously to [13, Lemma 1], so we omit the details.

**Lemma 3.2.** *A closed set $\mathcal{K}$ in $\mathcal{B}_{\beta,0}^{\varphi}$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1} \sup_{f\in\mathcal{K}} \mu_{\beta}^{\varphi}(z)|f'(z)| = 0.$$

**Theorem 3.3.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$ and $\varphi : [0,\infty) \to [0,\infty)$ be an $\mathcal{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$, with $\max\{\|\phi\|_{\infty}, \|\psi\|_{\infty}\} = 1$ and $g, h \in H(\mathbb{D})$ such that $P_{\phi}^g - P_{\psi}^h : A_{\alpha}^p \to \mathcal{B}_{\beta,0}^{\varphi}$ is bounded, then the following statements are equivalent:*

(i) $P_{\phi}^g - P_{\psi}^h : A_{\alpha}^p \to \mathcal{B}_{\beta,0}^{\varphi}$ *is compact*;

(ii)
$$\limsup_{|z|\to 1} \mathcal{T}_{\gamma}^{\beta}(g\phi)(z)\rho(z) + \limsup_{|z|\to 1} \mathcal{T}_{\gamma}^{\beta}(h\psi)(z)\rho(z) + \limsup_{|z|\to 1} \left|\mathcal{T}_{\gamma}^{\beta}(g\phi)(z) - \mathcal{T}_{\gamma}^{\beta}(h\psi)(z)\right| = 0. \tag{3.5}$$

Proof. (i) $\Rightarrow$ (ii). Suppose that $P_{\phi}^g - P_{\psi}^h : A_{\alpha}^p \to \mathcal{B}_{\beta,0}^{\varphi}$ is compact. Choose a sequence $\{z_k\}_{k\in\mathbb{N}}$ such that $|z_k| \to 1$ as $k \to \infty$. Define the function

$$\hat{f}_k(z) = \frac{(1 - |\phi(z_k)|^2)^{\gamma}}{(1 - \overline{\phi(z_k)}z)^{2\gamma}} \cdot \frac{\psi(z_k) - z}{1 - \overline{\psi(z_k)}z}.$$

It is clear that $\hat{f}_k \in A_{\alpha}^p$, then $(P_{\phi}^g - P_{\psi}^h)\hat{f}_k \in \mathcal{B}_{\beta,0}^{\varphi}$. That means that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \mu_{\beta}^{\varphi}(z_k)|((P_{\phi}^g - P_{\psi}^h)\hat{f}_k)'(z_k)| \\ &= \lim_{k\to\infty} \mu_{\beta}^{\varphi}(z_k)|\hat{f}_k(\phi(z_k))g(z_k) - \hat{f}_k(\psi(z_k))h(z_k)| = \lim_{k\to\infty} \frac{\mu_{\beta}^{\varphi}(z_k)|g(z_k)|}{(1 - |\phi(z_k)|^2)^{\gamma}}\rho(z_k), \end{aligned}$$

which implies

$$\limsup_{|z|\to 1} \mathcal{T}_\gamma^\beta(g\phi)(z)\rho(z) = \limsup_{|z|\to 1} \frac{\mu_\beta^\varphi(z)|g(z)|}{(1-|\phi(z)|^2)^\gamma}\rho(z) = 0. \tag{3.6}$$

Analogously, we can deduce that

$$\limsup_{|z|\to 1} \mathcal{T}_\gamma^\beta(h\psi)(z)\rho(z) = \limsup_{|z|\to 1} \frac{\mu_\beta^\varphi(z)|h(z)|}{(1-|\psi(z)|^2)^\gamma}\rho(z) = 0. \tag{3.7}$$

On the other hand, we define

$$f_k(z) = \frac{(1-|\phi(z_k)|^2)^\gamma}{(1-\overline{\phi(z_k)}z)^{2\gamma}},$$

then $f_k \in A_\alpha^p$ and $(P_\phi^g - P_\psi^h)f_k \in \mathcal{B}_{\beta,0}^\varphi$. Moreover, $f_k(\phi(z_k)) = 1/(1-|\phi(z_k)|^2)^\gamma$. Hence we formulate that

$$\begin{aligned}
&\mu_\beta^\varphi(z_k)|((P_\phi^g - P_\psi^h)f_k)'(z_k)| \\
&= \mu_\beta^\varphi(z_k)|f_k(\phi(z_k))g(z_k) - f_k(\psi(z_k))h(z_k)| \\
&= \mu_\beta^\varphi(z_k)\left|\frac{g(z_k)}{(1-|\phi(z_k)|^2)^\gamma} - \frac{h(z_k)}{(1-|\psi(z_k)|^2)^\gamma} + \frac{h(z_k)}{(1-|\psi(z_k)|^2)^\gamma} - f_k(\psi(z_k))h(z_k)\right| \\
&\geq \mu_\beta^\varphi(z_k)\left|\frac{g(z_k)}{(1-|\phi(z_k)|^2)^\gamma} - \frac{h(z_k)}{(1-|\psi(z_k)|^2)^\gamma}\right| \\
&\quad - \frac{\mu_\beta^\varphi(z_k)|h(z_k)|}{(1-|\psi(z_k)|^2)^\gamma}\left|(1-|\phi(z_k)|^2)^\gamma f_k(\phi(z_k)) - (1-|\psi(z_k)|^2)^\gamma f_k(\psi(z_k))\right| \\
&\succeq \mu_\beta^\varphi(z_k)\left|\frac{g(z_k)}{(1-|\phi(z_k)|^2)^\gamma} - \frac{h(z_k)}{(1-|\psi(z_k)|^2)^\gamma}\right| - \frac{\mu_\beta^\varphi(z_k)|h(z_k)|}{(1-|\psi(z_k)|^2)^\gamma}\rho(z_k),
\end{aligned}$$

which verifies

$$\begin{aligned}
&\mu_\beta^\varphi(z_k)\left|\frac{g(z_k)}{(1-|\phi(z_k)|^2)^\gamma} - \frac{h(z_k)}{(1-|\psi(z_k)|^2)^\gamma}\right| \\
&\quad \preceq \frac{\mu_\beta^\varphi(z_k)|h(z_k)|}{(1-|\psi(z_k)|^2)^\gamma}\rho(z_k) + \mu_\beta^\varphi(z_k)|((P_\phi^g - P_\psi^h)f_k)'(z_k)|.
\end{aligned}$$

The above displays together with (3.7) justify that

$$\begin{aligned}
&\limsup_{|z|\to 1} \left|\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)\right| \\
&\quad = \limsup_{|z|\to 1} \mu_\beta^\varphi(z)\left|\frac{g(z)}{(1-|\phi(z)|^2)^\gamma} - \frac{h(z)}{(1-|\psi(z)|^2)^\gamma}\right| = 0.
\end{aligned} \tag{3.8}$$

The formulas (3.6)–(3.8) entail the statement (ii).

(ii) $\Rightarrow$ (i). We recall that the operator $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_{\beta,0}^\varphi$ is compact if and only if it maps any bounded subset $B \subset A_\alpha^p$ into a relatively compact subset in $\mathcal{B}_{\beta,0}^\varphi$. Since $P_\phi^g - P_\psi^h : A_\alpha^p \to \mathcal{B}_{\beta,0}^\varphi$ is bounded, hence the norm closure of the set $(P_\phi^g - P_\psi^h)(B)$ is also bounded and closed in $\mathcal{B}_{\beta,0}^\varphi$. Furthermore, by Lemma 3.2, we only need to show $\lim\limits_{|z|\to 1} \sup\limits_{f\in B} \mu_\beta^\varphi(z)|[(P_\phi^g - P_\psi^h)f]'(z)| = 0$. The above display implies

$$\lim_{|z|\to 1} \sup_{f\in B} \mu_\beta^\varphi(z) \left| \left[ (P_\phi^g - P_\psi^h) \frac{f}{\|f\|_{A_\alpha^p}} \right]'(z) \right| = 0,$$

for $f \neq 0$. It is clear that for $f = 0$ the previous equation holds. Therefore, it is enough to show that

$$\lim_{z\to 1} \sup\{\mu_\beta^\varphi(z)|((P_\phi^g - P_\psi^h)f)'(z)| : f \in A_\alpha^p,\ \|f\|_{A_\alpha^p} \le 1\} = 0.$$

Now, given $f \in A_\alpha^p$ with $\|f\|_{A_\alpha^p} \le 1$, analogously to (2.24), we have that

$$\begin{aligned}
&\mu_\beta^\varphi(z)|((P_\phi^g - P_\psi^h)f)'(z)| \\
&\quad = \mu_\beta^\varphi(z)|f(\phi(z))g(z) - f(\psi(z))h(z)| \\
&\quad \preceq |\mathcal{T}_\gamma^\beta(g\phi)(z)|\rho(z) + |\mathcal{T}_\gamma^\beta(g\phi)(z) - \mathcal{T}_\gamma^\beta(h\psi)(z)| \to 0,\ |z| \to 1.
\end{aligned}$$

That means the operator $P_\phi^g – P_\psi^h : A_\alpha^p \to \mathcal{B}_{\beta,0}^\varphi$ is compact. The proof is finished. $\square$

## 4. Some corollaries

In this section, we employed the above theorems to provide some characterizations for the difference $T_g C_\phi - T_h C_\psi$ acting from weighted Bergman spaces to $\beta$-Bloch–Orlicz spaces.

Replacing $g, h \in H(\mathbb{D})$ by $g', h' \in H(\mathbb{D})$ in $P_\phi^g - P_\phi^h$, it turns out that $P_\phi^{g'} - P_\psi^{h'} = T_g C_\phi - T_h C_\psi$, which is defined as

$$(T_g C_\phi - T_h C_\psi)f(z) = \int_0^z f(\phi(t))g'(t)dt - \int_0^z f(\psi(t))h'(t)dt, \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

In addition, we denote the two notations below for the next two corollaries,

$$\mathcal{T}_\gamma^\beta(g'\phi)(z) = \frac{\mu_\beta^\varphi(z)g'(z)}{(1 - |\phi(z)|^2)^\gamma}, \qquad \mathcal{T}_\gamma^\beta(h'\psi)(z) = \frac{\mu_\beta^\varphi(z)h'(z)}{(1 - |\psi(z)|^2)^\gamma}.$$

**Corollary 4.1.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$ and $\varphi : [0, \infty) \to [0, \infty)$ be an $\mathcal{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$, then the following statements are equivalent:*

(i) $T_g C_\phi - T_h C_\psi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$ *is bounded;*

(ii) $$\sup_{z\in\mathbb{D}} |\mathcal{T}^\beta_\gamma(g'\phi)(z)|\rho(z) + \sup_{z\in\mathbb{D}} \left|\mathcal{T}^\beta_\gamma(g'\phi)(z) - \mathcal{T}^\beta_\gamma(h'\psi)(z)\right| < \infty,$$
$$\sup_{z\in\mathbb{D}} |\mathcal{T}^\beta_\gamma(h'\psi)(z)|\rho(z) + \sup_{z\in\mathbb{D}} \left|\mathcal{T}^\beta_\gamma(g'\phi)(z) - \mathcal{T}^\beta_\gamma(h'\psi)(z)\right| < \infty;$$

(iii) $$\sup_{a\in\mathbb{D}} \|(T_g C_\phi - T_h C_\psi) f_a\|_{\mathcal{B}^\varphi_\beta} + \sup_{a\in\mathbb{D}} \|(T_g C_\phi - T_h C_\psi) \hat{f}_a\|_{\mathcal{B}^\varphi_\beta} < \infty.$$

*Morever, if* $\sup_{n\in\mathbb{N}_0} n^\gamma \|g'\phi^n - h'\psi^n\|_{\mu^\varphi_\beta} < \infty$ *holds, then* (i) *is true.*

**Corollary 4.2.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$ and $\varphi : [0, \infty) \to [0, \infty)$ be an $\mathcal{N}$-function. Suppose $g, h \in H(\mathbb{D})$ and $\phi, \psi \in S(\mathbb{D})$ satisfying $\max\{\|\phi\|_\infty,\ \|\psi\|_\infty\} = 1$. If the operators $T_g C_\phi,\ T_h C_\psi : A^p_\alpha \to \mathcal{B}^\varphi_\beta$ are bounded, then the following relationships hold:*

$$\begin{aligned}
&\|T_g C_\phi - T_h C_\psi\|_{e, A^p_\alpha \to \mathcal{B}^\varphi_\beta} \\
&\approx \lim_{r\to 1} \sup_{|\phi(z)|>r} \left|\mathcal{T}^\beta_\gamma(g'\phi)(z)\right| \rho(z) + \lim_{r\to 1} \sup_{|\psi(z)|>r} \left|\mathcal{T}^\beta_\gamma(h'\psi)(z)\right| \rho(z) \\
&\quad + \lim_{r\to 1} \sup_{\min\{|\phi(z)|, |\psi(z)|\}>r} \left|\mathcal{T}^\beta_\gamma(g'\phi)(z) - \mathcal{T}^\beta_\gamma(h'\psi)(z)\right| \\
&\approx \limsup_{|a|\to 1} \|(T_g C_\phi - T_h C_\psi) f_a\|_{\mathcal{B}^\varphi_\beta} + \limsup_{|a|\to 1} \|(T_g C_\phi - T_h C_\psi) \hat{f}_a\|_{\mathcal{B}^\varphi_\beta} \\
&\preceq \limsup_{n\to\infty} n^\gamma \|g'\phi^n - h'\psi^n\|_{\mu^\varphi_\beta}.
\end{aligned}$$

In what follows, we will use the below conditions:

$$\lim_{|z|\to 1} \mu^\varphi_\beta(z)|g'(z) - h'(z)| = 0; \tag{4.1}$$

$$\lim_{|z|\to 1} \mu^\varphi_\beta(z)|\phi(z) - \psi(z)||g'(z)| = 0; \tag{4.2}$$

$$\lim_{|z|\to 1} \mu^\varphi_\beta(z)|\phi(z) - \psi(z)||h'(z)| = 0. \tag{4.3}$$

**Corollary 4.3.** *Let $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$ and $\varphi : [0, \infty) \to [0, \infty)$ be an $\mathcal{N}$-function. Suppose $\phi, \psi \in S(\mathbb{D})$ and $g, h \in H(\mathbb{D})$, then the following statements are equivalent:*

(i) $T_gC_\phi - T_hC_\psi : A^p_\alpha \to \mathcal{B}^\varphi_{\beta,0}$ *is bounded;*

(ii) $T_gC_\phi - T_hC_\psi : A^p_\alpha \to \mathcal{B}^\varphi_{\beta}$ *is bounded,* (4.1) *and* (4.2) *hold;*

(iii) $T_gC_\phi - T_hC_\psi : A^p_\alpha \to \mathcal{B}^\varphi_{\beta}$ *is bounded,* (4.1) *and* (4.3) *hold.*

**Corollary 4.4.** *Let* $1 < p < \infty$, $\alpha > -1$, $0 < \beta < \infty$ *and* $\varphi : [0,\infty) \to [0,\infty)$ *be an* $\mathcal{N}$*-function. Suppose* $\phi, \psi \in S(\mathbb{D})$ *with* $\max\{\|\phi\|_\infty,\ \|\psi\|_\infty\} = 1$ *and* $g, h \in H(\mathbb{D})$ *such that* $T_gC_\phi - T_hC_\psi : A^p_\alpha \to \mathcal{B}^\varphi_{\beta,0}$ *is bounded, then the following statements are equivalent:*

(i) $T_gC_\phi - T_hC_\psi : A^p_\alpha \to \mathcal{B}^\varphi_{\beta,0}$ *is compact*;

(ii) $$\limsup_{|z|\to 1} \mathcal{T}^\beta_\gamma(g'\phi)(z)\rho(z) + \limsup_{|z|\to 1} \mathcal{T}^\beta_\gamma(h'\psi)(z)\rho(z) + \limsup_{|z|\to 1} \left|\mathcal{T}^\beta_\gamma(g'\phi)(z) - \mathcal{T}^\beta_\gamma(h'\psi)(z)\right| = 0.$$

Using the above theorems, we can also easily deduce similar descriptions for the differences $T^g - T^h$ and $T_g - T_h$ acting from weighted Bergman spaces to $\beta$-Bloch–Orlicz spaces, hence we omitted them.

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