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Solution Sets of Quadratic Complementarity Problems

Jie Wang · Shenglong Hu · Zheng-Hai Huang

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Abstract In this paper, we study quadratic complementarity problems (QCP-s), which form a subclass of nonlinear complementarity problems (NCPs) with the nonlinear functions being quadratic polynomial mappings. QCPs serve as an important bridge linking linear complementarity problems and NCPs. Various properties on the solution set for a QCP, including existence, compactness and uniqueness are studied. The results are established from assumptions giv-

J. Wang
School of Mathematics, Tianjin University, Tianjin, 300350 China
ajiewang@tju.edu.cn

S. Hu, Corresponding author
School of Mathematics, Tianjin University, Tianjin, 300350 China
Tel: +86-22-27403615, Fax: +82-22-27403615
timhu@tju.edu.cn

Z.-H. Huang
School of Mathematics, Tianjin University, Tianjin, 300350 China
huangzhenghai@tju.edu.cn

en in terms of the comprising matrices of the underlying tensor, henceforth easily checkable. Examples are given to demonstrate that the results improve or generalize the corresponding QCP counterparts of the well-known NCP theory, and broaden the boundary knowledge of NCPs as well.

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1 Introduction

The classical *linear complementarity problems* (LCPs) (cf. [1]) have wide range of applications in applied science and technology, such as operations research, economics, engineering, to just name a few. The *nonlinear complementarity problems* (NCPs) are generalizations of LCPs by considering general nonlinear mappings (cf. [2]). Likewise, the so dubbed *quadratic complementarity problems* (QCPs) that will be studied in this article are generalizations of LCPs by considering quadratic polynomial mappings on one hand, and on the other hand more concrete realizations of NCPs.

One motivation for studying QCPs is the three person non-cooperative games [3, 4]; another is a generalized Markowitz portfolio problem whose first order optimality condition is a QCP [5]. The generalized Markowitz portfolio problem is in general NP-hard, and therefore properties on its KKT points are helpful and would be guidelines to design efficient algorithms to solve it.

Though QCPs form a subclass of NCPs [2], they deserve particular investigations with at least twofold reasons: (i) they would serve as a bridge between the LCPs and the general NCPs, a very first step towards the nonlinear cases, and of which a concrete case; and (ii) they give a unified model for several classes of optimization problems (e.g., the cubic polynomial minimizations over the nonnegative orthant, the generalized Markowitz portfolio problems, three person non-cooperative games, etc.) which should have their own specifically developed theory and numerical methodologies. Actually, our research will show that the study on QCPs can even broaden the boundary of the knowledge for NCPs (cf. Sections 3.2 and 4).

This study also comes from the recent trend on *tensor complementarity problems* (TCPs). QCPs encompass the third order TCPs in which the nonlinear mappings are the sum of quadratic forms and constant vectors [6]. In this field, Song and Qi [6] showed the existence of solutions for TCPs under (strict) semi-positivity; and presented some relations among several classes of tensors. (cf. [7]). Che, Qi and Wei [8] studied properties of TCPs with positive definite tensors. Song and Yu further studied S-tensors and properties of the solution sets of the corresponding TCPs [9]. Bai, Huang and Wang [10] showed that solution sets of TCPs with P tensors are nonempty and compact. Huang and Qi [3] reformulated a class of multilinear games as TCPs, providing examples for TCPs and establishing a bridge between these two classes of problems. For more research in this field and related, please refer to [6–8,11,12] and references therein.

This article will give a study on solution sets of QCPs using various tools from classical NCPs, tensor analysis, as well as some particularly designed techniques. We will organize the rest of this article as follows.

Basic notation and concepts will be presented in Section 2. A generalized Frank-Wolfe theorem for cubic polynomial optimization problems will be given in Section 3.1, which involves R_0 tensors. With this, existence of solutions to QCPs is given under mild assumptions. The compactness of the solution sets will be discussed in Section 3.2. C-strict copositivity of a tensor and K-positive semidefinite plus of a matrix will be introduced there, based on which a compactness result will be given. The result generalizes, actually broadens, the well known ones in the literature (cf. [2, Proposition 2.2.12]); and it is proven under a regularity condition (i.e., (15)), which is formulated geometrically. This regularity condition combines information on all of the tensor, the matrix and the vector. Examples will be presented to show these promised novelties.

Uniqueness of the solution set will be investigated in Section 4 in terms of the null spaces of a collection of matrices. The uniqueness theorem involves the above regularity condition. The result generalizes and broadens the literature– it can handle a tensor which is not strictly copositive. An example is given there. Final remarks is given in the last section to conclude this article.

2 Preliminaries

A (real) third order n-dimensional tensor (a.k.a. *hypermatrix* [13]) $\mathcal{A} = [a_{i_1 i_2 i_3}] \in \mathbb{R}^{n\times n\times n}$ is a third-way array, where $i_j \in \{1, \ldots, n\}$ and $j = 1, 2, 3$. The set of all third order n-dimensional tensors is denoted by $\mathrm{T}(\mathbb{R}^n,3)$, and the set of all $n\times n$ (symmetric) matrices is denoted by $(\mathrm{S}(\mathbb{R}^n,2))$ $\mathrm{T}(\mathbb{R}^n,2)$. A tensor $\mathcal{A}\in\mathrm{T}(\mathbb{R}^n,3)$ can be viewed as a concatenation of n matrices of size $n\times n$. More precisely, for $i\in\{1,\dots,n\}$, the i-th slice $\mathcal{A}_{i,\cdot,\cdot}$ of $\mathcal{A}$ refers to the matrix $[a_{ijk}]_{j,k=1}^n$.

Given vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}\in\mathbb{R}^n$, $\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}$ refers to a rank one tensor whose (i,j,k)-th component is $x_iy_jz_k$. $\mathbf{x}^{\otimes 3}$ simplifies the symmetric rank one tensor $\mathbf{x}\otimes\mathbf{x}\otimes\mathbf{x}$. $\mathbf{x}^{\otimes 2}$ is defined similarly.

Let $\mathcal{A}\in\mathrm{T}(\mathbb{R}^n,3)$ and $\mathbf{x}\in\mathbb{R}^n$, $\mathcal{A}\mathbf{x}^2$ is a vector with its ith component as

$$(\mathcal{A}\mathbf{x}^2)_i := \sum_{i_2,i_3=1}^n a_{ii_2i_3}x_{i_2}x_{i_3},\ \text{for } i\in\{1,\dots,n\}.$$

A tensor $\mathcal{A}\in\mathrm{T}(\mathbb{R}^n,3)$ is *copositive*, if $\mathcal{A}\mathbf{x}^3:=\mathbf{x}^\mathsf{T}(\mathcal{A}\mathbf{x}^2)\geq 0$ for all $\mathbf{x}\in\mathbb{R}^n_+$. It is called *strictly copositive*, if $\mathcal{A}\mathbf{x}^3>0$ for all $\mathbf{x}\in\mathbb{R}^n_+\setminus\{\mathbf{0}\}$.

The QCP refers to finding a vector $\mathbf{x}\in\mathbb{R}^n$ such that

$$\text{(QCP)}\quad \mathbf{x}\geq\mathbf{0},\ \mathcal{A}\mathbf{x}^2+B\mathbf{x}+\mathbf{c}\geq\mathbf{0} \text{ and } \mathbf{x}^\mathsf{T}(\mathcal{A}\mathbf{x}^2+B\mathbf{x}+\mathbf{c})=0, \tag{1}$$

in which $\mathcal{A}\in\mathrm{T}(\mathbb{R}^n,3)$ a given third order tensor, $B\in\mathbb{R}^{n\times n}$ a given matrix, and $\mathbf{c}\in\mathbb{R}^n$ a given vector; $\mathbf{x}\geq\mathbf{0}$ means $x_i\geq 0$ for all $i\in\{1,\dots,n\}$.

For QCP (1), sometimes it is without loss of any generality to consider tensors in the subspace $\mathbb{R}^n\otimes\mathrm{S}(\mathbb{R}^n,2)$ of $\mathrm{T}(\mathbb{R}^n,3)$. It is the set of tensors which have symmetric elements with respect to the second and the third indices, i.e., $\mathcal{A}\in\mathbb{R}^n\otimes\mathrm{S}(\mathbb{R}^n,2)$ means $\mathcal{A}_{i,\cdot,\cdot}\in\mathrm{S}(\mathbb{R}^n,2)$ for all $i\in\{1,\dots,n\}$. Denote by $A_i:=\mathcal{A}_{i,\cdot,\cdot}$ for $i=1,\dots,n$. Then, with $F(\mathbf{x}):=\mathcal{A}\mathbf{x}^2+B\mathbf{x}+\mathbf{c}$, we have

$$\nabla(\mathbf{x}^\mathsf{T}F(\mathbf{x}))=(\mathbf{x}^\mathsf{T}A_1\mathbf{x},\dots,\mathbf{x}^\mathsf{T}A_n\mathbf{x})^\mathsf{T}+2\sum_{i=1}^n x_iA_i\mathbf{x}+(B+B^\mathsf{T})\mathbf{x}+\mathbf{c}. \tag{2}$$

Given a tensor $\mathcal{A} \in \mathrm{T}(\mathbb{R}^n, 3)$, we define its *transpose* $\mathcal{A}^\mathsf{T}$ as the tensor in $\mathrm{T}(\mathbb{R}^n, 3)$ with entries

$$(\mathcal{A}^\mathsf{T})_{ijk} = a_{jik} \text{ for all } i, j, k \in \{1, \ldots, n\}.$$

Denote by $D_i := \left(\mathcal{A}^\mathsf{T}\right)_{i,\cdot,\cdot} \in \mathbb{R}^{n \times n}$ for $i = 1, \ldots, n$, and define $\mathcal{A}^\mathsf{T}\mathbf{x} := \sum_{i=1}^n x_i D_i$ for all $\mathbf{x} \in \mathbb{R}^n$.

3 Nonemptiness and Compactness

In the following, we will denote the solution set of QCP (1) by $\textsc{sol}(\mathcal{A}, B, \mathbf{c})$.

3.1 Nonemptiness via a Frank-Wolfe Type Theorem

A tensor $\mathcal{A} \in \mathrm{T}(\mathbb{R}^n, 3)$ is called an R_0 *tensor* if the system (cf. [7])

$$\mathbf{x}^\mathsf{T} A_i \mathbf{x} \geq 0, \text{ if } x_i = 0, \text{ and } \mathbf{x}^\mathsf{T} A_i \mathbf{x} = 0, \text{ if } x_i > 0 \text{ for all } i \in \{1, \ldots, n\}$$

does not have a solution in $\mathbb{R}^n_+ \setminus \{\mathbf{0}\}$.

Proposition 3.1 *Let $\mathcal{A} \in \mathrm{T}(\mathbb{R}^n, 3)$ be an R_0 tensor. Then, whenever QCP (1) is feasible, the minimization problem*

$$\inf \mathbf{x}^\mathsf{T} F(\mathbf{x}) \text{ s.t. } \mathbf{x} \geq \mathbf{0},\ F(\mathbf{x}) \geq \mathbf{0} \tag{3}$$

has an optimal solution.

Proof Let the feasible solution set be $S := \{\mathbf{x} \mid \mathbf{x} \geq \mathbf{0},\ F(\mathbf{x}) \geq \mathbf{0}\}$. By assumption, $S \neq \emptyset$. Denote by $v_* = \inf\{\mathbf{x}^\mathsf{T} F(\mathbf{x}) \mid \mathbf{x} \in S\}$. Let $B_\rho := \{\mathbf{x} \in \mathbb{R}^n \mid$

$\|\mathbf{x}\| \leq \rho\}$ be the ball centered at zero with radius ρ. Thus, there exists $\kappa > 0$ such that $S \cap B_\rho \neq \emptyset$ for all $\rho \geq \kappa$. Let

$$v_\rho := \min\{\mathbf{x}^\mathsf{T} F(\mathbf{x}) \mid \mathbf{x} \in S \cap B_\rho\} \text{ for all } \rho \geq \kappa. \tag{4}$$

The optimal solution set s_ρ of (4) is obvious compact. Thus, we define

$$\mathbf{x}_\rho \in \operatorname{argmin}\{\|\mathbf{x}\| \mid \mathbf{x} \in s_\rho\} \text{ for all } \rho \geq \kappa$$

as a minimum norm optimal solution. We claim that there exists $\gamma > \kappa$ such that

$$\|\mathbf{x}_\rho\| < \rho \text{ for all } \rho \geq \gamma. \tag{5}$$

Suppose on the contrary that there exists a sequence $\{\rho_k\}$ such that

$$\|\mathbf{x}_{\rho_k}\| = \rho_k \text{ with } \rho_k \to \infty.$$

Taking subsequence if necessary, we may assume without loss of generality that $\frac{\mathbf{x}_{\rho_k}}{\rho_k} \to \overline{\mathbf{x}}$. Obviously, the feasibility and the normalization imply that

$$\overline{\mathbf{x}} \geq \mathbf{0} \text{ and } \overline{\mathbf{x}} \neq \mathbf{0}. \tag{6}$$

Dividing each defining polynomial of F by ρ_k^2, we get with the feasibility that

$$\overline{\mathbf{x}}^\mathsf{T} A_i \overline{\mathbf{x}} \geq 0 \text{ for all } i = 1, \ldots, n. \tag{7}$$

By the nonemptiness of S, we conclude that $v_\rho \downarrow v_*$. Thus,

$$v_{\rho_k} = \mathbf{x}_{\rho_k}^\mathsf{T} F(\mathbf{x}_{\rho_k}) = \sum_{i=1}^n (\mathbf{x}_{\rho_k})_i \mathbf{x}_{\rho_k}^\mathsf{T} A_i \mathbf{x}_{\rho_k} + \mathbf{x}_{\rho_k}^\mathsf{T} B \mathbf{x}_{\rho_k} + \mathbf{c}^\mathsf{T} \mathbf{x}_{\rho_k}$$

should be bounded. Dividing the equation by ρ_k^3, we get that $\mathcal{A}\overline{\mathbf{x}}^3 = 0$. This, together with (6) and (7), violates the hypothesis that $\mathcal{A}$ is an R_0 tensor. Therefore, the claim (5) is proved.

In the next, we claim that there exists $\tau > \gamma$ such that

$$v_\rho = v_* \text{ for all } \rho \geq \tau. \tag{8}$$

Suppose on the contrary that $v_\rho > v_*$ for all $\rho \geq \kappa$. As $v_\rho \downarrow v_*$, we can find $\gamma < \rho_1 < \rho_2$ such that $v_{\rho_1} > v_{\rho_2}$. By the claim (5), we have that $\|\mathbf{x}_{\rho_2}\| < \rho_2$. Since $v_{\rho_1} > v_{\rho_2}$, we have $\|\mathbf{x}_{\rho_2}\| > \rho_1$. Let $\rho_3 = \|\mathbf{x}_{\rho_2}\|$. Then, $\gamma < \rho_1 < \rho_3 < \rho_2$, and $\|\mathbf{x}_{\rho_3}\| < \rho_3 = \|\mathbf{x}_{\rho_2}\|$. $\rho_3 < \rho_2$ implies that $v_{\rho_2} \leq v_{\rho_3}$.

If $v_{\rho_2} = v_{\rho_3}$, then $\mathbf{x}_{\rho_3}$ is an optimal solution to v_{ρ_2} with a strictly smaller norm than $\mathbf{x}_{\rho_2}$, which is a contradiction to the definition. If $v_{\rho_2} < v_{\rho_3}$, then it, together with $\rho_3 = \|\mathbf{x}_{\rho_2}\|$, gives a contradiction to $\mathbf{x}_{\rho_3}$ being an optimal solution. As a consequence, the claim (8) is proved. Thus, the proposition follows immediately. □

Proposition 3.1 is a generalization of the classical Frank-Wolfe theorem [14].

Definition 3.1 (Co-Semidefinite Pair) A tensor $\mathcal{A} \in \mathrm{T}(\mathbb{R}^n, 3)$ and a matrix $B \in \mathrm{T}(\mathbb{R}^n, 2)$ is called a *co-semidefinite pair* if the matrix pencil $\mathcal{A}^\mathsf{T}\mathbf{x} + B$ is positive semidefinite for all $\mathbf{x} \in \mathbb{R}^n_+$.

Obviously, $\mathcal{A}$ and B form a co-semidefinite pair if and only if B, D_1, $\dots$, D_n are all positive semidefinite matrices. Actually, the sufficiency is immediate. For the necessity, the matrix B should be positive semidefinite is apparent. For the positive semidefiniteness of the matrices D_i's, we can drive a proof by contradiction. Suppose D_1 is not positive semidefinite, then with sufficiently large t, the matrix $\mathcal{A}^\mathsf{T}(t\mathbf{e}_1) + B$ would not be positive semidefinite.

Proposition 3.2 *Let* $\mathbf{x}_*$ *be an optimal solution of* (3) *with* $\mathcal{A} \in \mathbb{R}^n \otimes \mathrm{S}(\mathbb{R}^n, 2)$. *If a constraint qualification is satisfied at* $\mathbf{x}_*$ *to ensure at which the Karush-Kuhn-Tucker condition holds, and* $\mathcal{A}$ *and* B *form a co-semidefinite pair, then* $\mathbf{x}_* \in \mathrm{SOL}(\mathcal{A}, B, \mathbf{c})$.

Proof It follows from the Karush-Kuhn-Tucker condition hypothesis that there exists $\mathbf{u}_* \geq \mathbf{0}$ such that (cf. (2) for gradients)

$$\begin{bmatrix} \mathbf{x}_*^\mathsf{T} A_1 \mathbf{x}_* \\ \vdots \\ \mathbf{x}_*^\mathsf{T} A_n \mathbf{x}_* \end{bmatrix} + 2\sum_{i=1}^n (x_*)_i A_i \mathbf{x}_* + (B+B^\mathsf{T})\mathbf{x}_* + \mathbf{c} - 2\sum_{i=1}^n (u_*)_i A_i \mathbf{x}_* - B^\mathsf{T}\mathbf{u}_* \geq \mathbf{0} \quad (9)$$

$$\mathbf{x}_*^\mathsf{T}\left(\begin{bmatrix} \mathbf{x}_*^\mathsf{T} A_1 \mathbf{x}_* \\ \vdots \\ \mathbf{x}_*^\mathsf{T} A_n \mathbf{x}_* \end{bmatrix} + 2\sum_{i=1}^n (x_*)_i A_i \mathbf{x}_* + (B+B^\mathsf{T})\mathbf{x}_* \right.$$
$$\left. + \mathbf{c} - 2\sum_{i=1}^n (u_*)_i A_i \mathbf{x}_* - B^\mathsf{T}\mathbf{u}_* \right) = 0 \quad (10)$$

$$\mathbf{u}_*^\mathsf{T} F(\mathbf{x}_*) = 0,\ \mathbf{x}_* \geq \mathbf{0},\ \mathbf{u}_* \geq \mathbf{0}.\ F(\mathbf{x}_*) \geq \mathbf{0}. \quad (11)$$

Thus, multiplying (9) by $\mathbf{u}_*$ and using the complementarity of $\mathbf{u}_*$ (cf. (11)), we have

$$\mathbf{u}_*^\mathsf{T}\left(-2\sum_{i=1}^n (x_*)_i A_i \mathbf{x}_* - B^\mathsf{T}\mathbf{x}_* + 2\sum_{i=1}^n (u_*)_i A_i \mathbf{x}_* + B^\mathsf{T}\mathbf{u}_* \right) \leq 0,$$

and, using the feasibility of $\mathbf{x}_*$, we have from (10) that

$$\mathbf{x}_*^\mathsf{T}\left(2\sum_{i=1}^n (x_*)_i A_i \mathbf{x}_* + B^\mathsf{T}\mathbf{x}_* - 2\sum_{i=1}^n (u_*)_i A_i \mathbf{x}_* - B^\mathsf{T}\mathbf{u}_* \right) \leq 0 \quad (12)$$

Therefore,

$$2\langle \mathcal{A}, \mathbf{u}_* \otimes \mathbf{u}_* \otimes \mathbf{x}_* - \mathbf{x}_* \otimes \mathbf{u}_* \otimes \mathbf{x}_* + \mathbf{x}_*^{\otimes 3} - \mathbf{u}_* \otimes \mathbf{x}_* \otimes \mathbf{x}_* \rangle + \langle B, (\mathbf{u}_* - \mathbf{x}_*)^{\otimes 2} \rangle \leq 0,$$

which is equivalent to

$$2\langle \mathcal{A}^\mathsf{T} \mathbf{x}_*, (\mathbf{x}_* - \mathbf{u}_*)^{\otimes 2} \rangle + \langle B, (\mathbf{u}_* - \mathbf{x}_*)^{\otimes 2} \rangle \leq 0.$$

Since $\mathcal{A}$ and B form a co-semidefinite pair, we have

$$2\langle \mathcal{A}^\mathsf{T} \mathbf{x}_*, (\mathbf{x}_* - \mathbf{u}_*)^{\otimes 2} \rangle + \langle B, (\mathbf{u}_* - \mathbf{x}_*)^{\otimes 2} \rangle = 0.$$

Thus, (12) should be an equality, which together with (10), further implies that $\mathbf{x}_*^\mathsf{T} F(\mathbf{x}_*) = 0$. Thus, $\mathbf{x}_* \in \textsc{sol}(\mathcal{A}, B, \mathbf{c})$. □

Whenever the optimal value of (3) is zero, we have complementarity of $\mathbf{x}_*$ and $F(\mathbf{x}_*)$. Recall that $\nabla F(\mathbf{x}) = 2[A_1\mathbf{x}, \ldots, A_n\mathbf{x}]^\mathsf{T} + B$. Therefore, the *linear independence constraint qualification* (LICQ) holds generically.

If $\mathcal{A} \in \mathbb{R}^n \otimes \mathrm{S}(\mathbb{R}^n, 2)$ is an R_0 tensor, then Proposition 3.1 guarantees an optimal solution for (3) whenever (1) is feasible. The following example comes from Propositions 3.1 and 3.2.

Example 3.1 Let $\mathcal{A} \in \mathbb{R}^2 \otimes \mathrm{S}(\mathbb{R}^2, 2)$ with $a_{111} = a_{222} = 1$ and all other $a_{i_1 i_2 i_3} = 0$, and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. It is easy to verify that $\mathcal{A}$ is an R_0 tensor, $D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $D_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are all positive semidefinite matrices, and hence $\mathcal{A}$ and B form a co-semidefinite pair. It is also easy to see that the corresponding QCP (1) is feasible for any $\mathbf{c} \in \mathbb{R}^2$. The LICQ holds at any optimal solution in this case. By Proposition 3.1, the solution set is nonempty.

For this example, the nonemptyness can also be checked by direct calculation. Actually, if $c_1 \geq 0$, then we can take $x_1 = 0$; otherwise, $x_1 = \sqrt{-c_1} > 0$. If $c_2 \geq 0$, then we can take $x_2 = 0$ as well; otherwise, $x_2 = \frac{\sqrt{1-4c_2}-1}{2} > 0$.

3.2 Compact Solution Sets

In this section, we will study the compactness/existence of the solution set $\mathrm{SOL}(\mathcal{A}, B, \mathbf{c})$. Obviously, the set $\mathrm{SOL}(\mathcal{A}, B, \mathbf{c})$ is closed by continuity.

Definition 3.2 (C-strict copositivity) Let $C \subseteq \mathbb{R}^n_+$ be a nonempty closed cone. A tensor $\mathcal{A} \in \mathrm{T}(\mathbb{R}^n, 3)$ is called *C-strictly copositive*, if $\mathcal{A}$ is copositive and *strictly copositive* on the cone C, i.e.,

$$\mathcal{A}\mathbf{x}^3 \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n_+, \text{ and } \mathcal{A}\mathbf{x}^3 > 0 \text{ for all } \mathbf{x} \in C \setminus \{\mathbf{0}\}.$$

Obviously, $\{\mathbf{0}\}$-strict copositivity is the copositivity, and $\mathbb{R}^n_+$-strict copositivity is the strict copositivity in the usual sense respectively. There are plenty of examples of C-strictly copositive tensors with C being a face of $\mathbb{R}^n_+$, i.e.,

$$C = \{\mathbf{x} \in \mathbb{R}^n_+ \mid x_i = 0 \text{ for all } i \in I\}$$

for some subset $I \subseteq \{1, \ldots, n\}$. In the following, we give an example with C being not a face of $\mathbb{R}^n_+$.

Example 3.2 Let $\mathcal{A} = (a_{i_1 i_2 i_3}) \in \mathrm{T}(\mathbb{R}^2, 3)$ and $a_{111} = 1$, $a_{112} = -1$, $a_{211} = 1$ and $a_{i_1 i_2 i_3} = 0$ for all the other $i_1, i_2, i_3 \in \{1, 2\}$. Let $C := \mathbb{R}^2_{\geq} = \{\mathbf{x} \in \mathbb{R}^2_+ \mid x_1 \geq x_2\}$ be the cone of vectors with nonincreasing components. Then $\mathcal{A}$ is C-strictly copositive. Actually, $\mathcal{A}\mathbf{x}^3 = x_1^3$. Clearly, $\mathcal{A}$ is copositive and strictly copositive on the cone C, while $\mathcal{A}$ is not strictly copositive.

Definition 3.3 **(K-positive semidefinite plus)** Let $K \subseteq \mathbb{R}^n$ be a nonempty closed cone. A matrix $B \in \mathbb{R}^{n\times n}$ is called *K-positive semidefinite plus*, if B is positive semidefinite plus on K, i.e.,

$$\mathbf{x}^\mathsf{T} B\mathbf{x} \geq 0, \text{ for all } \mathbf{x} \in K, \text{ and}$$
$$\text{whenever } \mathbf{x}^\mathsf{T} B\mathbf{x} = 0 \text{ for } \mathbf{x} \in K, \text{ it follows } B\mathbf{x} = \mathbf{0}. \quad (13)$$

In Definition 3.3, the subset K can be a linear subspace. If $K = \mathbb{R}^n$, we call B simply *positive semidefinite plus*. Given a point $\mathbf{x} \in \mathbb{R}^n$, $\mathbb{R}_+\mathbf{x}$ is the cone $\{\alpha\mathbf{x} : \alpha \in \mathbb{R}_+\}$, and

$$(\mathbb{R}_+\mathbf{x})^\diamond := \begin{cases} (\mathbb{R}_+\mathbf{x})^*, \text{ if } \mathbf{x} \neq \mathbf{0}, \\ \{\mathbf{0}\}, \text{ otherwise}, \end{cases} \quad (14)$$

where K^* means the dual cone of a given cone K.

Proposition 3.3 **(Compact Solution Set)** *Let $\mathcal{A} \in \mathrm{T}(\mathbb{R}^n, 3)$ be C-strictly copositive for a nonempty closed cone $C \subseteq \mathbb{R}^n_+$, $B \in \mathbb{R}^{n\times n}$ a K-positive semidefinite plus matrix for a nonempty closed cone $K \subseteq \mathbb{R}^n$, and $\mathbf{c} \in \mathbb{R}^n$ a vector. Let the intersection of the kernel of B and the linear subspace* $\mathrm{lin}(K)$ *generated by K be $L \subseteq \mathbb{R}^n$. Suppose that $K^\complement \cap \mathbb{R}^n_+ \subseteq C$, and*

$$L \cap \mathbb{R}^n_+ \subseteq C \cup \left[\mathrm{int}\left((\mathbb{R}_+\mathbf{c})^\diamond\right) \cap \mathbb{R}^n_+\right]. \quad (15)$$

Then, the QCP (1) *has a nonempty bounded solution set.*

Proof It is sufficient to show that the set

$$L_\leq := \{\mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{x}^\mathsf{T}(\mathbf{c} + B\mathbf{x} + \mathcal{A}\mathbf{x}^2) \leq 0\}$$

is bounded, by [2, Proposition 2.2.3]. In the following, we assume on the contrary that the set $L_{\leq}$ is unbounded and from which we will derive a contradiction.

Suppose that $\{\mathbf{x}^k\} \subseteq L_{\leq}$ is an unbounded sequence. Without loss of generality, we assume that

$$\lim_{k\to\infty} \|\mathbf{x}^k\| = \infty, \quad \lim_{k\to\infty} \frac{\mathbf{x}^k}{\|\mathbf{x}^k\|} = \mathbf{d} \in \mathbb{R}^n_+.$$

Obviously, $\mathbf{0} \neq \mathbf{d} \geq \mathbf{0}$. Further taking subsequence if necessary, we can assume that either

$$\{\mathbf{x}^k\} \subset K, \text{ or } \{\mathbf{x}^k\} \subset K^{\complement}.$$

Since C is closed, it follows from $K^{\complement} \cap \mathbb{R}^n_+ \subseteq C$ that if $\{\mathbf{x}^k\} \subset K^{\complement}$, then $\mathbf{d} \in C$.

On the other hand, it follows from

$$(\mathbf{x}^k)^{\mathsf{T}}(\mathbf{c} + B\mathbf{x}^k + \mathcal{A}(\mathbf{x}^k)^2) \leq 0, \tag{16}$$

and the copositivity of $\mathcal{A}$ that $\mathcal{A}\mathbf{d}^3 = 0$. Thus,

$$\mathbf{d} \notin C \tag{17}$$

by the C-strict copositivity of $\mathcal{A}$. Consequently,

$$\{\mathbf{x}^k\} \subset K, \tag{18}$$

and henceforth

$$\mathbf{d} \in K \cap \mathbb{R}^n_+. \tag{19}$$

It follows from (16) and the copositivity of $\mathcal{A}$ that

$$(\mathbf{x}^k)^{\mathsf{T}}(\mathbf{c} + B\mathbf{x}^k) \leq 0, \text{ for all } k. \tag{20}$$

Therefore, by dividing the inequality by $\|\mathbf{x}^k\|^2$ and taking limitation, we conclude that $\mathbf{d}^\mathsf{T} B\mathbf{d} \leq 0$, which, together with (19) and the fact that B is K-positive semidefinite plus, implies that

$$B\mathbf{d} = 0, \text{ or equivalently } \mathbf{d} \in L.$$

Thus,

$$\mathbf{d} \in L \cap \mathbb{R}^n_+. \tag{21}$$

By (18) and (20), as well as the K-positive semidefiniteness plus of B, we have that $\mathbf{c}^\mathsf{T}\mathbf{d} \leq 0$, which implies that $\mathbf{d} \notin \operatorname{int}\big((\mathbb{R}_+\mathbf{c})^\diamond\big)$.

This, together with (17) and (21), gives a contradiction to (15). $\square$

Condition (15) is called a *regularity* for QCP (1).

Proposition 3.4 *Let $B \in \mathbb{R}^{n\times n}$ be a positive semidefinite plus matrix with the kernel being $L \subseteq \mathbb{R}^n$, and $\mathbf{c} \in \mathbb{R}^n$ be a vector. Then, the following statements are equivalent:*

1. *There exists a $\mathbf{y} \in \mathbb{R}^n$ such that*

$$\mathbf{c} + B\mathbf{y} \in \mathbb{R}^n_{++}. \tag{22}$$

2. *The following regularity holds*

$$L \cap \mathbb{R}^n_+ \subseteq \{\mathbf{0}\} \cup \big[\operatorname{int}\big((\mathbb{R}_+\mathbf{c})^\diamond\big) \cap \mathbb{R}^n_+\big]. \tag{23}$$

Proof Suppose that there exists a $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{c} + B\mathbf{y} \in \mathbb{R}^n_{++}$. Then,

$$\mathbf{c} \in L^\perp + \mathbb{R}^n_{++},$$

since the range space of B is $L^\perp$. Thus (cf. [15])

$$\mathbf{c} \in \operatorname{int}((L \cap \mathbb{R}^n_+)^*) = \operatorname{int}(L^\perp + \mathbb{R}^n_{++}) = L^\perp + \mathbb{R}^n_{++},$$

which implies

$$\langle \mathbf{c}, \mathbf{d} \rangle > 0 \text{ for all } \mathbf{d} \in L \cap \mathbb{R}^n_+ \setminus \{\mathbf{0}\}.$$

Therefore, either $L \cap \mathbb{R}^n_+ = \{\mathbf{0}\}$ or

$$L \cap \mathbb{R}^n_+ \setminus \{\mathbf{0}\} \subseteq \left[\operatorname{int}\left((\mathbb{R}_+\mathbf{c})^\diamond\right) \cap \mathbb{R}^n_+\right].$$

While, both cases are covered by (23).

If $L \cap \mathbb{R}^n_+ = \{\mathbf{0}\}$, then we have (cf. [15])

$$\mathbb{R}^n = (L \cap \mathbb{R}^n_+)^* = L^\perp + \mathbb{R}^n_+.$$

We must have a point $\mathbf{z} \in \mathbb{R}^n$ such that $B\mathbf{z} \in \mathbb{R}^n_{--}$. For any $\mathbf{c} \in \mathbb{R}^n$, we can find a $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{c} + B\mathbf{y} \in \mathbb{R}^n_+$. Therefore,

$$\mathbf{c} + B(\mathbf{y} - \mathbf{z}) \in \mathbb{R}^n_{++}.$$

So, (22) follows.

Suppose in the following that $L \cap \mathbb{R}^n_+$ has dimension being strictly larger than zero. The condition (23) is equivalent to

$$\langle \mathbf{c}, \mathbf{d} \rangle > 0 \text{ for all } \mathbf{d} \in L \cap \mathbb{R}^n_+ \setminus \{\mathbf{0}\}.$$

Therefore,

$$\mathbf{c} \in \operatorname{int}((L \cap \mathbb{R}^n_+)^*) = \operatorname{int}(L^\perp + \mathbb{R}^n_{++}) = L^\perp + \mathbb{R}^n_{++}.$$

Consequently, we have that there exists a $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{c} + B\mathbf{y} \in \mathbb{R}^n_{++}$. $\square$

We know from Proposition 3.4 that (22) is equivalent to a special case (23) of (15). Thus, Proposition 3.3 generalizes [2, Proposition 2.2.12]; and [8, Theorem 4.5(b)] for TCPs with third order tensors:

(i) Taking $C = \{\mathbf{0}\}$ in Proposition 3.3, together with Proposition 3.4, Proposition 3.3 generalizes [2, Proposition 2.2.12(a)].

(ii) Taking $C = \mathbb{R}^n_+$ in Proposition 3.3, $\mathcal{A}$ is then strictly copositive, which implies that the coercivity condition in [2, Proposition 2.2.12(b)] is satisfied, then Proposition 3.3 generalizes [2, Proposition 2.2.12(b)].

(iii) Taking $C = \mathbb{R}^n_+$ and $B = 0$ in Proposition 3.3, we recover [8, Theorem 4.5(b)] for TCPs with third order tensors, since (15) is satisfied for any given $\mathbf{c}$.

Whenever C is a nontrivial proper subcone in $\mathbb{R}^n_+$, the standard NCP and TCP theory is helpless. While, Proposition 3.3 may provide a solution.

Example 3.3 Let $\mathcal{A} = (a_{i_1i_2i_3}) \in \mathrm{T}(\mathbb{R}^2, 3)$ and $a_{122} = -1$, $a_{221} = 1$, $a_{222} = 1$ and $a_{i_1i_2i_3} = 0$ for all the other $i_1, i_2, i_3 \in \{1, 2\}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. Let $C = \{\mathbf{x} \in \mathbb{R}^n_+ \mid x_1 \leq x_2\}$ and $K = \mathbb{R}^2_+$. It can be checked that $\mathcal{A}$ is C-strictly copositive, but fails to be strictly copositive; and B is K-positive semidefinite plus. With the notation as in Proposition 3.3, $L = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 = 0\}$, and $L \cap \mathbb{R}^2_+ \subset C$. Thus, for any $\mathbf{c} \in \mathbb{R}^2$, $\textsc{sol}(\mathcal{A}, B, \mathbf{c})$ is nonempty and compact.

We shall show the nonemptiness and compactness through direct calculation. If $c_2 \geq 0$, then $x_2 = 0$, and hence $2x_1^2 + c_1x_1 = 0$. Thus, the solution set is nonempty ($x_1 = 0$ is a solution) and always compact. If $c_2 < 0$, there is a solution with $x_1 = 0$ and $x_2 > 0$ for any c_1. On the other hand, in this case, solutions must be with $x_2 > 0$. So, $x_2^2 + x_1x_2 + c_2 = 0$. Since $x_1 \geq 0$, x_2 cannot go to infinity. Thus, x_2 is always bounded. If x_1 goes to infinity, then x_2 should go to zero to maintain the equality $x_2^2 + x_1x_2 + c_2 = 0$, while $-x_2^2 + 2x_1 + c_1 = 0$ as $x_1 > 0$, which is a contradiction for any fixed c_1.

The next example is a modification of Example 3.3 in which $\mathbf{c}$ plays a role.

Example 3.4 Let $\mathcal{A} = (a_{i_1i_2i_3}) \in \mathrm{T}(\mathbb{R}^2, 3)$ and $a_{122} = -1$, $a_{221} = 1$, and $a_{i_1i_2i_3} = 0$ for all the other $i_1, i_2, i_3 \in \{1, 2\}$, the matrix B is as the previous one, and $\mathbf{c} = (c_1, c_2)^\mathsf{T}$ with some $c_2 > 0$. Let $C = \{\mathbf{0}\}$, $K = \mathbb{R}^2_+$.

In this case, $L \cap \mathbb{R}^2_+ = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 = 0, x_2 \geq 0\}$ as well. While, it is easy to see pictorially that $L \cap \mathbb{R}^n_+ \subset \mathrm{int}\left((\mathbb{R}_+\mathbf{c})^\diamond\right) \cap \mathbb{R}^2_+$. Thus, the regularity (15) holds as well and hence the solution set is nonempty and compact by Proposition 3.3. It can also be checked that the corresponding solution set is nonempty and compact in this case as Example 3.3.

In the following, we note a connection between the regularity (15) and the existence for QCP (1) via a generalized Frank-Wolfe theorem by Andronov, Belousov, and Shironin [16]. A tensor is symmetric if the entries are invariant when permuting their indices.

The existence for QCP (1) with symmetric $\mathcal{A}$ and B satisfying (15) follows from this generalized Frank-Wolfe theorem for cubic polynomial objective under linear constraints. In this case, we consider the optimization problem

$$\min \left\{ \frac{1}{3}\mathcal{A}\mathbf{x}^3 + \frac{1}{2}\mathbf{x}^\mathsf{T}B\mathbf{x} + \mathbf{c}^\mathsf{T}\mathbf{x} \mid \mathbf{x} \geq \mathbf{0} \right\}. \tag{24}$$

It is proved in [16] that whenever the objective function is bounded below over the feasible set, there is an optimal solution. Obviously, the LICQ holds at any optimal solution, which further implies the existence of a KKT point. It is straightforward to write down the KKT system for (24) and it is the same as QCP (1). Whenever the regularity (15) is satisfied, an almost the same proof as that for Proposition 3.3 will show that the objective function is indeed bounded from below.

4 Uniqueness

If $A \in \mathbb{R}^{n\times n}$ is positive semidefinite, we define the *null* $\mathrm{null}(A)$ of A as the set of vectors in $\mathbb{R}^n$ such that $\mathbf{x}^\mathsf{T} A\mathbf{x} = 0$, i.e.,

$$\mathrm{null}(A) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^\mathsf{T} A\mathbf{x} = \mathbf{0}\}.$$

It is easy to see that $\mathrm{null}(A)$ is a linear subspace. Whenever furthermore $A \in \mathrm{S}(\mathbb{R}^n, 2)$, i.e., A is symmetric, then $\mathrm{null}(A)$ is the kernel of A, i.e.,

$$\mathrm{null}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\},$$

while in general this is not true.

Let $\{D_1, \dots, D_n\} \subset \mathbb{R}^{n\times n}$ be a collection of n matrices. If each D_i is positive semidefinite, then the nulls of the matrix pencils

$$x_1D_1 + \cdots + x_nD_n \text{ for } \mathbf{x} \in \mathbb{R}^n_+$$

form a partially ordered finite set $\mathrm{W}(D_1, \dots, D_n)$. Actually, whenever each D_i is positive semidefinite, the null of the matrix $x_1D_1 + \cdots + x_nD_n$ for $\mathbf{x} \in \mathbb{R}^n_+$

is equal to the null of the matrix

$$\mathrm{sign}(x_1)D_1 + \cdots + \mathrm{sign}(x_n)D_n,$$

where

$$\mathrm{sign}(\alpha) = 1 \ \text{ if } \alpha > 0; \text{ or } \ \mathrm{sign}(\alpha) = 0 \ \text{ if } \alpha = 0; \text{ or } \ \mathrm{sign}(\alpha) = -1 \ \text{ if } \alpha < 0.$$

In fact, the null of $\mathrm{sign}(x_1)D_1 + \cdots + \mathrm{sign}(x_n)D_n$ is equal to

$$\bigcap_{i=1}^{n} \mathrm{null}\big(\mathrm{sign}(x_i)D_i\big).$$

Therefore, with respect to the set inclusion, $\mathrm{W}(D_1, \ldots, D_n)$ is a set with the maximal elements being $\{\mathrm{null}(D_i) \mid i \in \{1, \ldots, n\}\}$, and the unique minimum element being $\bigcap_{i=1}^{n} \mathrm{null}(D_i)$. A *pseudo-maximal element* in $\mathrm{W}(D_1, \ldots, D_n)$ is defined as an element of the form

$$\mathrm{null}(D_i) \cap \mathrm{null}(D_j) \text{ for some } 1 \le i < j \le n.$$

Given a tensor $\mathcal{A} \in \mathrm{T}(\mathbb{R}^n, 3)$, recall $\mathcal{A}^\mathsf{T}$ is the tensor by transposing the first and the second indices. Recall that $D_i := \big(\mathcal{A}^\mathsf{T}\big)_{i,\cdot,\cdot}$ for $i = 1, \ldots, n$ are the slices of $\mathcal{A}^\mathsf{T}$.

Lemma 4.1 *Let $\mathcal{A} \in \mathrm{T}(\mathbb{R}^n, 3)$ and $B \in \mathbb{R}^{n\times n}$. Under either of the following conditions, the mapping $F(\mathbf{x}) := \mathcal{A}\mathbf{x}^2 + B\mathbf{x} + \mathbf{c}$ is strictly monotone on $\mathbb{R}^n_+$ for an arbitrary $\mathbf{c} \in \mathbb{R}^n$:*

1. *$\mathcal{A}$ is C-strictly copositive for a nonempty closed cone $C \subseteq \mathbb{R}^n_+$, $B \in \mathbb{R}^{n\times n}$ a positive semidefinite matrix which is positive definite on a nonempty cone*

$P \subseteq \mathbb{R}^n$ such that $\mathbb{R}^n_+ \subseteq P \cup C$, and the matrices $D_1, \ldots, D_n$ are positive semidefinite with

$$\mathrm{null}(B) \cap W = \{\mathbf{0}\}$$

for every pseduo-maximal element $W \in \mathrm{W}(D_1, \ldots, D_n)$;

2. *the matrices $D_1, \ldots, D_n$ and B are positive semidefinite with*

$$\mathrm{null}(B) \cap W = \{\mathbf{0}\}$$

for every maximal element $W \in \mathrm{W}(D_1, \ldots, D_n)$.

Proof We have for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$

$$\begin{aligned}\langle F(\mathbf{x}) - F(\mathbf{y}), \mathbf{x} - \mathbf{y}\rangle &= \langle \mathcal{A}\mathbf{x}^2 - \mathcal{A}\mathbf{y}^2, \mathbf{x} - \mathbf{y}\rangle + \langle B(\mathbf{x} - \mathbf{y}), \mathbf{x} - \mathbf{y}\rangle \\ &= \langle \mathcal{A}^{\mathsf{T}}(\mathbf{x} + \mathbf{y}) + B, (\mathbf{x} - \mathbf{y})(\mathbf{x} - \mathbf{y})^{\mathsf{T}}\rangle. \end{aligned} \tag{25}$$

It is easy to see that under either hypothesis, the tensor $\mathcal{A}$ is copositive.

Suppose, without loss of generality, that $\mathbf{y} = \mathbf{0}$ at first. Then, (25) becomes

$$\langle \mathcal{A}^{\mathsf{T}}\mathbf{x} + B, \mathbf{x}\mathbf{x}^{\mathsf{T}}\rangle. \tag{26}$$

If the hypothesis 1 is satisfied, then (26) is nonnegative since $\mathcal{A}$ is copositive and B is positive semidefintie. If $\mathbf{x}^{\mathsf{T}}B\mathbf{x} = 0$, then $\mathbf{x} \notin P$, and hence $\mathbf{x} \in C$, which further implies $\langle \mathcal{A}^{\mathsf{T}}\mathbf{x}, \mathbf{x}\mathbf{x}^{\mathsf{T}}\rangle > 0$. If the other hypothesis 2 is satisfied and $\mathbf{x}^{\mathsf{T}}B\mathbf{x} = 0$, then $\mathbf{x} \in \mathrm{null}(B)$. Since $\mathbf{x} \neq \mathbf{0}$ and $\mathrm{null}(B) \cap W = \{\mathbf{0}\}$ for every maximal element $W \in \mathrm{W}(D_1, \ldots, D_n)$, it follows that $\langle \mathcal{A}^{\mathsf{T}}\mathbf{x}, \mathbf{x}\mathbf{x}^{\mathsf{T}}\rangle > 0$.

In the following, we suppose that both $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, and at least two elements of $\mathbf{x} + \mathbf{y}$ are nonzero, since the case when $\mathbf{x} + \mathbf{y}$ has only one nonzero component can be proved similarly as the previous argument. Then, at least two matrices D_i's are involved in $\mathcal{A}^\mathsf{T}(\mathbf{x}+\mathbf{y})$. Thus, the null of the matrix $\mathcal{A}^\mathsf{T}(\mathbf{x}+\mathbf{y})$ is contained in a pseduo-maximal element $W \in \mathrm{W}(D_1,\dots,D_n)$. Since $\mathbf{x} \neq \mathbf{y}$, and $\mathrm{null}(B) \cap W = \{\mathbf{0}\}$ for every pseduo-maximal element $W \in \mathrm{W}(D_1,\dots,D_n)$ under either hypothesis, (25) is positive. $\square$

Proposition 4.1 *If $F(\mathbf{x}) = \mathcal{A}\mathbf{x}^2 + B\mathbf{x} + \mathbf{c}$ is strictly monotone on $\mathbb{R}^n_+$, then the solution set $\mathrm{SOL}(\mathcal{A},B,\mathbf{c})$ of QCP (1) has at most one element.*

Proof The proof is by contradiction. Suppose that $\mathbf{x},\mathbf{y} \in \mathrm{SOL}(\mathcal{A},B,\mathbf{c})$ and $\mathbf{x} \neq \mathbf{y}$. Then,

$$\mathbf{x}^\mathsf{T} F(\mathbf{x}) = \mathbf{y}^\mathsf{T} F(\mathbf{y}) = 0.$$

Therefore, it follows from $\mathbf{x},\mathbf{y},F(\mathbf{x}),F(\mathbf{y}) \in \mathbb{R}^n_+$ that

$$(\mathbf{x}-\mathbf{y})^\mathsf{T}(F(\mathbf{x})-F(\mathbf{y})) = -\mathbf{y}^\mathsf{T} F(\mathbf{x}) - \mathbf{x}^\mathsf{T} F(\mathbf{y}) \leq 0.$$

However, the strict monotonicity of F implies that

$$(\mathbf{x}-\mathbf{y})^\mathsf{T}(F(\mathbf{x})-F(\mathbf{y})) > 0,$$

since $\mathbf{x} \neq \mathbf{y}$. Thus, a promised contradiction is derived. $\square$

The next theorem on the uniqueness follows from Propositions 3.3 and 4.1, and Lemma 4.1.

Theorem 4.1 *Let $\mathcal{A} \in \mathrm{T}(\mathbb{R}^n,3)$ and $B \in \mathbb{R}^{n\times n}$. Suppose that $\mathcal{A}$ is C-strictly copositive for a nonempty closed cone $C \subseteq \mathbb{R}^n_+$, B a K-positive semidefinite plus matrix for a nonempty closed cone $K \subseteq \mathbb{R}^n$. Let the intersection of the* *kernel of* B *and the linear subspace* $\operatorname{lin}(K)$ *generated by* K *be* $L \subseteq \mathbb{R}^n$*. Suppose that* $K^{\complement} \cap \mathbb{R}^n_+ \subseteq C$*, and*

$$L \cap \mathbb{R}^n_+ \subseteq C \cup \left[\operatorname{int}\left((\mathbb{R}_+ \mathbf{c})^{\diamond} \right) \cap \mathbb{R}^n_+ \right].$$

Then, under either of the following conditions:

1. $B \in \mathbb{R}^{n\times n}$ *a positive semidefinite matrix which is positive definite on a cone* $P \subseteq \mathbb{R}^n$ *such that* $\mathbb{R}^n_+ \subseteq P \cup C$*, and the matrices* $D_1, \dots, D_n$ *are positive semidefinite with*

$$\operatorname{null}(B) \cap W = \{\mathbf{0}\}$$

   for every pseduo-maximal element $W \in \mathrm{W}(D_1, \dots, D_n)$*;*
2. *the matrices* $D_1, \dots, D_n$ *and* B *are positive semidefinite with*

$$\operatorname{null}(B) \cap W = \{\mathbf{0}\}$$

   for every maximal element $W \in \mathrm{W}(D_1, \dots, D_n)$*,*

the QCP (1) *has a unique solution.*

The next example utilizes the second hypothesis in Theorem 4.1.

Example 4.1 Let $\mathcal{A} \in \mathrm{T}(\mathbb{R}^2, 3)$ and $a_{111} = a_{121} = 1$ and $a_{i_1 i_2 i_3} = 0$ for all the other $i_1, i_2, i_3 \in \{1, 2\}$. $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

From the given data, $D_1 = D_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is easy to verify that D_1, D_2, B are positive semidefinite. Let $C = \{\mathbf{x} \in \mathbb{R}^2_+ \mid x_1 \geq x_2\}$. Then, $\mathcal{A}$ is C-strictly copositive, and B is $K = \mathbb{R}^2_+$-positive semidefinite plus. Similar as

Example 3.3, the regularity condition holds for any $\mathbf{c} \in \mathbb{R}^2$. It is easy to see that any maximal element of $\mathrm{W}(D_1, D_2)$ is $\{\mathbf{x} \in \mathbb{R}^2 \mid x_1 = 0\}$, which intersects $\mathrm{null}(B)$ trivially. Therefore, the corresponding QCP has a unique solution.

We shall show the uniqueness by direct calculation. The system is

$$\begin{cases} 0 \le x_1^2 + x_1 x_2 + c_1 \perp x_1 \ge 0, \\ \\ 0 \le x_2 + c_2 \perp x_2 \ge 0. \end{cases}$$

If $c_2 \ge 0$, then $x_2 = 0$. Consequently, $x_1 = 0$ when $c_1 \ge 0$; and $x_1 = \sqrt{-c_1}$ when $c_1 < 0$. If $c_2 < 0$, then $x_2 = -c_2$. Consequently, $x_1 = 0$ when $c_1 \ge 0$; and $x_1 = \frac{c_2 + \sqrt{c_2^2 - 4c_1}}{2}$. Thus, we have uniqueness for each case.

The next example is a modification of Example 4.1 in which the first hypothesis in Theorem 4.1 is conducted.

Example 4.2 Let $\mathcal{A} \in \mathrm{T}(\mathbb{R}^2, 3)$ and $a_{111} = 1$ and $a_{i_1 i_2 i_3} = 0$ for all the other $i_1, i_2, i_3 \in \{1, 2\}$. Then $D_1 = \begin{bmatrix} 1 & 0 \\ \\ 0 & 0 \end{bmatrix}$ and $D_2 = 0$. Let $B = \begin{bmatrix} 0 & 0 \\ \\ 0 & 1 \end{bmatrix}$. We can take $P = \{\mathbf{x} \in \mathbb{R}^2 \mid x_2 \neq 0\}$. All the other settings are similar to the previous example. The unique pseduo-maximal element in $\mathrm{W}(D_1, D_2)$ is $\{\mathbf{x} \in \mathbb{R}^2 \mid x_1 = 0\}$. All the hypotheses in Theorem 4.1 are satisfied then. Likewise, the uniqueness follows.

5 Conclusion

In this article, we studied existence, compactness and uniqueness of the solution sets of QCPs. Assumptions to guarantee these results are mostly presented in terms of matrices, which should be more tractable. Interestingly, the results in this article generalize the well-known ones in the literature and even broaden the boundary of known knowledge (e.g., Sections 3.2 and 4). These demonstrate that research on QCPs shall be interesting and meaningful for both QCPs and general NCPs. In particular, the study on QCPs would provide fruitful insights on investigations for NCPs.

We conclude this article with remarks that the proposed C-strictly copositivity of a tensor and K-positive semidefiniteness plus of a matrix can be applied to the generalized Markowitz portfolio problem. Actually, the matrix in the QCP reformulation of its optimality condition is K-positive semidefinite plus over the kernel K of that matrix; and the tensor is C-strictly copositive for a proper subcone of the nonnegative orthant. Details and further investigations will be carried out in the coming study.

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