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On mean values of some arithmetic functions involving different number fields

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Abstract In 2008, Deza and Varukhina established asymptotic formula for the mean value of the arithmetic function $\tau_{k_1}^K(n)\tau_{k_2}^K(n)\cdots\tau_{k_l}^K(n)$, where K is a quadratic or cyclotomic field, and $\tau_k^K(n)$ is the k-dimensional divisor function in the number field K. Recently, Lü generalized their results to any Galois extension K of the rational field. It seems interesting to deal with similar problems which involve different number fields. In this paper, we are concerned with the mean value of the arithmetic function $\tau_{k_1}^{K_1}(n)\tau_{k_2}^{K_2}(n)\cdots\tau_{k_l}^{K_l}(n)$, where K_j are number fields whose discriminants are relatively prime.

Keywords Divisor function · Number field · Dedekind zeta function · Compositum

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1 Introduction and main results

Arithmetic functions play important roles in number theory and discrete mathematics. Since the behavior of many arithmetic functions is very irregular, we often try to study the average order of arithmetic functions by establishing the asymptotic formulae of

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their mean value. As an example, we choose to mention the k-dimensional divisor problem, which studies the behavior of the mean value of $\tau_k(n)$. Here, as usual $\tau_k(n)$ denotes the number of the representation of n as a product of k natural numbers. See Chap. 13 in Ivić [6] and the references therein for detailed explanation.

Let K be an algebraic number field of finite degree d over the rational field $\mathbb{Q}$. Denote the number of integral ideals in K with norm n by $a_K(n)$. Chandraseknaran and Good [1] showed that $a_K(n)$ is a multiplicative function, and satisfies

$$a_K(n) \le \tau(n)^d, \tag{1.1}$$

where $\tau(k)$ is the divisor function, and $d = [K : \mathbb{Q}]$. Although the number of integral ideals $\sum_{n\le x} a_K(n)$ appeals to many authors [5,8,10,11], it was Chandraseknaran and Narasimhan [2] who first considered the second moment of $a_K(n)$ for a general extension $K/\mathbb{Q}$ of degree d. They proved that

$$\sum_{n\le x} a_K(n)^2 \ll x(\log x)^{d-1}. \tag{1.2}$$

Later, Chandraseknaran and Good [1] showed that if K is a Galois extension of $\mathbb{Q}$ of degree d, then for any $\varepsilon > 0$ and any integer $l \ge 2$, we have

$$\sum_{n\le x} a_K(n)^l = xP_K(\log x) + O(x^{1-\frac{2}{d^l}+\varepsilon}), \tag{1.3}$$

where P_K denotes a suitable polynomial of degree $d^{l-1} - 1$.

Let $\mathfrak{a}$ (with or without subscripts) denote an integral ideal in number field K. The problem to obtain an asymptotic formula for the mean value of the arithmetic function

$$\tau_k^K(n) = \sum_{N(\mathfrak{a}_1\mathfrak{a}_2\ldots\mathfrak{a}_k)=n} 1 = \sum_{n=n_1n_2\ldots n_k} a_K(n_1)a_K(n_2)\ldots a_K(n_k) \tag{1.4}$$

is known as the k-dimensional divisor problem in the field K. Namely, we are interested in the average behavior of the sum

$$\sum_{n\le x} \tau_k^K(n) = \sum_{N(\mathfrak{a}_1\mathfrak{a}_2\ldots\mathfrak{a}_k)\le x} 1.$$

In 1988, Panteleeva [12] considered the divisor problem in the quadratic field $\mathbb{Q}(\sqrt{D})$ and the cyclotomic field $\mathbb{Q}(\zeta)$ ($\zeta^t = 1$). Let D be a squarefree number, $|D| \le \log^2 x$, and $K = \mathbb{Q}(\sqrt{D})$ a quadratic field. Then, for any $k \ge 1$, she proved that

$$\sum_{n\le x} \tau_k^K(n) = xP_k(\log x) + \theta x^{1-\frac{10}{133}k^{-\frac{2}{3}}}(C\log x)^{2k},$$

where P_k is a polynomial of degree $k-1$, $|\theta| \le 1$, and $C > 0$ is an absolute constant. For the cyclotomic field $K = \mathbb{Q}(\zeta)$ ($\zeta^t = 1$), she proved that for any $k \ge 1$

$$\sum_{n\leq x}\tau_k^K(n)=xP_k(\log x)+\theta x^{1-\frac{1}{12}(\varphi(t)k)^{-\frac{2}{3}}}(C\log x)^{\varphi(t)k},$$

where P_k is a polynomial of degree $k-1$, θ is a complex number, $|\theta|\leq 1$, $C>0$ is an absolute constant, and $\varphi(t)$ is the Euler's function.

In 1994, Panteleeva [13] further studied the asymptotic behavior of the product function of several multi-dimensional divisor functions, i.e., $\tau_{k_1}(n)\tau_{k_2}(n)\ldots\tau_{k_l}(n)$, where $l\geq 1$, $k_1,k_2,\ldots,k_l\geq 2$ are integers. Based on some deep results in analytic number theory, she was able to prove

$$\sum_{n\leq x}\tau_{k_1}(n)\tau_{k_2}(n)\ldots\tau_{k_l}(n)=xP_m(\log x)+\theta x^{1-\frac{2}{31}m^{-\frac{2}{x3}}}(C\log x)^m, \tag{1.5}$$

where $l\geq 1$, $k_1,k_2,\ldots,k_l\geq 2$ are integer, $m=k_1k_2\ldots k_l$, $m\leq\log x$, P_m is a polynomial of degree $m-1$, θ is a complex number, $|\theta|\leq 1$, and $C>0$ is an absolute constant.

In 2008, Deza and Varukhina [3] considered the generalized problems of (1.5) in number fields, namely

$$\sum_{n\leq x}\tau_{k_1}^K(n)\tau_{k_2}^K(n)\ldots\tau_{k_l}^K(n). \tag{1.6}$$

They established asymptotic formulae for (1.6) in quadratic and cyclotomic fields. More precisely, they proved that for the quadratic field $K=\mathbb{Q}(\sqrt{D})$,

$$\sum_{n\leq x}\tau_{k_1}^K(n)\tau_{k_2}^K(n)\ldots\tau_{k_l}^K(n)=xP_m(\log x)+\theta x^{1-\frac{1}{15}m^{-\frac{2}{3}}}(C\log x)^{2m}, \tag{1.7}$$

where $l\geq 1$, $k_1,k_2,\ldots,k_l\geq 2$ are integers, $m=k_1k_2\ldots k_l$, $m\leq(\log x)^{\frac{5}{6}}$, P_m is a polynomial of degree $m-1$, θ is a complex number, $|\theta|\leq 1$, and $C>0$ is an absolute constant. For the cyclotomic field $K=\mathbb{Q}(\zeta)$ ($\zeta^t=1$), they proved

$$\sum_{n\leq x}\tau_{k_1}^K(n)\tau_{k_2}^K(n)\ldots\tau_{k_l}^K(n)=xP_m(\log x)+\theta x^{1-\frac{1}{13}(\varphi(t)m)^{-\frac{2}{3}}}(C\log x)^{\varphi(t)m}, \tag{1.8}$$

where $l\geq 1$, $k_1,k_2,\ldots,k_l\geq 2$ are integer, $m=k_1k_2\ldots k_l$, $m\leq(\log x)^{\frac{5}{6}}$, P_m is a polynomial of degree $m-1$, θ is a complex number, $|\theta|\leq 1$, $C>0$ is an absolute constant, and $\varphi(t)$ is the Euler's function.

Recently, Lü [9] proved that the true degrees of the polynomials $P_m(t)$'s in (1.7) and (1.8) are $m=k_1k_2\ldots k_l2^{l-1}$ and $m=k_1k_2\ldots k_l\varphi(t)^{l-1}$, respectively. Then, Lü proved a slightly general result, which states that for a Galois extension $K/\mathbb{Q}$ of degree d and any positive integer l, we have

$$\sum_{n\leq x}\tau_{k_1}^K(n)\tau_{k_2}^K(n)\ldots\tau_{k_l}^K(n)=xP_m(\log x)+O(x^{1-\frac{3}{md+6}+\varepsilon}),\tag{1.9}$$

where $k_1,k_2,\ldots,k_l\geq 2$ are integers, $m=k_1k_2\ldots k_ld^{l-1}$, P_m is a polynomial of degree $m-1$, and $\varepsilon>0$ is an arbitrarily small constant. At the same time, it was shown that for any Abelian extension $K/\mathbb{Q}$, the error term in (1.9) can be strengthened to have the same quality as those in (1.7) and (1.8).

It seems interesting to consider similar problems involving different fields. In this paper, we shall investigate the average behavior of the sum

$$\sum_{n\leq x}\tau_{k_1}^{K_1}(n)\tau_{k_2}^{K_2}(n)\ldots\tau_{k_l}^{K_l}(n),\tag{1.10}$$

where $K_j/\mathbb{Q}$ are number fields with degrees d_j, $j=1,2,\ldots,l$.

First, we choose to give an upper bound in a general setting by a simple argument, which can be regarded as a generalization of Chandraseknaran and Narasimhan's result (1.2).

Theorem 1.1 *Let $K_j/\mathbb{Q}$ be number fields whose degrees satisfy $d_1\leq d_2\leq\cdots\leq d_l$. Then we have*

$$\sum_{n\leq x}\tau_{k_1}^{K_1}(n)\tau_{k_2}^{K_2}(n)\ldots\tau_{k_l}^{K_l}(n)\ll x(\log x)^{\prod_{j=1}^{l}k_j\prod_{j=1}^{l-1}d_j-1}.$$

Obviously, when $l=2$, $K_1=K_2$, $d_1=d_2=d$, and $k_1=k_2=1$, our result coincides with (1.2). When $K_1=K_2=\cdots=K_l=K/\mathbb{Q}$ is a Galois extension of degree d, the upper bound in Theorem 1.1 coincides with the first term in the asymptotic formula (1.9).

In order to simplify matters for (1.10), it seems natural to consider number fields whose discriminants are relatively prime.

Theorem 1.2 *Let $K_j/\mathbb{Q}$ be number fields whose discriminants D_j are relatively prime, i.e., $(D_i,D_j)=1$, $1\leq i\neq j\leq l$. Then we have*

$$\sum_{n\leq x}\tau_{k_1}^{K_1}(n)\tau_{k_2}^{K_2}(n)\ldots\tau_{k_l}^{K_l}(n)=xP_m(\log x)+O(x^{1-\frac{3}{md_1d_2\ldots d_l+6}+\varepsilon}),$$

where $m=k_1k_2\ldots k_l$, P_m is a polynomial of degree $m-1$, and $\varepsilon>0$ is an arbitrarily small constant.

From Theorem 1.2, when $k_1=k_2=\cdots=k_l=1$, we have the following corollary.

Corollary 1.3 *Under the same conditions as Theorem 1.2, we have*

$$\sum_{n\leq x}a_{K_1}(n)a_{K_2}(n)\ldots a_{K_l}(n)=c_{K_1,K_2,\ldots,K_l}x+O(x^{1-\frac{3}{d_1d_2\ldots d_l+6}+\varepsilon}).$$

By combining the arguments in Theorems 1.1 and 1.2, one easily finds the following result.

Corollary 1.4 *Let $K_j/\mathbb{Q}$ be number fields whose degrees satisfy $d_1 \le d_2 \le \cdots \le d_l$. For an integer i_0 satisfying $1 \le i_0 \le l$, the discriminants of K_j $(i_0 \le j \le l)$ are relatively prime: $(D_i, D_j) = 1$, $i_0 \le i \ne j \le l$. Then we have*

$$\sum_{n\le x} \tau_{k_1}^{K_1}(n)\tau_{k_2}^{K_2}(n)\cdots\tau_{k_l}^{K_l}(n) \ll x(\log x)^{\prod_{j=1}^{l} k_j \prod_{j=1}^{i_0-1} d_j - 1}.$$

2 Preliminaries

Let K be an algebraic number field of degree d, and $\zeta_K(s)$ the Dedekind zeta function of the field K. Then, for $\Re s > 1$, it is defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}} N_{K/\mathbb{Q}}(\mathfrak{a})^{-s}, \tag{2.1}$$

where the sum is extended over all integral ideals $\mathfrak{a}$ of the field K, and $N_{K/\mathbb{Q}}(\mathfrak{a})$ is the absolute norm of $\mathfrak{a}$. Then, we can rewrite it as

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}, \tag{2.2}$$

where $a_K(n)$ denotes the number of integral ideals in K with norm n. Since $a_K(n)$ is a multiplicative function, for $\Re s > 1$,

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s} = \prod_p \left(1 + \frac{a_K(p)}{p^s} + \frac{a_K(p^2)}{p^{2s}} + \cdots\right). \tag{2.3}$$

From the definition of $\tau_k^K(n)$ in (1.4), we have that for $\Re s > 1$

$$\zeta_K(s)^k = \sum_{\mathfrak{a}_1}\sum_{\mathfrak{a}_2}\cdots\sum_{\mathfrak{a}_k} (N_{K/\mathbb{Q}}(\mathfrak{a}_1)N_{K/\mathbb{Q}}(\mathfrak{a}_2)\cdots N_{K/\mathbb{Q}}(\mathfrak{a}_k))^{-s} = \sum_{n=1}^{\infty} \frac{\tau_k^K(n)}{n^s}. \tag{2.4}$$

Since $\tau_k^K(n)$ is also a multiplicative function, and

$$\tau_k^K(n) = \sum_{n=n_1n_2\cdots n_k} a_K(n_1)\cdots a_K(n_k) \ll \sum_{n=n_1n_2\cdots n_k} (\tau(n_1)\cdots\tau(n_k))^{d-1} \ll n^{\varepsilon} \tag{2.5}$$

for any $\varepsilon > 0$, we can rewrite (2.4) for $\Re s > 1$ as

$$\zeta_K(s)^k = \prod_p \left(1 + \frac{\tau_k^K(p)}{p^s} + \frac{\tau_k^K(p^2)}{p^{2s}} + \cdots\right). \tag{2.6}$$

On the other hand, from (2.3), we have

$$\begin{aligned}\zeta_K(s)^k &= \left(\sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}\right)^k = \prod_p \left(1 + \frac{a_K(p)}{p^s} + \frac{a_K(p^2)}{p^{2s}} + \cdots\right)^k \\ &= \prod_p \left(1 + \frac{k a_K(p)}{p^s} + \frac{k a_K(p^2) + \frac{k(k-1)}{2} a_K(p)}{p^{2s}} + \cdots\right).\end{aligned} \tag{2.7}$$

By comparing (2.6) with (2.7), we have the following lemma.

Lemma 2.1 *Let K be an algebraic number field of degree d, and the function $\tau_k^K(n)$ defined in (1.4). Then, for any prime number p, we have*

$$\tau_k^K(p) = k a_K(p),$$

where $a_K(p)$ denotes the number of integral ideals in K with norm p.

Our next lemma gives an upper bound for $a_K(n)$, which improves (1.1).

Lemma 2.2 *Let $K/\mathbb{Q}$ be a number field of degree d. Then we have*

$$a_K(n) \le \tau_d(n).$$

Proof Since $a_K(n)$ and $\tau_d(n)$ are both multiplicative functions, it suffices to prove $a_K(p^r) \le \tau_d(p^r)$, where p is a rational prime number, and r is a natural number. Let $\mathfrak{O}_K$ be the ring of algebraic integers in K. Denote by $p\mathfrak{O}_K$ the principal ideal in K generated by p. It is well known that $p\mathfrak{O}_K$ can be uniquely factorized into a product of prime ideals in K

$$p\mathfrak{O}_K = \mathfrak{B}_1^{e_1} \mathfrak{B}_2^{e_2} \cdots \mathfrak{B}_g^{e_g},$$

where $\mathfrak{B}_j$ are distinct prime ideals in K with norm p^{f_j}, $j = 1, 2 \ldots, g$. These e_j's and f_j's satisfy the relation

$$e_1 f_1 + e_2 f_2 + \cdots + e_g f_g = d. \tag{2.8}$$

If $\mathfrak{a}$ is a non-zero integral ideal satisfying $N_{K/\mathbb{Q}}(\mathfrak{a}) = p^r$, so that we have

$$\mathfrak{a} = \mathfrak{B}_1^{r_1} \mathfrak{B}_2^{r_2} \cdots \mathfrak{B}_g^{r_g}.$$

By checking the norms on both sides, we have

$$r_1 f_1 + r_2 f_2 + \cdots + r_g f_g = r.$$

Then, it is easy to find that $a_K(p^r)$, the number of integral ideals with norm p^r, equals the number of solutions of the Diophantine equation

$$f_1 x_1 + f_2 x_2 + \cdots + f_g x_g = r. \tag{2.9}$$

Obviously, the number of solutions of (2.9) is not more than the number of solutions of Diophantine equation (note that $g \leq d$)

$$x_1 + x_2 + \cdots + x_d = r. \tag{2.10}$$

It is well known that the number of solutions of (2.10) equals

$$\binom{r+d-1}{d-1} = \tau_d(p^r).$$

Hence, we have $a_K(p^r) \leq \tau_d(p^r)$. This completes the proof of Lemma 2.2. □

Let KL be the compositum of two number fields K and L, whose discriminants are relatively prime. We would like to establish a relationship between $a_{KL}(p)$ and $a_K(p)a_L(p)$.

Lemma 2.3 *Suppose K and L are two field extensions of $\mathbb{Q}$ such that*

$$[K : \mathbb{Q}] = m, [L : \mathbb{Q}] = n, (D_K, D_L) = 1,$$

where D_K and D_L are discriminants of K and L respectively. Hence, for any prime number p, we have that

$$a_{KL}(p) = a_K(p)a_L(p).$$

Proof Since the discriminants D_K and D_L are relatively prime, their compositum KL is a field of degree mn, i.e., $[KL : \mathbb{Q}] = [K : \mathbb{Q}][L : \mathbb{Q}]$. In this case, the inclusion $\mathfrak{O}_K\mathfrak{O}_L \subseteq \mathfrak{O}_{KL}$ is an equality.

Let $A_1 \subset \mathfrak{O}_K$ and $A_2 \subset \mathfrak{O}_L$ be two integral ideals with $N_{K/\mathbb{Q}}(A_1) = N_{L/\mathbb{Q}}(A_2) = p$. By the fundamental theorem of Abel group, we could find integral bases $\{\omega_1, \ldots, \omega_m\}$, $\{\upsilon_1, \ldots, \upsilon_n\}$ of $\mathfrak{O}_K$, $\mathfrak{O}_L$ such that

$$\mathfrak{O}_K = \mathbb{Z}\omega_1 \oplus \cdots \oplus \mathbb{Z}\omega_m,\ A_1 = \mathbb{Z}\alpha_1\omega_1 \oplus \cdots \oplus \mathbb{Z}\alpha_m\omega_m;$$
$$\mathfrak{O}_L = \mathbb{Z}\upsilon_1 \oplus \cdots \oplus \mathbb{Z}\upsilon_n,\ A_2 = \mathbb{Z}\beta_1\upsilon_1 \oplus \cdots \oplus \mathbb{Z}\beta_n\upsilon_n$$

where $\alpha_i, \beta_j \in \mathbb{Z}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Hence, we know

$$N_{K/\mathbb{Q}}(A_1) = |\prod_i \alpha_i| = N_{L/\mathbb{Q}}(A_2) = |\prod_j \beta_j| = p.$$

On noting that $\mathfrak{O}_K\mathfrak{O}_L = \mathfrak{O}_{KL}$, we learn that $\{\omega_1\upsilon_1, \ldots, \omega_1\upsilon_n, \ldots, \omega_m\upsilon_n\}$ is an integral basis of $\mathfrak{O}_{KL}$. Take $A = A_1\mathfrak{O}_L + A_2\mathfrak{O}_K \subset \mathfrak{O}_{KL}$. We have

$$\begin{aligned} N_{KL/\mathbb{Q}}(A) &= |\mathfrak{O}_{KL}/A| = |\mathfrak{O}_K\mathfrak{O}_L/(A_1\mathfrak{O}_L + A_2\mathfrak{O}_K)| \\ &= |\oplus_{i,j}\,\mathbb{Z}\omega_i\upsilon_j/(\oplus_{i,j}\mathbb{Z}\alpha_i\omega_i\upsilon_j + \oplus_{i,j}\mathbb{Z}\beta_j\omega_i\upsilon_j)| \\ &= |\prod_{i,j}(\alpha_i, \beta_j)| \\ &= p. \end{aligned}$$

Hence, we have

$$a_{KL}(p) \geq a_K(p)a_L(p). \tag{2.11}$$

On the other hand, for any integral ideal $A \subset \mathfrak{O}_{KL}$ with $N_{KL/\mathbb{Q}}(A) = p$, let

$$A_1 = N_{KL/K}(A),\ A_2 = N_{KL/L}(A).$$

Note that $N_{KL/\mathbb{Q}}(A) = N_{K/\mathbb{Q}}(N_{KL/K}(A)) = N_{L/\mathbb{Q}}(N_{KL/L}(A))$, we have that

$$N_{K/\mathbb{Q}}(A_1) = N_{L/\mathbb{Q}}(A_2) = p.$$

What is more, if there is another integral ideal A' in $\mathfrak{O}_{KL}$ such that

$$A_1 = N_{KL/K}(A) = N_{KL/K}(A'),\ A_2 = N_{KL/L}(A) = N_{KL/L}(A'),$$

we would have that $A, A' | A_1\mathfrak{O}_L, A_2\mathfrak{O}_K$. Then we have

$$A, A' \supset A_1\mathfrak{O}_L + A_2\mathfrak{O}_K.$$

Together with

$$N_{KL/\mathbb{Q}}(A) = N_{KL/\mathbb{Q}}(A') = N_{KL/\mathbb{Q}}(A_1\mathfrak{O}_L + A_2\mathfrak{O}_K) = p,$$

we have $A = A' = A_1\mathfrak{O}_L + A_2\mathfrak{O}_K$.

Hence, we have

$$a_{KL}(p) \leq a_K(p)a_L(p). \tag{2.12}$$

By (2.11) and (2.12), we have that for any prime number p,

$$a_{KL}(p) = a_K(p)a_L(p).$$

This completes the proof of Lemma 2.3. □

In order to control the error terms in Theorem 1.2, we take a short cut to cite one result in Heath-Brown [4].

Lemma 2.4 *Let K be an algebraic number field of degree n, Then*

$$\zeta_K(1/2+it) \ll_K t^{\frac{n}{6}+\varepsilon}, \qquad (t \geq 1)$$

for any fixed $\varepsilon > 0$.

3 Proof of Theorem 1.1.

Let $K/\mathbb{Q}$ be a number field of degree d. Recall that

$$\tau_k^K(n) = \sum_{n=n_1 n_2 \cdots n_k} a_K(n_1) \cdots a_K(n_k). \tag{3.1}$$

By Lemma 2.2, we have

$$\tau_k^K(n) \leq \sum_{n=n_1 n_2 \cdots n_k} \tau_d(n_1) \cdots \tau_d(n_k) = \tau_{dk}(n). \tag{3.2}$$

Then we have

$$\sum_{n\leq x} \tau_{k_1}^{K_1}(n)\tau_{k_2}^{K_2}(n) \cdots \tau_{k_l}^{K_l}(n) \leq \sum_{n\leq x} \tau_{d_1k_1}(n)\tau_{d_2k_2}(n) \cdots \tau_{d_{l-1}k_{l-1}}(n)\tau_{k_l}^{K_l}(n). \tag{3.3}$$

Define an L-function associated to the function $\tau_{d_1k_1}(n) \cdots \tau_{d_{l-1}k_{l-1}}(n)\tau_{k_l}^{K_l}(n)$ in the half-plane $\Re s > 1$

$$L_{k_1,k_2,\ldots,k_l}^{K_1,K_2,\ldots,K_l}(s) = \sum_{n=1}^{\infty} \frac{\tau_{d_1k_1}(n)\tau_{d_2k_2}(n) \cdots \tau_{d_{l-1}k_{l-1}}(n)\tau_{k_l}^{K_l}(n)}{n^s}, \tag{3.4}$$

for it is absolutely convergent in this region. Since $\tau_{d_1k_1}(n)\tau_{d_2k_2}(n) \cdots \tau_{d_{l-1}k_{l-1}}(n)$ $\tau_{k_l}^{K_l}(n)$ is multiplicative, for $\Re s > 1$, we can write

$$L_{k_1,k_2,\ldots,k_l}^{K_1,K_2,\ldots,K_l}(s) = \prod_p \Bigg(1 + \frac{\tau_{d_1k_1}(p) \cdots \tau_{d_{l-1}k_{l-1}}(p)\tau_{k_l}^{K_l}(p)}{p^s} + \frac{\tau_{d_1k_1}(p^2) \cdots \tau_{d_{l-1}k_{l-1}}(p^2)\tau_{k_l}^{K_l}(p^2)}{p^{2s}} + \cdots\Bigg),$$

where the product is over all primes. By Lemma 2.1, we have

$$L_{k_1,k_2,\ldots,k_l}^{K_1,K_2,\ldots,K_l}(s) = \prod_p \left(1 + \frac{k_1k_2 \cdots k_{l-1}d_1d_2 \cdots d_{l-1}k_l a_{K_l}(p)}{p^s} + \cdots\right). \tag{3.5}$$

On the other hand, from (2.3), we have

$$\zeta_{K_l}(s)^{k_1k_2\cdots k_l d_1 d_2\cdots d_{l-1}} = \prod_p \left(1 + \frac{k_1k_2\cdots k_l d_1 d_2\cdots d_{l-1} a_{K_l}(p)}{p^s} + \cdots\right). \quad (3.6)$$

From (3.5) and (3.6), we have

$$L^{K_1,K_2,\ldots,K_l}_{k_1,k_2,\ldots,k_l}(s) = \zeta_{K_l}(s)^{\prod_{j=1}^{l} k_j \prod_{j=1}^{l-1} d_j} U(s), \quad (3.7)$$

where $U(s)$ denotes a Dirichlet series, which is absolutely convergent for $\sigma > \frac{1}{2}$ and uniformly convergent for $\sigma > \frac{1}{2} + \varepsilon$. Therefore, $L^{K_1,K_2,\ldots,K_l}_{k_1,k_2,\ldots,k_l}(s)$ admits a meromorphic continuation to the half-plane Re$s > \frac{1}{2}$, and only has a pole $s = 1$ of order $\prod_{j=1}^{l} k_j \prod_{j=1}^{l-1} d_j$ in this region.

By (3.4) and (3.7), we easily find that

$$\sum_{n\le x} \tau_{d_1k_1}(n)\tau_{d_2k_2}(n)\cdots\tau_{d_{l-1}k_{l-1}}(n)\tau^{K_l}_{k_l}(n) \sim cx(\log x)^{\prod_{j=1}^{l} k_j \prod_{j=1}^{l-1} d_j - 1}. \quad (3.8)$$

This, together with (3.3), completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.2.

Let $K_j/\mathbb{Q}$ be number fields whose discriminants D_j are relatively prime, i.e., $(D_i, D_j) = 1$, $1 \le i \ne j \le l$. Define

$$L^{K_1,K_2,\ldots,K_l}_{k_1,k_2,\ldots,k_l}(s) = \sum_{n=1}^{\infty} \frac{\tau^{K_1}_{k_1}(n)\tau^{K_2}_{k_2}(n)\cdots\tau^{K_l}_{k_l}(n)}{n^s}, \quad (4.1)$$

Since $\tau^{K_1}_{k_1}(n)\tau^{K_2}_{k_2}(n)\cdots\tau^{K_l}_{k_l}(n)$ is multiplicative, for $\Re s > 1$, we can write

$$L^{K_1,K_2,\ldots,K_l}_{k_1,k_2,\ldots,k_l}(s) = \prod_p \left(1 + \frac{\tau^{K_1}_{k_1}(p)\tau^{K_2}_{k_2}(p)\cdots\tau^{K_l}_{k_l}(p)}{p^s} + \frac{\tau^{K_1}_{k_1}(p^2)\tau^{K_2}_{k_2}(p^2)\cdots\tau^{K_l}_{k_l}(p^2)}{p^{2s}} + \cdots\right),$$

where the product is over all primes. By Lemmas 2.2 and 2.3, we have

$$L^{K_1,K_2,\ldots,K_l}_{k_1,k_2,\ldots,k_l}(s) = \prod_p \left(1 + \frac{k_1k_2\cdots k_l a_{K_1}(p)a_{K_2}(p)\cdots a_{K_l}(p)}{p^s} + \cdots\right) \quad (4.2)$$

$$= \prod_p \left(1 + \frac{k_1k_2\cdots k_l a_{K_1K_2\cdots K_l}(p)}{p^s} + \cdots\right). \quad (4.3)$$

From (2.3), we have

$$L_{k_1,k_2,\ldots,k_l}^{K_1,K_2,\ldots,K_l}(s) = \zeta_{K_1K_2\cdots K_l}(s)^{\prod_{j=1}^{l} k_j} U_1(s), \tag{4.4}$$

where $U_1(s)$ denotes a Dirichlet series, which is absolutely convergent for $\sigma > \frac{1}{2}$ and uniformly convergent for $\sigma > \frac{1}{2}+\varepsilon$. Therefore, $L_{k_1,k_2,\ldots,k_l}^{K_1,K_2,\ldots,K_l}(s)$ admits a meromorphic continuation to the half-plane $\mathrm{Re}s > \frac{1}{2}$, and only has a pole $s = 1$ of order $\prod_{j=1}^{l} k_j :=$ m in this region.

By Perron's formula (see Proposition 5.54 in [7]), we have

$$\sum_{n\le x} \tau_{k_1}^{K_1}(n)\tau_{k_2}^{K_2}(n)\cdots\tau_{k_l}^{K_l}(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} L_{k_1,k_2,\ldots,k_l}^{K_1,K_2,\ldots,K_l}(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{4.5}$$

where $b = 1+\varepsilon$ and $1 \le T \le x$ is a parameter to be chosen later.

Next, we move the integration to the parallel segment with $\mathrm{Re}s = \frac{1}{2}+\varepsilon$. By Cauchy's residue theorem, we have

$$\begin{aligned}
&\sum_{n\le x} \tau_{k_1}^{K_1}(n)\tau_{k_2}^{K_2}(n)\cdots\tau_{k_l}^{K_l}(n)\\
&= \frac{1}{2\pi i}\left\{\int_{\frac{1}{2}+\varepsilon-iT}^{\frac{1}{2}+\varepsilon+iT} + \int_{\frac{1}{2}+\varepsilon+iT}^{b+iT} + \int_{b-iT}^{\frac{1}{2}+\varepsilon-iT}\right\} L_{k_1,k_2,\ldots,k_l}^{K_1,K_2,\ldots,K_l}(s)\frac{x^s}{s}ds\\
&\quad + \mathrm{Res}_{s=1} L_{k_1,k_2,\ldots,k_l}^{K_1,K_2,\ldots,K_l}(s)\frac{x^s}{s} + O\left(\frac{x^{1+\varepsilon}}{T}\right)\\
&:= xP_m(\log x) + J_1 + J_2 + J_3 + O\left(\frac{x^{1+\varepsilon}}{T}\right),
\end{aligned} \tag{4.6}$$

where $P_m(t)$ denotes a suitable polynomial in t of degree $m-1$.

By Lemma 2.4, and the Phragmen-Lindelöf principle for a strip (see e.g. Theorem 5.53 in Iwaniec and Kowalski [7]), we have that for $\frac{1}{2} \le \sigma \le 1+\varepsilon$

$$\zeta_{K_1K_2\cdots K_l}(\sigma+it) \ll (1+|t|)^{\frac{d_1d_2\cdots d_l}{3}(1-\sigma)+\varepsilon}. \tag{4.7}$$

Therefore, we have for $\frac{1}{2} \le \sigma \le 1+\varepsilon$

$$\zeta_{K_1K_2\cdots K_l}^{m}(\sigma+it) \ll (|t|+1)^{\frac{md_1d_2\cdots d_l}{3}(1-\sigma)+\varepsilon}. \tag{4.8}$$

For J_1, by (4.4), we have

$$
\begin{aligned}
J_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \int_1^T \left| L_{k_1,k_2,\ldots,k_l}^{K_1,K_2,\ldots,K_l}(s) \right| t^{-1} dt \\
&\ll x^{\frac{1}{2}+\varepsilon} + x^{1/2+\varepsilon} \int_1^T \left| \zeta_{K_1 K_2 \cdots K_l}^m (1/2+\varepsilon+it) \right| t^{-1} dt,
\end{aligned}
$$

where we have used that $U_1(s)$ is absolutely convergent in the region $\mathrm{Re} s \geqslant 1/2+\varepsilon$ and behaves as $O(1)$ there. Then, by (4.8), we have

$$
\begin{aligned}
J_1 &\ll x^{\frac{1}{2}+\varepsilon} + x^{1/2+\varepsilon} \log T \max_{T_1 \le T} \left\{ T_1^{-1} \int_{T_1/2}^{T_1} \left| \zeta_{K_1 K_2 \cdots K_l}(1/2+\varepsilon+it) \right|^m dt \right\} \\
&\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} \log T \max_{T_1 \le T} \left\{ T_1^{-1} \int_{T_1/2}^{T_1} t^{\frac{md_1 d_2 \cdots d_l}{6}+\varepsilon} dt \right\} \\
&\ll x^{\frac{1}{2}+\varepsilon} + x^{\frac{1}{2}+\varepsilon} T^{\frac{md_1 d_2 \cdots d_l}{6}+\varepsilon}.
\end{aligned} \tag{4.9}
$$

For the integrals over the horizontal segments, we have

$$
\begin{aligned}
J_2 + J_3 &\ll \int_{\frac{1}{2}+\varepsilon}^b x^\sigma \left| \zeta_{K_1 K_2 \cdots K_l}^m (\sigma+iT) \right| T^{-1} d\sigma \\
&\ll \max_{\frac{1}{2}+\varepsilon \le \sigma \le b} x^\sigma T^{\frac{md_1 d_2 \cdots d_l}{3}(1-\sigma)+\varepsilon} T^{-1} \\
&= \max_{\frac{1}{2}+\varepsilon \le \sigma \le b} \left(\frac{x}{T^{\frac{md_1 d_2 \cdots d_l}{3}}} \right)^\sigma T^{\frac{md_1 d_2 \cdots d_l}{3}-1+\varepsilon} \\
&\ll \frac{x^{1+\varepsilon}}{T} + x^{\frac{1}{2}+\varepsilon} T^{\frac{md_1 d_2 \cdots d_l}{6}-1+\varepsilon}.
\end{aligned} \tag{4.10}
$$

From (4.6), (4.9), and (4.10), we have

$$
\begin{aligned}
\sum_{n \le x} \tau_{k_1}^{K_1}(n) \tau_{k_2}^{K_2}(n) \cdots \tau_{k_l}^{K_l}(n) &= x P_m(\log x) + O(x^{1+\varepsilon} T^{-1}) \\
&\quad + O(x^{\frac{1}{2}+\varepsilon} T^{\frac{md_1 d_2 \cdots d_l}{6}+\varepsilon}).
\end{aligned} \tag{4.11}
$$

On taking $T = x^{\frac{3}{md_1 d_2 \cdots d_l + 6}}$ in (4.11), we have

$$
\sum_{n \le x} \tau_{k_1}^K(n) \tau_{k_2}^K(n) \cdots \tau_{k_l}^K(n) = x P_m(\log x) + O(x^{1-\frac{3}{md_1 d_2 \cdots d_l+6}+\varepsilon}).
$$

This completes the proof of Theorem 1.2. □

At the end of this section, we complete the proof of Corollary 1.4.

Proof of Corollary 1.4 We have

$$\sum_{n\le x}\tau_{k_1}^{K_1}(n)\tau_{k_2}^{K_2}(n)\cdots\tau_{k_l}^{K_l}(n)\le\sum_{n\le x}\tau_{d_1k_1}(n)\cdots\tau_{d_{i_0-1}k_{i_0-1}}(n)\tau_{k_{i_0}}^{K_{i_0}}(n)\cdots\tau_{k_l}^{K_l}(n).$$

Then, the generating function of the right-hand side is

$$\zeta_{K_{i_0}K_{i_0+1}\cdots K_l}(s)^{\prod_{j=1}^{l}k_j\prod_{j=1}^{i_0-1}d_j}U_2(s),$$

where $U_2(s)$ denotes a Dirichlet series, which is absolutely convergent for $\sigma>\frac{1}{2}$ and uniformly convergent for $\sigma>\frac{1}{2}+\varepsilon$. This gives the proof of Corollary 1.4. □

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