Global Uniqueness and Solvability of Tensor Variational

Inequalities

Yong Wang · Zheng-Hai Huang · Liqun Qi

Communicated by Guoyin Li

Received: date / Accepted: date

Abstract In this paper, we consider a class of variational inequalities, where the involved function is the sum of an arbitrary given vector and a homogeneous polynomial defined by a tensor; and we call it the tensor variational inequality. The tensor variational inequality is a natural extension of the affine variational inequality and the tensor complementarity problem. We show that a class of multi-person noncoop-

Yong Wang

School of Mathematics, Tianjin University, Tianjin 300072, P.R. China.

E-mail: wang_yong@tju.edu.cn

Zheng-Hai Huang

Corresponding Author. School of Mathematics, Tianjin University, Tianjin 300072, P.R. China.

Tel.: +86-22-27403615 Fax: +86-22-27403615

E-mail: huangzhenghai@tju.edu.cn

Liqun Qi

Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong. E-mail: maqilq@polyu.edu.hk erative games can be formulated as a tensor variational inequality. In particular, we investigate the global uniqueness and solvability of the tensor variational inequality. To this end, we first introduce two classes of structured tensors and discuss some related properties; and then, we show that the tensor variational inequality has the property of global uniqueness and solvability under some assumptions, which is different from the existing result for the general variational inequality.

Keywords Tensor variational inequality · Global uniqueness and solvability · Noncooperative game · Strictly positive definite tensor · Exceptionally family of elements

Mathematics Subject Classification (2000) MSC 90C33 · 90C30 · 65H10

1 Introduction

The finite dimensional variational inequality (VI) has been studied extensively due to its wide applications in many fields [1,2]. It is called an affine variational inequality if the involved function is linear. The existence and uniqueness of solution to the VI is a basic and important issue in the studies of the VI. It is well known that the VI has at most one solution when the involved function is strictly monotone [1-3]; and a unique solution when the involved function is strongly monotone [1,2].

It is well known that complementarity problem (CP) is an important subclass of the VIs, which has been studied extensively due to its wide applications [4,5]. Recently, a specific subclass of CPs, called the tensor complementarity problem (TCP) [6], has attracted much attention; and many theoretical results about the properties of the solution set of TCP have been developed, including existence of solution [7–11], global uniqueness of solution [11, 12], boundedness of solution set [8, 13–16], stability of solution [17], sparsity of solution [18], and so on. In addition, an application of the TCP was given in [19].

Inspired by the development of the TCP, we consider a subclass of the VIs, where the involved function is the sum of an arbitrary given vector and a homogeneous polynomial defined by a tensor; and we call it the tensor variational inequality (TVI). The concerned problem is a natural generalization of the TCP and the affine variational inequality. It is well known that the polynomial optimization problem is an important class of optimization problems, which has been studied extensively [20–22]. It is easy to see that the TVI is equivalent to a class of polynomial optimization problems. In addition, we show that a class of multi-person noncooperative games can be reformulated as a TVI. These are our motivations to consider the TVI.

In this paper, we mainly investigate the property of global uniqueness and solvability (GUS-property) of the TVI in the case that 0 belongs to the set involved in the TVI. In this case, we show that there is no strongly monotonously homogeneous polynomial whose degree is larger than 2. In order to investigate the GUS-property of the TVI, we first introduce two classes of structured tensors and discuss some related properties; and then, we show that the TVI has the GUS-property when the involved function is strictly monotone and the involved set contains 0, which is different from the existing result obtained in the case of the general variational inequality.

The rest of this paper is organized as follows. In Section 2, we recall some basic definitions and results. In Section 3, we introduce the TVI and reformulate a class of

multi-person noncooperative games as a TVI. In Section 4, we define two classes of structured tensors and discuss some related properties. In particular, we show that the TVI has the GUS-property under some assumptions. In Section 5, we propose some open problems. The conclusions are given in Section 6.

2 Preliminaries

In this section, we recall some basic concepts and results, which are useful for our subsequent analysis.

Given a nonempty set $X \subseteq \mathbb{R}^n$ and a function $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$, then the variational inequality, denoted by the VI(*X*,*F*), is to find a point $x^* \in X$ such that

$$\langle y - x^*, F(x^*) \rangle \ge 0$$
 for all $y \in X$. (1)

It is called an affine variational inequality when the function *F* is linear. Moreover, if the set *X* is the nonnegative orthant $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \ge 0\}$, then (1) reduces to

$$x \ge 0$$
, $F(x) \ge 0$, $x^{\top}F(x) = 0$,

which is called the complementarity problem, denoted by the CP(F).

In the theoretical studies of the nonlinear variational inequality and complementarity problem, some special types of functions play important roles. The following two classes of functions will be used in this paper.

Definition 2.1 A mapping $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to be

(i) **strictly monotone on** *X*, if and only if

$$\langle F(x) - F(y), x - y \rangle > 0$$
 for all $x, y \in X$ with $x \neq y$;

(ii) strongly monotone on X, if and only if there exists a constant c > 0 such that

$$\langle F(x) - F(y), x - y \rangle \ge c \|x - y\|^2 \quad \text{for all } x, y \in X.$$
(2)

Obviously, a strongly monotone function on $X \subseteq \mathbb{R}^n$ must be strictly monotone on X. Moreover, for $X = \mathbb{R}^n$ and an affine mapping, i.e., F(x) = Ax + q, where $A \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, F is strongly monotone if and only if it is strictly monotone, and if and only if A is positive definite [2]. However, such results do not hold for the general nonlinear function.

The exceptionally family of elements is a powerful tool to investigate the solvability of the VI(X, F) [23–27]. There are several different definitions for the exceptionally family of elements. In this paper, we use the following definition.

Definition 2.2 [26, Definition 3.1] Let $\hat{x} \in \mathbb{R}^n$ be an arbitrary given point. A sequence $\{x^r\}_{r>0}$ is said to be an exceptionally family of elements for the VI(*X*,*F*) with respect to \hat{x} if the following conditions are satisfied:

- (i) $||x^r|| \to \infty$ as $r \to \infty$;
- (ii) $x^r \hat{x} \in X$;

(iii) there exists $\alpha_r \in]0,1[$ such that, for any $r \ge \|P_X(0) - \hat{x}\|$,

$$-[F(x^r-\hat{x})+(1-\alpha_r)(x^r-\hat{x})] \in \mathscr{N}_X(x^r-\hat{x}),$$

where $\mathcal{N}_X(x^r - \hat{x})$ denotes the normal cone of *X* at $x^r - \hat{x}$ and $P_X(\cdot)$ is the projection operator on *X*.

The normal cone of *X* at *x* is defined by (

$$\mathcal{N}_{X}(x) = \begin{cases} \{z \in \mathbb{R}^{n} : z^{\top}(y-x) \leq 0, \forall y \in X\}, \text{ if } x \in X, \\ \emptyset, & \text{otherwise.} \end{cases}$$
(3)

About the relationship between the exceptionally family of elements and the solution of the VI(X, F), we will use the following lemma whose proof can be found in [26].

Lemma 2.1 [26, Theorem 3.1] Let X be a nonempty, closed and convex set in \mathbb{R}^n and $F : X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Then, either the VI(X,F) has a solution or, for any point $\hat{x} \in \mathbb{R}^n$, there exists an exceptionally family of elements for the VI(X,F) with respect to \hat{x} .

Throughout this paper, for any given positive integer *n*, we use [n] to denote the set $\{1, 2, ..., n\}$. For any given positive integers $m, r_1, ..., r_{m-1}$ and r_m , an *m*-order $r_1 \times r_2 \times \cdots \times r_m$ -dimensional real tensor can be denoted by $\mathscr{A} = (a_{i_1 i_2 \cdots i_m})$ with $a_{i_1 i_2 \cdots i_m} \in \mathbb{R}$ for any $i_j \in [r_j]$ and $j \in [m]$. Furthermore, if $r_j = n$ for all $j \in [m]$, then \mathscr{A} is called an *m*-order *n*-dimensional real tensor; and we denote the set of all *m*-order *n*-dimensional real tensors by $\mathbb{T}_{m,n}$. In particular, $\mathscr{A} \in \mathbb{T}_{m,n}$ is called a symmetric tensor if the entries $a_{i_1 i_2 \cdots i_m}$ are invariant under any permutation of their indices. For any $\mathscr{A} \in \mathbb{T}_{m,n}$ and $x \in \mathbb{R}^n$, $\mathscr{A} x^{m-1} \in \mathbb{R}^n$ is a vector defined by

$$(\mathscr{A}x^{m-1})_i := \sum_{i_2, i_3, \cdots, i_m=1}^n a_{ii_2 \cdots i_m} x_{i_2} x_{i_3} \cdots x_{i_m}, \quad \forall i \in [n]$$

3 The TVI and an Application

In this section, we first introduce the TVI and discuss the relationship between it and a class of polynomial optimization problems; and then, give an application of the TVI. For any $\mathscr{A} \in \mathbb{T}_{m,n}$, $q \in \mathbb{R}^n$ and a nonempty set $X \subseteq \mathbb{R}^n$, the TVI we considered is given specifically in the following way: Find a vector $x^* \in X$ such that

$$\langle y - x^*, \mathscr{A}(x^*)^{m-1} + q \rangle \ge 0 \quad \text{for all } y \in X,$$
 (4)

which is denoted by the $\text{TVI}(X, \mathscr{A}, q)$. From the relationship between the variational inequalities and the complementarity problems, which is also described at the beginning of Section 2, it is easy to see that, when $X = \mathbb{R}^n_+$, the $\text{TVI}(X, \mathscr{A}, q)$ is equivalent to the tensor complementarity problem: Find a vector $x^* \ge 0$ such that

$$\mathscr{A}(x^*)^{m-1} + q \ge 0$$
, and $(x^*)^T [\mathscr{A}(x^*)^{m-1} + q] = 0$.

It should be noted that Song and Qi [6] proposed a $\text{TVI}(X, \mathscr{A}, q)$ with q = 0 in a question related to applications of structured tensors; but to the best of our knowledge, the $\text{TVI}(X, \mathscr{A}, q)$ has not been studied so far even in the case of q = 0.

The $\text{TVI}(X, \mathscr{A}, q)$ arises in a natural way in the framework of polynomial optimization problems, which is given as follows:

Proposition 3.1 For any given symmetric tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ and $q \in \mathbb{R}^n$, suppose that $f(x) = \frac{1}{m} \mathscr{A} x^m + q^\top x$ is a convex function and $X \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set. Then, x^* solves the $TVI(X, \mathscr{A}, q)$ if and only if x^* is an optimal solution of the optimization problem $\min\{f(x) : x \in X\}$.

Proof Since \mathscr{A} is symmetric, it follows that $\nabla f(x) = \mathscr{A}x^{m-1} + q$. Then, the result is straightforward from [28, Page 10].

In the following, we consider an application of the $\text{TVI}(X, \mathscr{A}, q)$ related to the problem of *m*-person noncooperative game. For any $k \in [m]$, let $x^k \in \mathbb{R}^{r_k}$ and $X_k \subseteq \mathbb{R}^{r_k}$

be player k's strategy and strategy set, respectively. We denote

$$[m]_{-k} := [m] \setminus \{k\}, \quad n := \sum_{j \in [m]} r_j, \quad n_{-k} := \sum_{j \in [m]_{-k}} r_j,$$
$$x := (x^j)_{j \in [m]} \in \mathbb{R}^{r_1} \times \dots \times \mathbb{R}^{r_m} = \mathbb{R}^n,$$
$$x^{-k} := (x^j)_{j \in [m]_{-k}} \in \mathbb{R}^{r_1} \times \dots \times \mathbb{R}^{r_{k-1}} \times \mathbb{R}^{r_{k+1}} \times \dots \times \mathbb{R}^{r_m} = \mathbb{R}^{n_{-k}},$$

and

$$X := \prod_{j \in [m]} X_j \subseteq \mathbb{R}^{r_1} \times \dots \times \mathbb{R}^{r_m} = \mathbb{R}^n.$$
(5)

Then, for any $k \in [m]$, the *k*th player decides his own strategy by solving the following optimization problem with the opponents' strategy x^{-k} fixed:

$$\min f_k(y^k, x^{-k}) \quad \text{s.t. } y^k \in X_k,$$

where $f_k : \mathbb{R}^{r_1} \times \cdots \times \mathbb{R}^{r_m} \to \mathbb{R}$ denotes player *k*'s cost function.

A tuple $x^* := ((x^1)^*, (x^2)^*, \dots, (x^m)^*)$ satisfying

$$(x^k)^* \in \arg\min_{y^k \in X_k} f_k(y^k, x^{-k}), \quad \forall k \in [m]$$

is called a Nash equilibrium of the *m*-person noncooperation game.

Proposition 3.2 [2, Proposition 1.4.2] Let every $X_i \subseteq \mathbb{R}^{r_i}$ be closed and convex. Suppose that for each fixed $x := ((x^1), (x^2), \dots, (x^m)) \in X$, the function $f_k(y^k, x^{-k})$ is convex and continuously differentiable in y^k . Then $x^* := ((x^1)^*, (x^2)^*, \dots, (x^m)^*)$ is a Nash equilibrium of the m-person noncooperation game if and only if x^* is a solution of the VI(X, F) with $F(x) \equiv (\nabla_{x^k} f_k(x))_{k \in [m]}$, where $\nabla_{x^k} f_k(x)$ is the gradient of the function $f_k(x)$ with respect to x^k .

In this paper, for any $k \in [m]$, we use $\mathscr{A}^k = (a_{i_1i_2\cdots i_m}^k)$ to denote player *k*'s payoff tensor and assume that player *k*'s cost function f_k is given by

$$f_k(x^k, x^{-k}) = \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_m=1}^{r_m} a^k_{i_1 i_2 \cdots i_m} x^1_{i_1} x^2_{i_2} \cdots x^{k-1}_{i_{k-1}} x^k_{i_k} x^{k+1}_{i_{k+1}} \cdots x^m_{i_m}.$$
 (6)

Then, the function $F(x) \equiv (\nabla_{x^k} f_k(x))_{k \in [m]}$ defined in Proposition 3.2 is a homogeneous polynomial function with the degree m-1, which can be defined by a tensor. To this end, we first introduce the following symbols: for any tensor $\mathscr{B} \in \mathbb{T}_{m,n}$ and $u^k \in \mathbb{R}^{r_k}$ with $k \in [m]_{-1}$, we denote

$$\mathscr{B}u^{2}\cdots u^{m} = \begin{pmatrix} \sum_{i_{2}=1}^{r_{2}}\cdots\sum_{i_{m}=1}^{r_{m}}b_{1i_{2}\cdots i_{n}}u_{i_{2}}^{2}\cdots u_{i_{m}}^{m} \\ \sum_{i_{2}=1}^{r_{2}}\cdots\sum_{i_{m}=1}^{r_{m}}b_{2i_{2}\cdots i_{n}}u_{i_{2}}^{2}\cdots u_{i_{m}}^{m} \\ \vdots \\ \sum_{i_{2}=1}^{r_{2}}\cdots\sum_{i_{m}=1}^{r_{m}}b_{r_{1}i_{2}\cdots i_{n}}u_{i_{2}}^{2}\cdots u_{i_{m}}^{m} \end{pmatrix};$$

and, for any $k \in [m]$, by using the payoff tensor $\mathscr{A}^k = (a_{i_1 i_2 \cdots i_m}^k)$, we define a new tensor $\overline{\mathscr{A}}^k = (\overline{a}_{i_1 i_2 \cdots i_m}^k)$ with $\overline{a}_{i_1 i_2 \cdots i_m}^k = a_{i_k i_1 \cdots i_{k-1} i_{k+1} \cdots i_m}^k$ for any $i_j \in [r_j]$ and $j \in [m]$. Furthermore, we construct a new tensor

$$\mathscr{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,n},\tag{7}$$

where for any $i_j \in [n]$ with $j \in [m]$,

$$a_{i_{1}i_{2}\cdots i_{m}} = \begin{cases} a_{i_{1}(i_{2}-r_{1})\cdots(i_{m}-\sum_{j=1}^{m-1}r_{j})}^{1}, & \text{if } i_{1} \in [r_{1}], i_{2} \in [r_{1}+r_{2}] \setminus [r_{1}], \dots, i_{m} \in [\sum_{j=1}^{m}r_{j}] \setminus [\sum_{j=1}^{m-1}r_{j}], \\ a_{(i_{1}-r_{1})i_{2}(i_{3}-r_{1}-r_{2})\cdots(i_{m}-\sum_{j=1}^{m-1}r_{j})}^{2}, & \text{if } i_{1} \in [r_{1}+r_{2}] \setminus [r_{1}], i_{2} \in [r_{1}], \\ & i_{3} \in [\sum_{j=1}^{3}r_{j}] \setminus [r_{1}+r_{2}], \dots, i_{m} \in [\sum_{j=1}^{m}r_{j}] \setminus [\sum_{j=1}^{m-1}r_{j}], \\ a_{(i_{1}-\sum_{j=1}^{k-1}r_{j})i_{2}(i_{3}-r_{1})\cdots(i_{k-1}-\sum_{j=1}^{k-3}r_{j})i_{k}(i_{k+1}-\sum_{j+1}^{k}r_{j})\cdots(i_{m}-\sum_{j=1}^{m-1}r_{j}), \\ & \text{if } k \in [m] \setminus \{1,2\}, \text{ and for any given } k, i_{1} \in [\sum_{j=1}^{k}r_{j}] \setminus [\sum_{j=1}^{k-2}r_{j}], \\ & i_{2} \in [r_{1}], i_{3} \in [r_{1}+r_{2}] \setminus [r_{1}], \dots, i_{k} \in [\sum_{j=1}^{k-1}r_{j}] \setminus [\sum_{j=1}^{k-2}r_{j}], \\ & i_{k+1} \in [\sum_{j=1}^{k+1}r_{j}] \setminus [\sum_{j=1}^{k}r_{j}], \dots, i_{m} \in [\sum_{j=1}^{m}r_{j}] \setminus [\sum_{j=1}^{m-1}r_{j}], \\ 0, \text{ otherwise.} \end{cases}$$

Then, it is not difficult to get that

$$\mathscr{A}x^{m-1} = \begin{pmatrix} \mathscr{A}^{1}x^{2}\cdots x^{m} \\ \vdots \\ \mathscr{A}^{k}x^{1}\cdots x^{k-1}x^{k+1}\cdots x^{m} \\ \vdots \\ \mathscr{A}^{m}x^{1}x^{2}\cdots x^{m-1} \end{pmatrix} = \begin{pmatrix} \nabla_{x^{1}}f_{1}(x^{1},x^{-1}) \\ \vdots \\ \nabla_{x^{k}}f_{k}(x^{k},x^{-k}) \\ \vdots \\ \nabla_{x^{m}}f_{m}(x^{m},x^{-m}) \end{pmatrix} = F(x).$$
(8)

Therefore, from Proposition 3.2 and (8) we can obtain the the following result.

Proposition 3.3 For any $k \in [m]$, we assume the function f_k is defined by (6) and every set $X_i \subseteq \mathbb{R}^{r_i}$ is closed and convex, then a tuple $x^* := ((x^1)^*, (x^2)^*, \dots, (x^m)^*)$ is a Nash equilibrium of the m-person noncooperation game if and only if x^* is a solution of the $TVI(X, \mathcal{A}, q)$ with q = 0 and \mathcal{A} being defined by (7).

4 GUS-Property of the TVI

The tensor variational inequality (4) is said to have the GUS-property if it has a unique solution for every $q \in \mathbb{R}^n$. Such an important property has been investigated for variational inequalities [1,2] and complementarity problems [29–32]. In this section, we discuss the GUS-property of the TVI(X, \mathscr{A}, q).

For the general VI, the following results come from [1,2].

Lemma 4.1 Let $X \subseteq \mathbb{R}^n$ be nonempty, closed and convex and $F : X \to \mathbb{R}^n$ be contin*uous*.

(i) If F is strictly monotone on X, then the VI(X,F) has at most one solution;

(ii) If F is strongly monotone on X, then the VI(X, F) has a unique solution.

Let $F: X \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$F(x) := \mathscr{A} x^{m-1} + q, \tag{9}$$

where $\mathscr{A} \in \mathbb{T}_{m,n}$ with m > 2 and $q \in \mathbb{R}^n$. Then, we have the following observation.

Proposition 4.1 For any tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ with m > 2 and $q \in \mathbb{R}^n$, let the function F be defined by (9). Suppose that $0 \in X \subseteq \mathbb{R}^n$, then the function F is not strongly monotone on X.

Proof Suppose that there exist a vector $q \in \mathbb{R}^n$ and a tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ with m > 2 such that the function F defined by (9) is strongly monotone on X, then there exists a positive constant c such that (2) holds for any $x, y \in X$. Let $y = 0 \in X$, then we get from (2) that

$$\mathscr{A}x^m \ge c \|x\|^2 \quad \text{for any } x \in X.$$
⁽¹⁰⁾

For any $x \neq 0$, it follows from (10) that

$$\mathscr{A}\left(\frac{x}{\|x\|}\right)^{m} \ge c \left\|\left(\frac{x}{\|x\|}\right)\right\|^{2} \|x\|^{2-m}.$$
(11)

Since $\left\|\frac{x}{\|x\|}\right\| = 1$, it follows that the left-hand side of the inequality (11) is bounded; but when $\|x\| \to 0$, it is obvious that the right-hand side of the inequality (11) tends to ∞ , which leads to a contradiction. Therefore, there exists no the strongly monotone function *F* in the form of $\mathscr{A}x^{m-1} + q$ for any $q \in \mathbb{R}^n$ and $\mathscr{A} \in \mathbb{T}_{m,n}$ with m > 2. \Box

From Lemma 4.1 (ii) and Proposition 4.1, a natural question is whether or not the VI(X, F) has the GUS-property when $0 \in X$ and the function F is defined by (9) where $\mathscr{A} \in \mathbb{T}_{m,n}$ with m > 2 and $q \in \mathbb{R}^n$. In this section, we answer this question. To this end, we first introduce two new classes of tensors in the next subsection and discuss the relationship between them.

4.1 Relationship of Two Classes of Tensors

In this subsection, we introduce two new classes of structured tensors and discuss the relationship between them.

Definition 4.1 Given a nonempty set $X \subseteq \mathbb{R}^n$. A tensor $\mathscr{A} \in \mathbb{T}_{m,n}$ is said to be

- (i) **positive definite on** *X*, if and only if $\mathscr{A}x^m > 0$ for any $x \in X$ and $x \neq 0$;
- (ii) strictly positive definite on *X*, if and only if

$$(x-y)^{\top}(\mathscr{A}x^{m-1}-\mathscr{A}y^{m-1}) > 0$$
 for any $x, y \in X$ with $x \neq y$

 $\mathscr{A} \in \mathbb{T}_{m,n}$ is said to be a strictly positive definite tensor if it is strictly positive definite on \mathbb{R}^n .

When $X = \mathbb{R}^n$, the positive definite tensor on *X* defined by Definition 4.1 (i) is just the positive definite tensor defined in [33]; When $X = \mathbb{R}^n_+$, the positive definite tensor on *X* defined by Definition 4.1 (i) is just the strictly copositive tensor defined in [34]. From Definitions 2.1 and 4.1, it is easy to see that the function *F* defined by (9) is strictly monotone on *X* if and only if the tensor \mathscr{A} is strictly positive definite on *X*.

A basic question is whether or not there exists a strictly positive definite tensor on some subset of \mathbb{R}^n . The following example gives a positive answer to this question.

Example 4.1 Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{T}_{4,2}$, where $a_{1111} = a_{2222} = 1$, and the others equal to zero. Then, \mathscr{A} is a strictly positive definite tensor on any non-single-point subset X of \mathbb{R}^2 .

It only needs to prove that \mathscr{A} is strictly positive definite on \mathbb{R}^2 .

Since

$$\mathscr{A}x^3 = \begin{pmatrix} x_1^3 \\ x_2^3 \end{pmatrix},$$

it follows that for any $x, y \in \mathbb{R}^2$,

$$(x_1 - y_1)[(\mathscr{A}x^3)_1 - (\mathscr{A}y^3)_1] = (x_1 - y_1)(x_1^3 - y_1^3)$$
$$= (x_1 - y_1)^2(x_1^2 + x_1y_1 + y_1^2);$$
(12)

$$(x_2 - y_2)[(\mathscr{A}x^3)_2 - (\mathscr{A}y^3)_2] = (x_2 - y_2)(x_2^3 - y_2^3)$$
$$= (x_2 - y_2)^2(x_2^2 + x_2y_2 + y_2^2).$$
(13)

For any $s, t \in \mathbb{R}$, we discuss the following three cases.

(I) $|s| \neq |t|$. In this case, we have

$$s^{2} + st + t^{2} > 2|s||t| + st = \begin{cases} 3st \ge 0, \text{ if } st \ge 0, \\ -st > 0, \text{ if } st < 0, \end{cases}$$

which implies that $s^2 + st + t^2 > 0$.

(II) s = t. In this case, we have

$$(s-t)^2(s^2+st+t^2) = 0$$

(III) $s = -t \neq 0$. In this case, we have

$$(s-t)^2(s^2+st+t^2) = 4s^4 > 0.$$

Now, for any $x, y \in \mathbb{R}^2$ and $x \neq y$, it follows that either $x_1 \neq y_1$ or $x_2 \neq y_2$. Therefore, by combining cases (I)-(III) with (12) and (13) we have

$$(x-y)^{\top}(\mathscr{A}x^3 - \mathscr{A}y^3) = \sum_{i=1}^{2} (x_i - y_i)^2 (x_i^2 + x_i y_i + y_i^2) > 0,$$

which demonstrates that \mathscr{A} is a strictly positive definite tensor on \mathbb{R}^2 .

In what follows, we discuss the relationship between two classes of tensors defined by Definition 4.1.

Proposition 4.2 Suppose that $0 \in X \subseteq \mathbb{R}^n$. Then, a strictly positive definite tensor on *X* must be positive definite on *X*.

Proof Given a tensor $\mathscr{A} \in \mathbb{T}_{m,n}$. Take $y = 0 \in X$, it follows from Definition 4.1(ii) that for any $x \in X$ with $x \neq 0$,

$$\mathscr{A}x^m = (x-0)^\top \left(\mathscr{A}x^{m-1} - \mathscr{A}0^{m-1} \right) > 0,$$

which, together with Definition 4.1(i), implies that \mathscr{A} is positive definite on *X*. \Box

However, if m > 2, a positive definite tensor on X is not necessary a strictly positive definite tensor on X, which can be seen in the following example.

Example 4.2 Let $\mathscr{A} = (a_{ijkl}) \in \mathbb{T}_{4,2}$, where $a_{1111} = a_{2222} = a_{2112} = 1$, $a_{1122} = -1$, and the others equal to zero. Denote $X := \mathbb{R}^2_+$. Then, \mathscr{A} is positive definite on X but not strictly positive definite on X.

First, we show that \mathscr{A} is positive definite on X. Since

$$\mathscr{A}x^{3} = \begin{pmatrix} x_{1}^{3} - x_{1}x_{2}^{2} \\ x_{2}^{3} + x_{1}^{2}x_{2} \end{pmatrix},$$

it follows that for any $x \in \mathbb{R}^2 \setminus \{0\}$,

$$x^{\top} \mathscr{A} x^{3} = x_{1}^{4} - x_{1}^{2} x_{2}^{2} + x_{2}^{4} + x_{1}^{2} x_{2}^{2} = x_{1}^{4} + x_{2}^{4} > 0.$$

Hence, \mathscr{A} is positive definite on \mathbb{R}^2 . Of course, \mathscr{A} is positive definite on *X*.

Second, we show that \mathscr{A} is not a strictly positive definite tensor on *X*. To this end, for any $\mu \in \mathbb{R}_+$ with $\mu \neq 0$, let $x = (2\mu, 3\mu)^\top$ and $y = (\mu, 3\mu)^\top$, then $x, y \in X, x \neq y$ and

$$\begin{aligned} (x-y)^{\top} (\mathscr{A}x^3 - \mathscr{A}y^3) &= (x_1 - y_1) [(\mathscr{A}x^3)_1 - (\mathscr{A}y^3)_1] + (x_2 - y_2) [(\mathscr{A}x^3)_2 - (\mathscr{A}y^3)_2] \\ &= (2\mu - \mu) [(2\mu)^3 - 2\mu (3\mu)^2 - (\mu^3 - \mu (3\mu)^2)] + 0 \\ &= -2\mu^4 \\ &< 0. \end{aligned}$$

Therefore, \mathscr{A} is not strictly positive definite on *X*.

4.2 Uniqueness of Solution to the TVI

In this subsection, we investigate the GUS-property of the $TVI(X, \mathcal{A}, q)$.

Theorem 4.1 Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set and $\mathscr{A} \in \mathbb{T}_{m,n}$ be a strictly positive definite tensor on X. Then, for any given $q \in \mathbb{R}^n$, the $TVI(X, \mathscr{A}, q)$ has at most one solution.

Proof Since \mathscr{A} is a strictly positive definite tensor on *X*, it follows from Definition 4.1 (ii) that the function $\mathscr{A}x^{m-1} + q$ is strictly monotone on *X* for any $q \in \mathbb{R}^n$. So, the desired result holds from Lemma 4.1 (i).

Theorem 4.2 Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set with $0 \in X$ and $\mathscr{A} \in \mathbb{T}_{m,n}$ be a positive definite tensor on X. Then, for any given $q \in \mathbb{R}^n$, the solution set of the $TVI(X, \mathscr{A}, q)$ is nonempty and compact.

Proof If the set *X* is bounded, then the result is obvious from [1,35]. In what follows, we assume that the set *X* is unbounded.

Suppose that the $\text{TVI}(X, \mathscr{A}, q)$ has no solution, then for $\hat{x} = 0 \in \mathbb{R}^n$, it follows from Lemma 2.1 that there exists an exceptionally family of elements $\{x^r\}_{r>0}$ for the $\text{TVI}(X, \mathscr{A}, q)$ with respect to 0. That is, we have

- (a) $||x^r|| \to \infty$ as $r \to \infty$;
- (b) $x^r \in X$ for any positive integer *r*;
- (c) there exists $\alpha_r \in]0,1[$ such that, for any $r \ge ||P_X(0)||$,

$$-[\mathscr{A}(x^r)^{m-1}+(1-\alpha_r)x^r]\in\mathscr{N}_X(x^r).$$

From the above (c) and the definition of the normal cone, we have

$$[\mathscr{A}(x^r)^{m-1} + (1 - \alpha_r)x^r]^\top (y - x^r) \ge 0 \quad \text{for any } y \in X,$$

which can be rewritten as

$$[\mathscr{A}(x^r)^{m-1}]^{\top}(y-x^r) \ge (\alpha_r - 1)(x^r)^{\top}(y-x^r) \quad \text{for any } y \in X.$$
(14)

From the above (a), it holds that $||x^r|| > 0$ for sufficiently large *r*. So, by dividing $||x^r||^m$ in both sides of the inequality (14), we get

$$\left[\mathscr{A}\frac{(x^{r})^{m-1}}{\|x^{r}\|^{m-1}}\right]^{\top}\left(\frac{y}{\|x^{r}\|}-\frac{x^{r}}{\|x^{r}\|}\right) \geq \frac{\alpha_{r}-1}{\|x^{r}\|^{m-2}}\left(\frac{x^{r}}{\|x^{r}\|}\right)^{\top}\left(\frac{y}{\|x^{r}\|}-\frac{x^{r}}{\|x^{r}\|}\right).$$

Let $z^r = \frac{x^r}{\|x^r\|}$, then the above inequality becomes

$$[\mathscr{A}(z^{r})^{m-1}]^{\top}\left(\frac{y}{\|x^{r}\|}-z^{r}\right) \geq \frac{\alpha_{r}-1}{\|x^{r}\|^{m-2}}(z^{r})^{\top}\left(\frac{y}{\|x^{r}\|}-z^{r}\right).$$
(15)

Since the sequence $\{z^r\}$ is bounded, there exists a convergent subsequence. Without lose of generality, we denote this subsequence by $\{z^r\}$ and its limit point by z^* . Noting that $\alpha_r \in]0,1[$ and $y \in X$ is an arbitrary given vector, by letting $r \to \infty$, it follows from (15) that $[\mathscr{A}(z^*)^{m-1}]^{\top}(-z^*) \ge 0$, i.e.,

$$\mathscr{A}(z^*)^m \le 0. \tag{16}$$

Next, we show that $z^* \in X$. Since $||x^r|| \to \infty$ as $r \to \infty$, it follows that $\frac{1}{||x^r||} < 1$ with sufficiently large *r*. Furthermore, since $0 \in X$ and *X* is convex, it follows from the above (b) that for sufficiently large *r*,

$$z^{r} = \frac{x^{r}}{\|x^{r}\|} = \left(1 - \frac{1}{\|x^{r}\|}\right)0 + \frac{1}{\|x^{r}\|}x^{r} \in X.$$

Thus, by the fact that the set *X* is closed, we get

 $z^* \in X$.

This, together with (16), contradicts that \mathscr{A} is a positive definite tensor on *X*. Therefore, the $\text{TVI}(X, \mathscr{A}, q)$ has at least one solution when \mathscr{A} is a positive definite tensor on *X*.

Denote the solution set of the $\text{TVI}(X, \mathscr{A}, q)$ by $\text{SOL}(X, \mathscr{A}, q)$. Suppose that the sequence $\{x^k\} \subseteq \text{SOL}(X, \mathscr{A}, q)$ and $x^k \to x^*$ as $k \to \infty$, then it follows that

$$(y-x^k)^{\top} \left[\mathscr{A}(x^k)^{m-1} + q \right] \ge 0 \quad \text{for all } y \in X.$$

Thus, let $k \to \infty$, then we get

$$(y-x^*)^\top \left[\mathscr{A}(x^*)^{m-1} + q \right] \ge 0 \quad \text{for all } y \in X.$$

That is, $x^* \in SOL(X, \mathscr{A}, q)$. So, the solution set of the $TVI(X, \mathscr{A}, q)$ is closed.

Suppose that the solution set of the $\text{TVI}(X, \mathscr{A}, q)$ is unbounded, then there exists a sequence $\{x^k\} \subseteq \text{SOL}(X, \mathscr{A}, q)$ such that $||x^k|| \to \infty$ as $k \to \infty$. Since

$$(y-x^k)^{\top} \left[\mathscr{A}(x^k)^{m-1} + q \right] \ge 0 \quad \text{for all } y \in X,$$

which leads to

$$\left(\frac{y}{\|x^k\|} - \frac{x^k}{\|x^k\|}\right)^\top \left[\mathscr{A}\left(\frac{x^k}{\|x^k\|}\right)^{m-1} + \frac{q}{\|x^k\|^{m-1}}\right] \ge 0$$

Let $k \to \infty$ and denote $x^* = \lim_{k \to \infty} \frac{x^k}{\|x^k\|}$, then we have that

$$x^* \in X$$
 and $-\mathscr{A}(x^*)^m \ge 0$,

which contradicts the condition that \mathscr{A} is a positive definite tensor on *X*. So, the solution set of the $TVI(X, \mathscr{A}, q)$ is bounded.

The proof is complete.

Corollary 4.1 Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set with $0 \in X$ and $\mathscr{A} \in \mathbb{T}_{m,n}$ be a strictly positive definite tensor on X. Then, for any given $q \in \mathbb{R}^n$, the solution set of the $TVI(X, \mathscr{A}, q)$ is nonempty and compact.

Proof Since $0 \in X$, it follows from Proposition 4.2 that a strictly positive definite tensor on *X* is necessary a positive definite tensor on *X*. Thus, the result is obvious from Theorem 4.2.

Theorem 4.3 Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set with $0 \in X$ and $\mathscr{A} \in \mathbb{T}_{m,n}$ be a strictly positive definite tensor on X. Then, for any given $q \in \mathbb{R}^n$, the $TVI(X, \mathscr{A}, q)$ has a unique solution.

Proof By virtue of Theorem 4.1 and Corollary 4.1, the result is straightforward. \Box

Equivalently, we have the following result.

Corollary 4.2 Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set with $0 \in X$ and $\mathscr{A} \in \mathbb{T}_{m,n}$. Suppose that the function $F(x) := \mathscr{A}x^{m-1} + q$ is strictly monotone on X, then the VI(X, F) has a unique solution for any $q \in \mathbb{R}^n$.

Remark 4.1 Suppose that 0 belongs to the strategy set *X* defined by (5) and the set *X* is closed and convex. Then, we can know from Theorem 4.3 that when the tensor \mathscr{A} defined by (7) is strictly positive definite, the *m*-person noncooperative game has a unique Nash equilibrium.

Let $X \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set and the function F be given by $F(x) = \mathscr{A}x^{m-1} + q$, where $\mathscr{A} \in \mathbb{T}_{m,n}$ and $q \in \mathbb{R}^n$. We have showed that, in the case of $0 \in X$, the VI(X, F) has the GUS-property if the function F is strictly monotone on *X*. What would happen if $0 \notin X$? From Lemma 4.1, we know that the VI(*X*,*F*) has the GUS-property if the function *F* is strongly monotone on *X*. A natural question is whether or not there exists a strongly monotone function $F(x) = \mathscr{A}x^{m-1} + q$ (with m > 2) on *X* with $0 \notin X$. The following example gives a positive answer to this question.

Example 4.3 Let

$$X := \{ (u,1)^\top : u \in \mathbb{R}, u \ge 1 \},$$
(17)

and $\mathscr{A} \in \mathbb{T}_{m,n}$ be defined in Example 4.1, then $F(x) := \mathscr{A}x^{m-1} + q$ with any $q \in \mathbb{R}^2$ is strongly monotone on *X*.

For any $x, y \in X$, it follows that there exist $u \ge 1$ and $v \ge 1$ such that $x = (u, 1)^{\top}$ and $y = (v, 1)^{\top}$. Furthermore, for any $q \in \mathbb{R}^2$, we have

$$(x-y)^{\top} [F(x) - F(y)] = (x-y)^{\top} (\mathscr{A} x^3 - \mathscr{A} y^3) = (u-v)^2 (u^2 + uv + v^2);$$

but for $\mu = 1$, we have

$$\mu \|x - y\|^2 = (u - v)^2.$$

Obviously,

$$(u-v)^2(u^2+uv+v^2) \ge 3uv(u-v)^2 \ge 3(u-v)^2 \ge (u-v)^2.$$

Thus, for any $x, y \in X$ and $q \in \mathbb{R}^2$, there exists a constant $\mu = 1$ such that

$$(x-y)^{\top} \left(\mathscr{A}x^3 - \mathscr{A}y^3 \right) \ge \mu ||x-y||^2.$$

So, the function F is strongly monotone on the set X defined by (17).

Therefore, when $X \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set with $0 \notin X$, from Lemma 4.1 (ii), we know that the $\text{TVI}(X, \mathscr{A}, q)$ has a unique solution on X if the function $\mathscr{A}x^{m-1} + q$ is strongly monotone on *X*. We do not know whether the condition of strong monotonicity can be weaken or not in this case.

Before the end of this section, we illustrate that a strictly monotone function $\mathscr{A}x^{m-1} + q$ on $X \subseteq \mathbb{R}^n$ is not necessarily strongly monotone on X when $0 \notin X$.

Example 4.4 Let $\mathscr{A} \in \mathbb{T}_{m,n}$ be defined in Example 4.1 and $X := \{(u,1)^{\top} : u \in \mathbb{R}\}$. Then, for any $q \in \mathbb{R}^2$, the function $\mathscr{A}x^{m-1} + q$ is strictly monotone on X but not strongly monotone on X.

First, from Example 4.1, it is obvious that the tensor \mathscr{A} is strictly positive definite on *X*. Therefore, the function $\mathscr{A}x^{m-1} + q$ is strictly monotone on *X*.

Second, we show that the function $\mathscr{A}x^{m-1} + q$ is not strongly monotone on *X*. Suppose that $\mathscr{A}x^{m-1} + q$ is strongly monotone on *X*, then there exists a scalar $\mu_0 > 0$ such that

$$(x-y)^{\top} \left(\mathscr{A} x^3 - \mathscr{A} y^3 \right) \ge \mu_0 ||x-y||^2 \quad \text{for any } x, y \in X.$$
(18)

Now, take $x^0 = (\sqrt{\mu_0}, 1)^\top \in X$ and $y^0 = (-\frac{\sqrt{\mu_0}}{2}, 1)^\top \in X$, then

$$\begin{aligned} \left(x^{0} - y^{0}\right)^{\top} \left[\mathscr{A}(x^{0})^{3} - \mathscr{A}(y^{0})^{3}\right] \\ &= \left(x_{1}^{0} - y_{1}^{0}\right)^{2} \left[(x_{1}^{0})^{2} + x_{1}^{0}y_{1}^{0} + (y_{1}^{0})^{2}\right] \\ &= \left[\sqrt{\mu_{0}} + \frac{\sqrt{\mu_{0}}}{2}\right]^{2} \left[(\sqrt{\mu_{0}})^{2} - \sqrt{\mu_{0}} \cdot \frac{\sqrt{\mu_{0}}}{2} + \left(\frac{\sqrt{\mu_{0}}}{2}\right)^{2}\right] \\ &= \left(\frac{3\sqrt{\mu_{0}}}{2}\right)^{2} \left(\mu_{0} - \frac{\mu_{0}}{2} + \frac{\mu_{0}}{4}\right) = \frac{27}{16}\mu_{0}^{2} \end{aligned}$$

and

$$\mu_0 \|x^0 - y^0\|^2 = \mu_0 \left[(x_1^0 - y_1^0)^2 + (x_2^0 - y_2^0)^2 \right] = \mu_0 \left[\sqrt{\mu_0} + \frac{\sqrt{\mu_0}}{2} \right]^2 = \frac{9}{4} \mu_0^2.$$

These yield that

$$(x^0 - y^0)^{\top} \left[\mathscr{A}(x^0)^3 - \mathscr{A}(y^0)^3 \right] < \mu_0 \|x^0 - y^0\|^2,$$

which contradicts the inequality (18). So, the function $\mathscr{A}x^{m-1} + q$ is not strongly monotone on *X*.

5 Perspectives and Open Problems

We have just done some initial research for the tensor variational inequality. Many questions need to be answered in the future. Here, we provide two questions as follows.

Question 5.1 How to design effective algorithms to solve the $TVI(X, \mathcal{A}, q)$ by using the specific structure of the tensor \mathcal{A} ?

Question 5.2 In [36], the author investigated the properties of the general polynomial complementarity problem denoted by the PCP(f) with

$$f(x) = \mathscr{A}_m x^{m-1} + \mathscr{A}_{m-1} x^{m-2} + \dots + \mathscr{A}_2 x + \mathscr{A}_1,$$
(19)

where \mathscr{A}_k is a tensor of order k and $\mathscr{A}_k x^{k-1}$ is a polynomial mapping for any $k \in [m]$. If we use the polynomial function f defined by (19) to replace the function $\mathscr{A} x^{m-1} + q$ in the $\text{TVI}(X, \mathscr{A}, q)$, i.e., find a vector $x^* \in X$ such that

$$\langle y - x^*, f(x^*) \rangle \ge 0$$
 for all $y \in X$,

then we call it the polynomial variational inequality, denoted by the PVI(X, f). What are the properties of solution to the PVI(X, f)?

6 Conclusions

In this paper, we studied the tensor variational inequality which is a subclass of the general variational inequality. For the general variational inequality, it is well known that the VI(X, F) has a unique solution if $X \subseteq \mathbb{R}^n$ is nonempty, closed and convex and the continuous function $F : X \to \mathbb{R}^n$ is strongly monotone. However, we proved that F is not strongly monotone when $F(x) = \mathscr{A}x^{m-1} + q$ with m > 2. Hence, the above result for the general variational inequality can not be applied to the tensor variational inequality, we defined two classes of structured tensors and discussed the relationship between them. Furthermore, we showed that the tensor variational inequality $TVI(X, \mathscr{A}, q)$ has the GUS-property when the function $F(x) := \mathscr{A}x^{m-1} + q$ is strictly monotone on X and $0 \in X$. It is possible that the method proposed in this paper can be applied to investigate the GUS-property of the polynomial variational inequality proposed in Question 5.2 of Section 5. In addition, for the study for the polynomial variational inequality given in Question 5.2 of Section 5, we believe that it is a good candidate to use the tool of degree theory.

Acknowledgements The first author's work is partially supported by the National Natural Science Foundation of China (Grant No. 71572125), the second author's work is partially supported by the National Natural Science Foundation of China (Grant No. 11431002), and the third author's work is partially supported by the Hong Kong Research Grant Council (Grant Nos. PolyU 502111, 501212, 501913 and 15302114).

References

- Harker, P.T., Pang, J.-S.: Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications. Math. Program. 48, 161-220 (1990)
- Facchinei, F., Pang, J.-S.: Finite-dimensional Variational Inequalities and Complementarity Problems. Volumes I and II, Springer, (2003)
- Lee, G.M., Tam, N.N., Yen, N.D.: Quadratic Programming and Affine Variational Inequalities A Qualitative Study. Volume 78 of the series Nonconvex Optimization and Its Applications, Springer, US (2005)
- Cottle, R.W., Pang, J.-S., Stone, R.E.: The Linear Complementarity Problem. Academic, Boston (1992)
- Han, J.Y., Xiu, N.H., Qi, H.D.: Nonlinear Complementarity Theory and Algorithms. Shanghai Science and Technology Press, Shanghai (2006). (in Chinese)
- Song, Y., Qi, L.: Properties of some classes of structured tensors. J. Optim. Theory Appl. 165, 854-873 (2015)
- Song, Y., Qi, L.: Properties of tensor complementarity problem and some classes of structured tensors. arXiv: 1412.0113v2, (2015)
- Wang, Y., Huang, Z.H., Bai, X.L.: Exceptionally regular tensors and tensor complementarity problems. Optim. Methods Softw., 31(4), 815-828 (2016)
- Song, Y., Qi, L.: Tensor complementarity problem and semi-positive tensors. J. Optim. Theory Appl. 169(3), 1069-1078 (2016)
- 10. Huang, Z.H., Suo, Y.Y., Wang, J.: On Q-tensors. To appear in Pac. J. Optim. (2016)
- Gowda, M.S., Luo, Z., Qi, L., Xiu, N.: Z-tensors and complementarity problems. arXiv: 1510.07933v1, (2015)
- Bai, X.L., Huang, Z.H., Wang, Y.: Global uniqueness and solvability for tensor complementarity problems. J. Optim. Theory Appl. 170, 72-84 (2016)
- Che, M., Qi, L., Wei, Y.: Positive definite tensors to nonlinear complementarity problems. J. Optim. Theory Appl. 168(2), 475-487 (2016)
- Song, Y., Yu, G.: Properties of solution set of tensor complementarity problem. J. Optim. Theory Appl. 170(1), 85-96 (2016)

- Song, Y., Qi, L.: Strictly semi-positive tensors and the boundedness of tensor complementarity problems. Optim. Lett., 11(7), 1407-1426 (2017)
- Ding, W., Luo, Z., Qi, L.: *P*-tensors, *P*₀-tensors, and tensor complementarity problem. arXiv:1507.06731 (2015)
- Yu, W., Ling, C., He, H.: On the properties of tensor complementarity problems. arXiv:1608.01735 (2016).
- Luo, Z., Qi, L., Xiu, N.: The sparsest solutions to Z-tensor complementarity problems. Optim. Lett. 11(3), 471-482 (2017)
- Huang, Z.H., Qi, L.: Formulating an *n*-person noncooperative game as a tensor complementarity problem, Comput. Optim. Appl. 66(3), 557-576 (2017)
- Lasserre, J.B.: Global optimization with polynomials and the problem of moments. SIAM J. Optim. 11, 796-817 (2001)
- Nie, J., Demmel, J., Sturmfels, B.: Minimizing polynomials via sum of squares over the gradient ideal. Math. Program. Ser. A, **106**(3), 587-606 (2006)
- Nie, J.: The hierarchy of local minimums in polynomial optimization. Math. Program. Ser. B, 151(2), 555-583, (2015)
- Isac, G., Zhao, Y. B.: Exceptional family of elements and the solvability of variational inequalities for unbounded sets in infinite-dimensional Hilbert spaces. J. Math. Anal. Appl. 246, 544-556 (2000)
- Zhao, Y.B., Han, J.Y.: Exceptional family of elements for a variational inequality problem and its applications. J. Global Optim. 14, 313-330 (1999)
- Zhao, Y.B., Han, J.Y., Qi, H.D.: Exceptional families and existence theorems for variational inequality problems. J. Optim. Theory Appl. 101(2), 475-495 (1999)
- Huang, Z.H.: Generalization of an existence theorem for variational inequalities. J. Optim. Theory Appl. 118(3), 567-585 (2003)
- Han, J.Y., Huang, Z.H., Fang, S.C.: Solvability of variational inequality problems. J. Optim. Theory Appl. 122(3), 501-520 (2004)
- Manicino, O.G., Stampacchia, G.: Convex programming and variational inequalities. J. Optim. Theory Appl. 9(1), 3-23, (1972)
- Samelson, H., Thrall, R.M., Wesler, O.: A partition theorem for euclidean *n*-space. P. Am. Math. Soc.
 9, 805-907 (1958)

- Megiddo, N., Kojima, M.: On the existence and uniqueness of solutions in nonlinear complementarity problems. Math. Program. 12, 110-130 (1977)
- Gowda, M.S., Sznajder, R.: Some global uniqueness and solvability results for linear complementarity problems over symmetric cones. SIAM J. Optim. 18, 461-481 (2007)
- Miao, X.H., Huang, Z.H.: GUS-property for Lorentz cone linear complementarity problems on Hilbert spaces. Sci. China Math. 54, 1259-1268 (2011)
- 33. Qi, L.: Eigenvalues of a real supersymmetric tensor. J. Symbolic Comput., 40, 1302-1324 (2005)
- Qi, L.: Symmetric nonnegative tensors and copositive tensors. Linear Algebra Appl., 439, 228-238 (2013)
- Hartman, P., Stampacchia, G.: On some nonlinear elliptic ditterential functional equations. Acta Math.
 115, 153-188 (1966)
- 36. Gowda, M.S.: Polynomial complementarity problems. Pac. J. Optim. 13(2), 227-241 (2017)