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GLOBAL SYNCHRONISING BEHAVIOR OF EVOLUTION EQUATIONS WITH EXPONENTIALLY GROWING NONAUTONOMOUS FORCING

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Abstract. This work is concerned with the following nonautonomous evolutionary system on a Banach space X,

$$x_t + Ax = f\left(x, h(t)\right), \tag{0.1}$$

where A is a hyperbolic sectorial operator on X, the nonlinearity $f \in C(X^\alpha \times X, X)$ is Lipschitz in the first variable, the nonautonomous forcing $h \in C(\mathbb{R}, X)$ is μ-subexponentially growing for some $\mu > 0$ (see (3.4) below for definition). Under some reasonable assumptions, we first establish an existence result for a unique nonautonomous hyperbolic equilibrium for the system in the framework of cocycle semiflows. We then demonstrate that the system exhibits a global synchronising behavior with the nonautonomous forcing h as time varies. Finally, we apply the abstract results to stochastic partial differential equations with additive white noise and obtain stochastic hyperbolic equilibria for the corresponding systems.

1. **Introduction.** The notion of equilibria is a fundamental concept in the study of the long time behaviour of dynamical systems. The study of the existence and stability of equilibria is of great interests in both mathematics and physics. In contrast to the autonomous dynamical systems, the existence of equilibria of nonautonomous or random dynamical systems is still a more difficult and subtle problem.

There are many works on the studies of dynamics of evolutionary systems under small bounded perturbations; see e.g. [4, 5, 7, 8, 11, 18, 20–22, 24, 25]. Generally, autonomous hyperbolic equilibria are locally structural stable under small bounded autonomous or nonautonomous perturbations; see e.g. [3–5]. This fact usually leads to the structural stability of gradient attractors; see e.g. [4, 5, 11]. However, it is also of great importance to study the effect of unbounded nonautonomous perturbation on terms of a dynamical system when we consider such systems as models of

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real phenomena, because there are many examples of unbounded nonautonomous perturbations such as (sub)linear and (sub)exponential ones.

In this paper, we first consider the nonautonomous system (0.1). To have a better understanding of the dynamics of (0.1), as usual we embed the system into the following cocycle system:

$$x_t + Ax = f\left(x, p(t)\right), \qquad p \in \mathcal{H}, \tag{1.1}$$

where $\mathcal{H} := \mathcal{H}[h]$ is the hull of h (see (3.7) below for definition). Then (1.1) generates a cocycle semiflow $\varphi = \varphi(t,p)x$ on X^α with base space $\mathcal{H}$ and driving system θ_t, where θ_t is the shift operator on $\mathcal{H}$ for each $t \in \mathbb{R}$. Suppose the Lipschitz constant of f and the subexponential growth rate μ of h are sufficiently small. We first prove that the system φ possesses a unique nonautonomous equilibrium $\Gamma \in C(\mathcal{H}, X^\alpha)$. Next we show the equilibrium is hyperbolic by proving the existences of the section unstable manifold $W^u(\Gamma,\cdot) : \mathcal{H} \to X^\alpha$ and the section stable manifold $W^s(\Gamma,\cdot) : \mathcal{H} \to X^\alpha$ of Γ. Meanwhile, we demonstrate that both $W^u(\Gamma,\cdot) : \mathcal{H} \to X^\alpha$ and $W^s(\Gamma,\cdot) : \mathcal{H} \to X^\alpha$ are continuous in the sense of Hausdorff distance. In consequence, if h is periodic (resp. pseudo periodic, almost periodic, uniformly almost automorphic), then $\Gamma(\theta_t h)$, $W^u(\Gamma, \theta_t h)$ and $W^s(\Gamma, \theta_t h)$ are also periodic (resp. pseudo periodic, almost periodic, uniformly almost automorphic). Moreover, we prove that $W^u(\Gamma, \theta_t h)$ exponentially forward attracts every point in X^α through φ. Accordingly, the system (0.1) exhibits a *global synchronising behavior* with the nonautonomous forcing h as time varies.

In the present paper, we also study the existence and asymptotic stability of stochastic hyperbolic equilibrium of the following stochastic equation with additive white noise

$$du + Audt = f(u)dt + dW(t) \tag{1.2}$$

on a Polish space X, where A is also a hyperbolic sectorial operator on X, $f : X^\alpha \to X$ is Lipschitz continuous, W is a X-valued Wiener process on the classic Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$. As far as I know, there are few papers to discuss the existence of stochastic hyperbolic equilibria for stochastic systems. When the sectorial operator A is positive, the existence of an exponentially stable non-trivial equilibrium was obtained by Caraballo et al. [6]. Here we treat a general case when the operator A is hyperbolic. We first need to construct a continuous stationary solution $Z(\vartheta.\omega) : \mathbb{R} \to X^\alpha$ for linear Langevin stochastic partial differential equation (SPDE) in X^α:

$$dz + Azdt = dW(t),$$

where $\{\vartheta_t\}_{t\in\mathbb{R}}$ is a family of measure preserving transformations on Ω. It is worth noting that the stationary solution $Z(\vartheta.\omega) : \mathbb{R} \to X^\alpha$ is *subexponentially growing* for each $\omega \in \Omega$. In other words, the random variable $Z : \Omega \to X^\alpha$ is *tempered* (see Definition 5.5 below).

By a transformation

$$v(t) = u(t) - Z(\vartheta_t\omega),$$

then for each fixed $\omega \in \Omega$, SPDE (1.2) becomes the following nonautonomous equation

$$\frac{dv}{dt} + Av = f\big(v + Z(\vartheta_t\omega)\big). \tag{1.3}$$

Then by applying the methods dealing with (0.1), the stochastic system (1.3) is proved to possess a unique tempered stochastic hyperbolic equilibrium $\Xi : \Omega \to X^\alpha$, whose ω-section $\Xi(\omega)$ not only backward attracts every point in its ω-section unstable manifold $W^u(\Xi, \omega)$ but also forward attracts every point in its ω-section stable manifold $W^s(\Xi, \omega)$. Other existence results on invariant manifolds for stochastic parabolic and hyperbolic differential equations with additive or multiplicative noise can be refereed to Duan et al. [15, 16], Lu and Schmalfuss [23], Brune and Schmalfuss [2]. We also point out that for each $\omega \in \Omega$, the original system (1.2) exhibits a global synchronising behavior with $h(t) = Z(\vartheta_t\omega)$, $t \in \mathbb{R}$ as time vaies.

This paper is organized as follows. In Section 2 and Section 3, we present basic definitions, and the mathematical setting of the system (1.1), respectively. Section 4 contains the proof of our main abstract results. In Section 5, we apply the abstract results to stochastic partial differential equations with additive white noise.

2. **Preliminaries.** In this section we introduce some basic definitions and notions [9, 10].

Let X be a complete metric space with metric $d(\cdot, \cdot)$. Given $M \subset X$, we denote $\overline{M}$, $\operatorname{int} M$, ∂M and M^c the *closure, interior, boundary and complement* of M of X, respectively. A set $U \subset X$ is called a *neighborhood* of $M \subset X$, if $\overline{M} \subset \operatorname{int} U$.

The *Hausdorff semidistance* and the *Hausdorff distance* in X are defined, respectively, as

$$H_X(M, N) = \sup_{x \in M} d(x, N), \qquad \forall\, M, N \subset X,$$
$$\delta_X(M, N) = \max\{H_X(M, N),\ H_X(M, N)\}, \qquad \forall\, M, N \subset X.$$

2.1. **Cocycle semiflows.** A nonautonomous system consists of a "base flow" and a "cocycle semiflow" that is in some sense driven by the base flow.

A *base flow* $\{\theta_t\}_{t\in\mathbb{R}}$ is a group of continuous transformations from a metric space Σ into itself such that

- $\theta_0 = id_\Sigma$,
- $\theta_t \circ \theta_s = \theta_{t+s}$ for all $t, s \in \mathbb{R}$,
- $\theta_t\Sigma = \Sigma$ for all $t \in \mathbb{R}$.

Definition 2.1. A *cocycle semiflow* φ on the phase space X over θ is a continuous mapping $\varphi : \mathbb{R}^+ \times \Sigma \times X \to X$ satisfying

- $\varphi(0, \sigma, x) = x$,
- $\varphi(t + s, \sigma, x) = \varphi(t, \theta_s\sigma, \varphi(s, \sigma, x))$ (cocycle property).

We usually denote $\varphi(t, \sigma)x := \varphi(t, \sigma, x)$. Then $\{\varphi(t, \sigma)\}_{t\geq 0,\, \sigma\in\Sigma}$ can be viewed as a family of continuous mappings on X.

2.2. **Nonautonomous equilibrium and its section invariant manifolds.** Consider a nonautonomous system $(\varphi, \theta)_{X,\Sigma}$.

For convenience in statement, a family of subsets $\mathcal{B} = \{B_\sigma\}_{\sigma\in\Sigma}$ of X is called a *nonautonomous set* in X.

Let $\mathcal{B} = \{B_\sigma\}_{\sigma\in\Sigma}$ be a nonautonomous set. For convenience, we will rewrite B_σ as $\mathcal{B}(\sigma)$, called the *σ-section* of $\mathcal{B}$. We also denote $\mathscr{P}(\mathcal{B})$ the union of the sets $\mathcal{B}(\sigma) \times \{\sigma\}$ $(\sigma \in \Sigma)$, i.e.,

$$\mathscr{P}(\mathcal{B}) = \bigcup_{\sigma\in\Sigma} \mathcal{B}(\sigma) \times \{\sigma\}.$$

Note that $\mathscr{P}(\mathcal{B})$ is a subset of $X \times \Sigma$.

A nonautonomous set $\mathcal{B}$ is said to be closed (resp. open, compact), if $\mathscr{P}(\mathcal{B})$ is closed (resp. open, compact) in $X \times \Sigma$.

A nonautonomous set $\mathcal{B}$ is said to be *invariant* under φ if

$$\varphi(t,\sigma)\mathcal{B}(\sigma)=\mathcal{B}(\theta_t\sigma),\qquad \sigma\in\Sigma.$$

Let $\mathcal{B}$ and $\mathcal{C}$ be two nonautonomous subsets of X. We say that $\mathcal{B}$ *pullback (resp. forward) attracts* $\mathcal{C}$ under φ if for any $\sigma\in\Sigma$,

$$\lim_{t\to\infty} H_X(\varphi(t,\theta_{-t}\sigma)\mathcal{C}(\theta_{-t}\sigma),\, \mathcal{B}(\sigma))=0.$$

$$\Big(resp.\ \lim_{t\to\infty} H_X(\varphi(t,\sigma)\mathcal{C}(\sigma),\, \mathcal{B}(\theta_t\sigma))=0.\Big)$$

Let $J\subset\mathbb{R}$ be an interval. A mapping $\gamma: J\to X$ is called a *solution* of φ on J, if there exists $\sigma\in\Sigma$ such that

$$\gamma(t)=\varphi(t-s,\theta_s\sigma)\gamma(s),\qquad \forall\, t,s\in J,\ \ t\ge s.$$

A solution on $J=\mathbb{R}$ is called a *full solution*.

Remark 1. We will also call a solution γ defined as above a *σ-solution* of φ to emphasize the dependence of γ on σ.

Definition 2.2. A nonautonomous set Γ is called a *nonautonomous equilibrium* of φ, if for each $\sigma\in\Sigma$, $\gamma(t):=\Gamma(\theta_t\sigma)$ is a full solution of φ.

Let Γ be a nonautonomous equilibrium of φ.

Definition 2.3. For each $\sigma\in\Sigma$, the *section unstable (resp. stable) manifold* of Γ at σ is defined to be the set

$$W^u(\Gamma,\sigma)=\left\{x\in X\ \middle|\ \begin{array}{l}\text{there is a }\sigma\text{-solution }\gamma(t)\text{ on }(-\infty,0]\text{ with }\gamma(0)=x\\ \text{such that }\lim_{t\to-\infty}H_X(\gamma(t),\Gamma(\theta_t\sigma))=0\end{array}\right\}.$$

$$\left(\text{resp. } W^s(\Gamma,\sigma)=\left\{x\in X\ \middle|\ \begin{array}{l}\text{there is a }\sigma\text{-solution }\gamma(t)\text{ on }[0,\infty)\text{ with }\gamma(0)=x\\ \text{such that }\lim_{t\to\infty}H_X(\gamma(t),\Gamma(\theta_t\sigma))=0\end{array}\right\}.\right)$$

In the following, we call $W^u(\Gamma,\sigma)$ (resp. $W^s(\Gamma,\sigma)$) the σ-section unstable and (resp. stable) manifold of Γ for short.

3. **Mathematical setting.**

Let X be a Banach space with norm $\|\cdot\|$, and let A be a sectorial operator in X. Pick a number $a>0$ sufficiently large so that

$$\mathrm{Re}\,\sigma(A+aI)>0.$$

Let $\Lambda=A+aI$. For each $\alpha\ge 0$, define $X^\alpha=D(\Lambda^\alpha)$. X^α is equipped with the norm $\|\cdot\|_\alpha$ defined by

$$\|x\|_\alpha=\|\Lambda^\alpha x\|,\quad x\in X^\alpha.$$

Note that the definition of X^α is independent of the choice of the number a.

A sectorial operator A is said to be *hyperbolic* if its spectrum $\sigma(A)$ has a decomposition $\sigma(A)=\sigma_u\cup\sigma_s$ with

$$\sigma_u=\sigma(A)\cap\{\mathrm{Re}\,\lambda<0\},\quad \sigma_s=\sigma(A)\cap\{\mathrm{Re}\,\lambda>0\}.$$

Accordingly, the space X has a direct sum decomposition: $X=X_u\bigoplus X_s$. Let

$$\Pi_i: X\to X_i,\qquad i=u,s$$

be the projection from X to X_i. Denote $A_u=A|_{X_u}$ and $A_s=A|_{X_s}$. By the basic knowledge on sectorial operators (see Henry [19]), we know that there exist $M\ge 1$, $\beta>0$ such that

$$\|\Lambda^\alpha e^{-A_u t}\|\le Me^{\beta t},\quad \|e^{-A_u t}\|\le Me^{\beta t},\qquad t\le 0, \tag{3.1}$$

$$\|\Lambda^\alpha e^{-A_s t}\Pi_s\Lambda^{-\alpha}\| \le Me^{-\beta t}, \quad \|\Lambda^\alpha e^{-A_s t}\| \le Mt^{-\alpha}e^{-\beta t}, \qquad t>0. \tag{3.2}$$

In the present paper, we will study the qualitative behaviour of a nonautonomous equation on X which have the form:

$$x_t + Ax = f\left(x, h(t)\right), \tag{3.3}$$

where the nonautonomous forcing $h \in C(\mathbb{R}, X)$ is *μ-subexponentially growing* for some $\mu > 0$, namely, $\lim_{t\to\pm\infty}\|h(t)\| = \infty$ and

$$\limsup_{t\to\pm\infty}\frac{\log^+\|h(t)\|}{|t|} = \mu_0 < \mu. \tag{3.4}$$

Note that it also covers the case when p is a periodic function, quasiperiodic function, almost periodic function or local almost periodic function [9, 21].

Suppose the nonlinearity $f \in C(X^\alpha \times X, X)$ satisfies

(1) Lipschitz condition:

$$\|f(x_1,y) - f(x_2,y)\| \le L_f\|x_1 - x_2\|_\alpha, \quad x_1, x_2 \in X^\alpha,\ y \in X. \tag{3.5}$$

(2) Linear growth condition: there exists a constant $C > 0$ such that

$$\|f(x,y)\| \le C\left(\|x\|_\alpha + \|y\| + 1\right), \qquad x \in X^\alpha,\ y \in X. \tag{3.6}$$

Denote by $C(\mathbb{R}, X)$ the set of continuous functions from $\mathbb{R}$ to X. Equip with $C(\mathbb{R}, X)$ the compact-open topology generated by the metric:

$$r(h_1,h_2) = \sum_{n=1}^{\infty}\frac{1}{2^n}\cdot\frac{\max_{t\in[-n,n]}\|h_1(t)-h_2(t)\|}{1+\max_{t\in[-n,n]}\|h_1(t)-h_2(t)\|}, \quad h_1, h_2 \in C(\mathbb{R}, X).$$

Then $C(\mathbb{R}, X)$ is a complete metric space. Define the hull of the nonautonomous forcing h as follows

$$\mathcal{H} := \mathcal{H}[h] = \overline{\{h(\tau+\cdot); \tau \in \mathbb{R}\}}^{C(\mathbb{R},X)}, \tag{3.7}$$

and define the shift operator on $\mathcal{H}$:

$$\theta_t : \mathcal{H} \to \mathcal{H},\ t \in \mathbb{R}, \quad \text{as } \theta_t p(\cdot) = p(t+\cdot).$$

It is clear that $\theta_t : \mathcal{H} \to \mathcal{H}$ is continuous.

Instead of (3.3), we will consider the more general cocycle system in X^α (where $\alpha \in [0,1)$):

$$x_t + Ax = f\left(x, p(t)\right), \qquad p \in \mathcal{H}. \tag{3.8}$$

For each $x_0 \in X^\alpha$ and $t_0 \in \mathbb{R}$, we denote by $x(t, t_0; x_0, p)$ the unique (strong) solution $x(t)$ of (3.8) with initial value $x(t_0) = x_0$.

Proposition 1 ([19]). *For each $x_0 \in X^\alpha$, there is a $T > t_0$ such that (3.8) has a unique solution $x(t) = x(t, t_0; x_0, p)$ on $[t_0, T)$ satisfying*

$$x(t) = e^{-A(t-t_0)}x_0 + \int_{t_0}^{t} e^{-A(t-\tau)}f\big(x(\tau), p(\tau)\big)d\tau, \quad t \in [t_0, T). \tag{3.9}$$

For convenience, we always assume that the unique solution (3.9) is globally defined. Define

$$\varphi(t,p)x_0 := x(t, 0; x_0, p), \qquad x_0 \in X^\alpha.$$

Then φ is a cocycle semiflow on X^α driven by the base flow θ on $\mathcal{H}$.

4. Nonautonomous hyperbolic equilibrium and global synchronising behavior of the forced nonautonomous system with h.

4.1. **Basics.** Let us first introduce several Banach spaces that will used throughout the paper. For $\mu \geq 0$, define

$$\mathscr{X}_\mu^{\pm} = \left\{ u \in C(\mathbb{R}^{\pm}; X^\alpha) : \sup_{t\in\mathbb{R}^{\pm}} e^{-\mu|t|}\|x(t)\|_\alpha < \infty \right\}.$$

$\mathscr{X}_\mu^{\pm}$ is equipped with the norm

$$\|x\|_{\mathscr{X}_\mu^{\pm}} = \sup_{t\in\mathbb{R}^{\pm}} e^{-\mu|t|}\|x(t)\|_\alpha, \qquad \forall\, x \in \mathscr{X}_\mu^{\pm}.$$

Replacing $\mathbb{R}^{\pm}$ with $\mathbb{R}$ in the above definition, one immediately obtains the definition of the space $\mathscr{X}_\mu$. Clearly

$$\mathscr{X}_\mu = C(\mathbb{R}; X^\alpha) \cap \mathscr{X}_\mu^{+} \cap \mathscr{X}_\mu^{-}.$$

The following lemma will play a basic role in the proof of our main theorem in the next section.

Lemma 4.1. *Let $\mu \in (0,\beta)$. Then the following assertions hold.*

(a) *Let $x \in \mathscr{X}_\mu^{-}$. Then x is the solution of (3.8) if and only if it solves the following integral equation*

$$\begin{aligned} x(t) = e^{-A_u t}\Pi_u x(0) &+ \int_0^t e^{-A_u(t-\tau)}\Pi_u f\big(x(\tau),p(\tau)\big)d\tau \\ &+ \int_{-\infty}^t e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau),p(\tau)\big)d\tau. \end{aligned} \tag{4.1}$$

(b) *Let $x \in \mathscr{X}_\mu^{+}$. Then x is a solution of (3.8) if and only if it solves the following integral equation*

$$\begin{aligned} x(t) = e^{-A_s t}\Pi_s x(0) &+ \int_0^t e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau),p(\tau)\big)d\tau \\ &- \int_t^{\infty} e^{-A_u(t-\tau)}\Pi_u f\big(x(\tau),p(\tau)\big)d\tau. \end{aligned} \tag{4.2}$$

(c) *Let $x \in \mathscr{X}_\mu$. Then x is the solution of (3.8) if and only if it solves the following integral equation*

$$\begin{aligned} x(t) = &\int_{-\infty}^t e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau),p(\tau)\big)d\tau \\ &- \int_t^{\infty} e^{-A_u(t-\tau)}\Pi_u f\big(x(\tau),p(\tau)\big)d\tau. \end{aligned} \tag{4.3}$$

Proof. (a) Let $x \in \mathscr{X}_\mu^{-}$ be a solution of (3.8). We write $x(t) = x_u(t) + x_s(t)$, where $x_u(t) := \Pi_u x(t)$, $x_s(t) := \Pi_s x(t)$. Then

$$x_u(t) = e^{-A_u t}x_u(0) + \int_0^t e^{-A_u(t-\tau)}\Pi_u f\big(x(\tau),p(\tau)\big)d\tau,$$

and

$$x_s(t) = e^{-A_s(t-t_0)}x_s(t_0) + \int_{t_0}^t e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau),p(\tau)\big)d\tau \tag{4.4}$$

for any $t_0 \le t$. Since

$$\begin{aligned}\|e^{-A_s(t-t_0)}x_s(t_0)\|_\alpha &\le Me^{-\beta(t-t_0)}\|x(t_0)\|_\alpha\\ &= Me^{-\beta t}e^{(\beta-\mu)t_0}\big(e^{\mu t_0}\|x(t_0)\|_\alpha\big)\\ &\le Me^{-\beta t}e^{(\beta-\mu)t_0}\|x\|_{\mathscr{X}_\mu^-} \to 0, \quad \text{as } t_0 \to -\infty,\end{aligned}$$

setting $t_0 \to -\infty$ in (4.4) we find that

$$x_s(t) = \int_{-\infty}^t e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau), p(\tau)\big)d\tau.$$

Consequently

$$\begin{aligned}x(t) &= x_u(t) + x_s(t)\\ &= e^{-A_u t}x_u(0) + \int_0^t e^{-A_u(t-\tau)}\Pi_u f\big(x(\tau), p(\tau)\big)d\tau\\ &\quad + \int_{-\infty}^t e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau), p(\tau)\big)d\tau.\end{aligned}$$

This is precisely what we desired in (4.1).

Conversely if x satisfies (4.1), then one can easily see that it solves (3.8) on $\mathbb{R}^-$.

($\mathbb{b}$) Let $x \in \mathscr{X}_\mu^+$ be a solution of (3.8). We write $x(t) = x_u(t) + x_s(t)$, where

$$x_u(t) = e^{-A_u t}x_u(t_0) + \int_{t_0}^t e^{-A_u(t-\tau)}\Pi_u f\big(x(\tau), p(\tau)\big)d\tau, \tag{4.5}$$

$$x_s(t) = e^{-A_s t}x_s(0) + \int_0^t e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau), p(\tau)\big)d\tau$$

for any $t_0 \ge t$. Observing that

$$\begin{aligned}\|e^{-A_u(t-t_0)}x_u(t_0)\|_\alpha &\le Me^{\beta(t-t_0)}\|x(t_0)\|_\alpha\\ &= Me^{\beta t}e^{-(\beta-\mu)t_0}e^{-\mu t_0}\|x(t_0)\|_\alpha\\ &\le Me^{\beta t}e^{-(\beta-\mu)t_0}\|x\|_{\mathscr{X}_\mu^+} \to 0, \quad \text{as } t_0 \to \infty,\end{aligned}$$

setting $t_0 \to \infty$ in (4.5) one immediately concludes that

$$x_u(t) = -\int_t^\infty e^{-A_u(t-\tau)}\Pi_u[f(x(\tau)) + p(\tau)]d\tau.$$

Thus

$$\begin{aligned}x(t) &= x_u(t) + x_s(t)\\ &= e^{-A_s t}x_s(0) + \int_0^t e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau), p(\tau)\big)d\tau\\ &\quad - \int_t^\infty e^{-A_u(t-\tau)}\Pi_u f\big(x(\tau), p(\tau)\big)d\tau.\end{aligned}$$

This is precisely what we desired in (4.2).

If x satisfies (4.2), then it clearly solves (3.8) on $\mathbb{R}^+$. The proof of ($\mathbb{b}$) is complete.

($\mathbb{c}$) Let $x \in \mathscr{X}_\mu$ be a solution of (3.8). Write $x(t) = x_u(t) + x_s(t)$, where

$$x_s(t) = e^{-A_s(t-t_0)}x_s(t_0) + \int_{t_0}^t e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau), p(\tau)\big)d\tau, \quad t \ge t_0,$$

$$x_u(t) = e^{-A_u(t-t_0)}x_u(t_0) + \int_{t_0}^{t} e^{-A_u(t-\tau)}\Pi_u f\big(x(\tau), p(\tau)\big)d\tau, \quad t \in \mathbb{R}.$$

Similar to ($\mathbb{a}$) and ($\mathbb{b}$), we have

$$x_s(t) = \int_{-\infty}^{t} e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau), p(\tau)\big)d\tau,$$

and

$$x_u(t) = -\int_{t}^{\infty} e^{-A_u(t-\tau)}\Pi_u f\big(x(\tau), p(\tau)\big)d\tau.$$

It follows that

$$\begin{aligned} x(t) = &\int_{-\infty}^{t} e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau), p(\tau)\big)d\tau \\ &- \int_{t}^{\infty} e^{-A_u(t-\tau)}\Pi_u f\big(x(\tau), p(\tau)\big)d\tau. \end{aligned}$$

Conversely, if x satisfies (4.3), it is trivial to check that x is a full solution of (3.8). □

4.2. **Main results.** Denote

$$X_i^\alpha := X_i \cap X^\alpha, \qquad i = u, s.$$

Our main results in the section can be summarized as

Theorem 4.2. *Let $h \in C(\mathbb{R}, X)$ be μ-subexponentially growing for some $\mu \in (0, \beta)$. Suppose the Lipschitz constant L_f of f is sufficiently small. Then*

($\mathbb{a}$) *The cocycle semiflow φ has a unique μ-subexponential equilibrium $\Gamma \in C(\mathcal{H}, X^\alpha)$, namely, the full solution $\Gamma(\theta.p) : \mathbb{R} \to X^\alpha$ is μ-subexponentially growing for each $p \in \mathcal{H}$.*

($\mathbb{b}$) *The equilibrium Γ is hyperbolic. Specifically, for each $p \in \mathcal{H}$, there exist two family of Lipschitz continuous mappings*

$$\xi_p : X_u^\alpha \to X_s^\alpha \qquad \zeta_p : X_s^\alpha \to X_u^\alpha$$

such that the p-section unstable and stable manifold of Γ are represented as

$$\begin{aligned} W^u(\Gamma, p) &= \{y + \xi_p(y) : y \in X_u^\alpha\} \\ W^s(\Gamma, p) &= \{\zeta_p(y) + y : y \in X_s^\alpha\} \end{aligned} \tag{4.6}$$

respectively, and

$$\lim_{t\to-\infty} \|\gamma^u(t) - \Gamma(\theta_t p)\|_\alpha = 0, \quad \gamma^u(0) \in W^u(\Gamma, p)$$

$$\lim_{t\to\infty} \|\gamma^s(t) - \Gamma(\theta_t p)\|_\alpha = 0, \quad \gamma^s(0) \in W^s(\Gamma, p)$$

exponentially fast.

($\mathbb{c}$) *Furthermore, the mappings*

$$\Xi : \mathcal{H} \to C\left(X_u^\alpha, X_s^\alpha\right) \text{ and } \Theta : \mathcal{H} \to C\left(X_s^\alpha, X_u^\alpha\right)$$

defined by

$$\Xi(p) = \xi_p \text{ and } \Theta(p) = \zeta_p, \qquad p \in \mathcal{H}$$

are continuous.

Remark 2. (1) The continuous dependence of ξ_p and ζ_p on p imply that

$$\lim_{p\to q}\delta_{X^\alpha}\big(W^u(\Gamma,p),\,W^u(\Gamma,q)\big)=0,\qquad \forall\, q\in\mathcal{H} \tag{4.7}$$

and

$$\lim_{p\to q}\delta_{X^\alpha}\big(W^s(\Gamma,p),\,W^s(\Gamma,q)\big)=0,\qquad \forall\, q\in\mathcal{H}, \tag{4.8}$$

respectively.

(2) It is clear that if h is periodic (resp. pseudo periodic, almost periodic, uniformly almost automorphic), then $\theta_t h$ is periodic (resp. pseudo periodic, almost periodic, uniformly almost automorphic). In consequence, the continuity of $\Gamma:\mathcal{H}\to X^\alpha$, (4.7) and (4.8) manifest that if h is periodic (resp. pseudo periodic, almost periodic, uniformly almost automorphic), then $\Gamma(\theta_t h)$, $W^u(\Gamma,\theta_t h)$ and $W^s(\Gamma,\theta_t h)$ are also periodic (resp. pseudo periodic, almost periodic, uniformly almost automorphic). This means $\Gamma(\theta_t h)$, $W^u(\Gamma,\theta_t h)$ and $W^s(\Gamma,\theta_t h)$ exhibit a *synchronising behavior* with the nonautonomous forcing h as time varies.

Proof of Theorem 4.2. (a) Suppose L_f is so small that

$$ML_f\int_0^\infty\big(1+\tau^{-\alpha}\big)e^{-\beta'\tau}d\tau<1, \tag{4.9}$$

where $\beta'=\beta-\mu$. For $p\in\mathcal{H}$, one can use the righthand side of equation (4.3) to define a contraction mapping $\mathcal{T}$ on $\mathscr{X}_\mu$ as follows:

$$\begin{aligned}\mathcal{T}x(t)=&\int_{-\infty}^t e^{-A_s(t-\tau)}\Pi_s f\big(x(\tau),p(\tau)\big)d\tau\\&-\int_t^\infty e^{-A_u(t-\tau)}\Pi_u f\big(x(\tau),p(\tau)\big)d\tau.\end{aligned}$$

We first verify that $\mathcal{T}$ maps $\mathscr{X}_\mu$ into itself. Let x be in $\mathscr{X}_\mu$. Then by (3.1),(3.2) and (3.6),

$$\begin{aligned}\|\mathcal{T}x(t)\|_\alpha\le{}& M\int_{-\infty}^t(t-\tau)^{-\alpha}e^{-\beta(t-\tau)}C\big(\|x(\tau)\|_\alpha+\|p(\tau)\|+1\big)d\tau\\&+M\int_t^\infty e^{-\beta(\tau-t)}C\big(\|x(\tau)\|_\alpha+\|p(\tau)\|+1\big)d\tau.\end{aligned}$$

One observes that

$$e^{-\mu|t|}=e^{-\mu|s+(t-s)|}\le e^{-\mu(|s|-|(t-s)|)}=e^{-\mu|s|}e^{\mu|t-s|},\qquad\forall\, t,s\in\mathbb{R}. \tag{4.10}$$

We have

$$
\begin{aligned}
& e^{-\mu|t|}\|\mathcal{T}x(t)\|_\alpha \\
\le\, & M\int_{-\infty}^{t}(t-\tau)^{-\alpha}e^{\mu|t-\tau|}e^{-\beta(t-\tau)}\big[e^{-\mu|\tau|}C\big(\|x(\tau)\|_\alpha+\|p(\tau)\|+1\big)\big]d\tau \\
& + M\int_{t}^{\infty}e^{\mu|t-\tau|}e^{-\beta(\tau-t)}\big[e^{-\mu|\tau|}C\big(\|x(\tau)\|_\alpha+\|p(\tau)\|+1\big)\big]d\tau \\
=\, & M\int_{-\infty}^{t}(t-\tau)^{-\alpha}e^{-\beta'(t-\tau)}\big[e^{-\mu|\tau|}C\big(\|x(\tau)\|_\alpha+\|p(\tau)\|+1\big)\big]d\tau \\
& + M\int_{t}^{\infty}e^{-\beta'(\tau-t)}\big[e^{-\mu|\tau|}C\big(\|x(\tau)\|_\alpha+\|p(\tau)\|+1\big)\big]d\tau \\
\le\, & MC\int_{0}^{\infty}\big(1+\tau^{-\alpha}\big)e^{-\beta'\tau}d\tau\Big(\|x\|_{\mathscr{X}_\mu}+\|p\|_{\infty,\mu}+1\Big) \\
:=\, & M_{\beta'}C\Big(\|x\|_{\mathscr{X}_\mu}+\|p\|_{\infty,\mu}+1\Big), \qquad \forall\, t\in\mathbb{R},
\end{aligned}
$$

where

$$M_{\beta'}:=M\int_0^\infty\big(1+\tau^{-\alpha}\big)e^{-\beta'\tau}d\tau,$$

$$\|p\|_{\infty,\mu}:=\sup_{t\in\mathbb{R}}e^{-\mu|t|}\|p(t)\|.$$

Notice from (3.4) that $\|p\|_{\infty,\mu}<\infty$. Hence $\|\mathcal{T}x\|_{\mathscr{X}_\mu}<\infty$, i.e. $\mathcal{T}x\in\mathscr{X}_\mu$.

Next, we check that $\mathcal{T}$ is a contraction mapping. Indeed, in a quite similar fashion as above, it can be shown that for any x, $x'\in\mathscr{X}_\mu$,

$$
\begin{aligned}
& e^{-\mu|t|}\|\mathcal{T}x(t)-\mathcal{T}x'(t)\|_\alpha \\
\le\, & ML_f\int_{-\infty}^{t}(t-\tau)^{-\alpha}e^{-\beta'(t-\tau)}\Big(e^{-\mu|\tau|}\|x(\tau)-x'(\tau)\|_\alpha\Big)d\tau \\
& + ML_f\int_{t}^{\infty}e^{\beta'(t-\tau)}\Big(e^{-\mu|\tau|}\|x(\tau)-x'(\tau)\|_\alpha\Big)d\tau \\
\le\, & M_{\beta'}L_f\|x-x'\|_{\mathscr{X}_\mu}, \qquad \forall\, t\in\mathbb{R}.
\end{aligned}
$$

$M_{\beta'}L_f<1$ by (4.9), it follows that $\mathcal{T}$ is contracting on $\mathscr{X}_\mu$.

Now thanks to the Banach fixed-point theorem, $\mathcal{T}$ has a unique fixed point $x_p\in\mathscr{X}_\mu$ which is precisely a full solution of (3.8) satisfying (4.3).

Define $\Gamma:\mathcal{H}\to X^\alpha$ by

$$\Gamma(p)=x_p(0), \quad p\in\mathcal{H}.$$

It is easy to verify that $\Gamma(\theta_t p)=x_p(t)$, $t\in\mathbb{R}$, hence Γ is an equilibrium of φ. In what follows we show that Γ is continuous.

Recall that for $p\in\mathcal{H}$, $x_p(t)=\Gamma(\theta_t p)$, $t\le 0$ satisfies

$$
\begin{aligned}
x_p(t)=&\int_{-\infty}^{t}e^{-A_s(t-\tau)}\Pi_s f\big(x_p(\tau),p(\tau)\big)d\tau \\
&-\int_t^\infty e^{-A_u(t-\tau)}\Pi_u f\big(x_p(\tau),p(\tau)\big)d\tau.
\end{aligned}
\tag{4.11}
$$

Let $p,\, q \in \mathcal{H}$. Then for $t \in \mathbb{R}$, by (4.10) and (4.11) we deduce that

$$\begin{aligned}
&e^{-\mu|t|}\|x_p(t) - x_q(t)\|_\alpha \\
&\le ML_f \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta(t-s)} e^{\mu|t-s|} \big(e^{-\mu|s|}\|x_p(s) - x_q(s)\|_\alpha\big) ds \\
&\quad + ML_f \int_{t}^{\infty} e^{-\beta(s-t)} e^{\mu|t-s|} \big(e^{-\mu|s|}\|x_p(s) - x_q(s)\|_\alpha\big) ds \\
&\quad + M \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta(t-s)} e^{\mu|t-s|} \Big(e^{-\mu|s|}\|p(s) - q(s)\|\Big) ds \\
&\quad + M \int_{t}^{\infty} e^{-\beta(s-t)} e^{\mu|t-s|} \Big(e^{-\mu|s|}\|p(s) - q(s)\|\Big) ds \\
&= ML_f \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta'(t-s)} \big(e^{-\mu|s|}\|x_p(s) - x_q(s)\|_\alpha\big) ds \\
&\quad + ML_f \int_{t}^{\infty} e^{-\beta'(s-t)} \big(e^{-\mu|s|}\|x_p(s) - x_q(s)\|_\alpha\big) ds \\
&\quad + M \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta'(t-s)} \Big(e^{-\mu|s|}\|p(s) - q(s)\|\Big) ds \\
&\quad + M \int_{t}^{\infty} e^{-\beta'(s-t)} \Big(e^{-\mu|s|}\|p(s) - q(s)\|\Big) ds.
\end{aligned} \tag{4.12}$$

Hence

$$\begin{aligned}
&e^{-\mu|t|}\|x_p(t) - x_q(t)\|_\alpha \\
&\le ML_f \left(\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta'(t-s)} ds + \int_{t}^{\infty} e^{-\beta'(s-t)} ds\right) \|x_p - x_q\|_{\mathscr{X}_\mu} \\
&\quad + M \left(\int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta'(t-s)} ds + \int_{t}^{\infty} e^{-\beta'(s-t)} ds\right) \|p - q\|_{\infty,\mu} \\
&\le M_{\beta'} L_f \|x_p - x_q\|_{\mathscr{X}_\mu} + M_{\beta'}\|p - q\|_{\infty,\mu}, \qquad \forall\, t \in \mathbb{R}.
\end{aligned} \tag{4.13}$$

Thus

$$\|x_p - x_q\|_{\mathscr{X}_\mu} \le \frac{M_{\beta'}}{1 - M_{\beta'} L_f}\|p - q\|_{\infty,\mu}.$$

Therefore

$$\|\Gamma(p) - \Gamma(q)\|_\alpha = \|x_p(0) - x_q(0)\|_\alpha \le \frac{M_{\beta'}}{1 - M_{\beta'} L_f}\|p - q\|_{\infty,\mu}. \tag{4.14}$$

We learn from (3.4) that $\|p - q\|_{\infty,\mu} \to 0$ as $p \to q$ (in $\mathcal{H}$). Then by (4.14) we have

$$\|\Gamma(p) - \Gamma(q)\|_\alpha \to 0, \quad \text{as } p \to q,$$

which completes the proof of the continuity of Γ.

($\mathbb{b}$) For each $p \in \mathcal{H}$ and $y \in X_u^\alpha$, the righthand side of (4.1) can define a mapping $\mathcal{T}^- = \mathcal{T}_{p,y}^-$ on $\mathscr{X}_\mu^-$ as follows:

$$\begin{aligned}
\mathcal{T}^- x(t) &= e^{-A_u t} y + \int_0^t e^{-A_u(t-\tau)} \Pi_u f\big(x(\tau), p(\tau)\big) d\tau \\
&\quad + \int_{-\infty}^{t} e^{-A_s(t-\tau)} \Pi_s f\big(x(\tau), p(\tau)\big) d\tau.
\end{aligned}$$

We first show that $\mathcal{T}^-$ maps $\mathscr{X}_\mu^-$ into itself. Let $x \in \mathscr{X}_\mu^-$. For any $t \le 0$,

$$\begin{aligned}
&e^{\mu t}\|\mathcal{T}^- x(t)\|_\alpha \\
&\le Me^{(\beta+\mu)t}\|y\|_\alpha + MC\int_t^0 e^{(\beta+\mu)(t-\tau)}e^{\mu\tau}\big(\|x(\tau)\|_\alpha + \|p(\tau)\| + 1\big)d\tau \\
&\quad + MC\int_{-\infty}^t (t-\tau)^{-\alpha}e^{-\beta'(t-\tau)}e^{\mu\tau}\big(\|x(\tau)\|_\alpha + \|p(\tau)\| + 1\big)d\tau \\
&\le M\|y\|_\alpha + M_{\beta'}C\Big(\|x\|_{\mathscr{X}_\mu^-} + \|p\|_{\infty,\mu} + 1\Big) < \infty.
\end{aligned}$$

Hence $\mathcal{T}^- u \in \mathscr{X}_\mu^-$.

Now we check that $\mathcal{T}^-$ is contracting on $\mathscr{X}_\mu^-$. For x, $x' \in \mathscr{X}_\mu^-$, we have for any $t \le 0$ that

$$\begin{aligned}
&e^{\mu t}\|\mathcal{T}^- x(t) - \mathcal{T}^- x'(t)\|_\alpha \\
&\le ML_f\int_t^0 e^{(\beta+\mu)(t-\tau)}e^{\mu\tau}\|x(\tau) - x'(\tau)\|_\alpha d\tau \\
&\quad + ML_f\int_{-\infty}^t (t-\tau)^{-\alpha}e^{-\beta'(t-\tau)}e^{\mu\tau}\|x(\tau) - x'(\tau)\|_\alpha d\tau \\
&\le M_{\beta'}L_f\|x - x'\|_{\mathscr{X}_\mu^-}.
\end{aligned}$$

Since $M_{\beta'}L_f < 1$, $\mathcal{T}^-$ is indeed a contraction mapping on $\mathscr{X}_\mu^-$.

By virtue of the Banach fixed-point theorem, $\mathcal{T}^-$ has a unique fixed point $x_{p,y}$ in $\mathscr{X}_\mu^-$. So $x_{p,y}(t)$ is precisely a solution of (3.8) on $\mathbb{R}^-$ with $\Pi_u x_{p,y}(0) = y$, which equivalently solves the following integral equation

$$\begin{aligned}
x_{p,y}(t) = e^{-A_u t}y &+ \int_0^t e^{-A_u(t-\tau)}\Pi_u f\big(x_{p,y}(\tau), p(\tau)\big)d\tau \\
&+ \int_{-\infty}^t e^{-A_s(t-\tau)}\Pi_s f\big(x_{p,y}(\tau), p(\tau)\big)d\tau.
\end{aligned} \tag{4.15}$$

We claim that $x_{p,y}(0)$ is Lipschitz continuous in y uniformly on $p \in \mathcal{H}$. Indeed, for $y, z \in X_u^\alpha$ and $t \le 0$,

$$\begin{aligned}
&e^{\mu t}\|x_{p,y}(t) - x_{p,z}(t)\|_\alpha \\
&\le Me^{(\beta+\mu)t}\|y - z\|_\alpha + ML_f\int_t^0 e^{(\beta+\mu)(t-\tau)}e^{\mu\tau}\|x_{p,y}(\tau) - x_{p,z}(\tau)\|_\alpha d\tau \\
&\quad + ML_f\int_{-\infty}^t (t-\tau)^{-\alpha}e^{-(\beta-\mu)(t-\tau)}e^{\mu\tau}\|x_{p,y}(\tau) - x_{p,z}(\tau)\|_\alpha d\tau \\
&\le M\|y - z\|_\alpha + M_{\beta'}L_f\|x_{p,y}(t) - x_{p,z}(t)\|_{\mathscr{X}_\mu^-}.
\end{aligned}$$

Hence,

$$\|x_{p,y}(0) - x_{p,z}(0)\|_\alpha \le \|x_{p,y} - x_{p,z}\|_{\mathscr{X}_\mu^-} \le \frac{M}{1 - M_{\beta'}L_f}\|y - z\|_\alpha,$$

and thus $x_{p,y}(0)$ is Lipschitz continuous in y uniformly on $p \in \mathcal{H}$.

Define

$$\xi_p(y) := \int_{-\infty}^0 e^{A_s\tau}\Pi_s f\big(x_{p,y}(\tau), p(\tau)\big)d\tau, \quad y \in X_u^\alpha. \tag{4.16}$$

Setting $t = 0$ in (4.15) leads to

$$x_{p,y}(0) = y + \xi_p(y), \qquad y \in X_u^\alpha. \tag{4.17}$$

The Lipschitz continuity of $x_{p,y}(0)$ in y then implies $\xi_p : X_u^\alpha \to X_s^\alpha$ is a Lipschitz continuous mapping.

Set

$$\mathcal{M}^u(p) = \{y + \xi_p(y) : y \in X_u^\alpha\}.$$

Then $\mathcal{M}^u(p)$ is homeomorphic to X_u^α. We claim that

$$W^u(\Gamma, p) = \mathcal{M}^u(p).$$

Lemma 4.1 implies $W^u(\Gamma, p) \subset \mathcal{M}^u(p)$. In what follows we check that

$$\mathcal{M}^u(p) \subset W^u(\Gamma, p). \tag{4.18}$$

Define a Banach space

$$\mathscr{W}_\mu^- := \left\{ w \in C(\mathbb{R}^-; X^\alpha) : \sup_{t \in \mathbb{R}^-} e^{-\mu t} \|w(t)\|_\alpha < \infty \right\}.$$

$\mathscr{W}_\mu^-$ is equipped with the norm

$$\|w\|_{\mathscr{W}_\mu^-} = \sup_{t \in \mathbb{R}^-} e^{-\mu t} \|w(t)\|_\alpha.$$

Let $x_p(t)$ be the full solution of φ satisfying (4.11). For each fixed $y \in X_u^\alpha$, define a mapping $\mathcal{G}^- = \mathcal{G}_y^- : \mathscr{W}_\mu^- \to \mathscr{W}_\mu^-$ as follows:

$$\begin{aligned}
\mathcal{G}^- w(t) &= e^{-A_u t}(y + z) \\
&\quad + \int_0^t e^{-A_u(t-\tau)} \Pi_u \big[f\big(x_p(\tau) + w(\tau), p(\tau)\big) - f\big(x_p(\tau), p(\tau)\big) \big] d\tau \\
&\quad + \int_{-\infty}^t e^{-A_s(t-\tau)} \Pi_s \big[f\big(x_p(\tau) + w(\tau), p(\tau)\big) - f\big(x_p(\tau), p(\tau)\big) \big] d\tau,
\end{aligned}$$

where

$$z = \int_0^\infty e^{A_u \tau} \Pi_u f\big(x_p(\tau), p(\tau)\big) d\tau.$$

(It is trivial to see that $\|z\|_\alpha < \infty$.) We first check that $\mathcal{G}^-$ is well defined. Indeed, let $w \in \mathscr{W}_\mu^-$. Then for any $t \le 0$,

$$\begin{aligned}
e^{-\mu t} \|\mathcal{G}^- w(t)\|_\alpha &\le M e^{\beta' t} \|y + z\|_\alpha + M L_f \int_t^0 e^{\beta'(t-\tau)} e^{-\mu \tau} \|w(\tau)\|_\alpha d\tau \\
&\quad + M L_f \int_{-\infty}^t (t-\tau)^{-\alpha} e^{-(\beta+\mu)(t-\tau)} e^{-\mu\tau} \|w(\tau)\|_\alpha d\tau \\
&\le M \|y + z\|_\alpha + M_{\beta'} L_f \|w\|_{\mathscr{W}_\mu^-} < \infty.
\end{aligned}$$

Thus $\mathcal{G}^-$ maps $\mathscr{W}_\mu^-$ into itself.

Now we show that $\mathcal{G}^-$ is contracting. Let $w,\, w' \in \mathscr{W}_\mu^-$. Then for $t \le 0$,

$$\begin{aligned}
&e^{-\mu t}\|\mathcal{G}^- w(t) - \mathcal{G}^- w'(t)\|_\alpha \\
&\quad\le ML_f \int_t^0 e^{\beta'(t-\tau)} e^{-\mu\tau} \|w(\tau) - w'(\tau)\|_\alpha d\tau \\
&\qquad + ML_f \int_{-\infty}^t (t-\tau)^{-\alpha} e^{-(\beta+\mu)(t-\tau)} e^{-\mu\tau} \|w(\tau) - w'(\tau)\|_\alpha d\tau \\
&\quad\le M_{\beta'} L_f \|w - w'\|_{\mathscr{W}_\mu^-}.
\end{aligned}$$

Since $M_{\beta'} L_f < 1$, $\mathcal{G}^-$ is a contraction mapping on $\mathscr{W}_\mu^-$.

Therefore $\mathcal{G}^-$ has a unique fixed point w in $\mathscr{W}_\mu^-$, which solves the following integral equation on $\mathbb{R}^-$:

$$\begin{aligned}
&w(t) = e^{-A_u t}(y+z) \\
&\quad + \int_0^t e^{-A_u(t-\tau)} \Pi_u \big[f\big(x_p(\tau) + w(\tau), p(\tau)\big) - f\big(x_p(\tau), p(\tau)\big)\big] d\tau \\
&\quad + \int_{-\infty}^t e^{-A_s(t-\tau)} \Pi_s \big[f\big(x_p(\tau) + w(\tau), p(\tau)\big) - f\big(x_p(\tau), p(\tau)\big)\big] d\tau.
\end{aligned} \tag{4.19}$$

(Note that $\|w(t)\|_\alpha$ tends to zero exponentially fast for $t \to -\infty$.)

Adding (4.11) to (4.19), we find

$$\begin{aligned}
x_p(t) + w(t) =& e^{-A_u t} y + \int_0^t e^{-A_u(t-\tau)} \Pi_u f\big(x_p(\tau) + w(\tau), p(\tau)\big) d\tau \\
&+ \int_{-\infty}^t e^{-A_s(t-\tau)} \Pi_s f\big(x_p(\tau) + w(\tau), p(\tau)\big) d\tau.
\end{aligned} \tag{4.20}$$

Set $\tilde{x}_{p,y}(t) = x_p(t) + w(t)$, $t \le 0$. The equation (4.20) means that $\tilde{x}_{p,y}(t)$, $t \le 0$ satisfies (4.15). By the uniqueness of the backward solution $x_{p,y}(t)$, $t \le 0$ with $\Pi_u x_{p,y}(0) = y$, one knows that

$$\tilde{x}_{p,y}(t) \equiv x_{p,y}(t), \qquad t \le 0.$$

We conclude that the backward solution $x_{p,y}(t)$ tends to the full solution $x_p(t)$ exponentially as $t \to -\infty$. Since $y \in X_u^\alpha$ is arbitrary, we prove (4.18) and complete the assertion.

($\mathfrak{c}$) Let $y \in X_u^\alpha$, and $p,\, q \in \mathcal{H}$. Then similar computations as in (4.12) and (4.13) show that

$$\|x_{p,y} - x_{q,y}\|_{\mathscr{X}_\mu^-} \le M_{\beta'} L_f \|x_{p,y} - x_{q,y}\|_{\mathscr{X}_\mu^-} + M_\mu \|p - q\|_{\infty,\mu}.$$

In particular,

$$\|x_{p,y}(0) - x_{q,y}(0)\|_\alpha \le \frac{M_{\beta'}}{1 - M_{\beta'} L_f} \|p - q\|_{\infty,\mu}.$$

This together with (4.17) shows that

$$\|\xi_p(y) - \xi_q(y)\|_\alpha \le \frac{M_{\beta'}}{1 - M_{\beta'} L_f} \|p - q\|_{\infty,\mu}.$$

Since $p \to q$ (in $\mathcal{H}$) implies that $\|p - q\|_{\infty,\mu} \to 0$, we conclude that

$$\lim_{p \to q} \sup_{y \in X_u^\alpha} \|\xi_p(y) - \xi_q(y)\|_\alpha = 0,$$

which verifies the Lipschitz continuity property of ξ_p in p.

The corresponding argument for stable manifold of Γ is completely similar to that for the unstable manifold, so is omitted. $\square$

4.3. **Dynamical completeness of hyperbolic equilibrium.** Here we show that the hyperbolic equilibrium Γ obtained in Theorem 4.2 will completely characterizes the dynamics of the cocycle semiflow φ by proving its unstable manifold is globally forward stable. Consequently, the original system (3.3) exhibits a *global synchronising behavior* (in the sense of Remark 2) with the forcing h.

Theorem 4.3. *Suppose the Lipschitz constant L_f of f is sufficiently small. Then the unstable manifold of Γ is exponentially forward attracting all points in X^α through φ, that is,*

$$\lim_{t\to+\infty} \|\varphi(t,p)x_0 - W^u(\Gamma,\theta_t p)\|_\alpha = 0, \qquad \forall x_0 \in X^\alpha.$$

Remark 3. Theorem 4.3 and Remark 2 indicate that the system φ exhibits a global synchronising behavior with the nonautonomous forcing h as time varies.

Proof of Theorem 4.3. The proof is adapted from that of Theorem 6.1.4 in Henry [19].

Let M, $\beta > 0$ be that in (3.1) and (3.2). Denote by L_ξ the Lipschitz constant of mapping $\xi_p : X_u^\alpha \to X_s^\alpha$ in Theorem 4.2. Note that L_ξ is independent on $p \in \mathcal{H}$ and $\lim_{L_f\to 0} L_\xi = M$. Suppose L_f is so small that

$$ML_f(1+L_\xi) < \beta \tag{4.21}$$

and that

$$ML_f\left(1+\frac{4ML_f(1+L_\xi)}{\beta}\right)\int_0^\infty \tau^{-\alpha}e^{-\tau}du < (\beta/2)^{1-\alpha}. \tag{4.22}$$

Suppose $x(t) = x_u(t) + x_s(t) \in X^\alpha$, $t \geq 0$ is a solution of (3.8) and let

$$\chi(t) = x_s(t) - \xi(t, x_u(t)),$$

where $\xi(t, x_u(t)) := \xi_{\theta_t p}(x_u(t))$. In the following, we show that there is a $0 < \gamma < \beta/2$ and a constant $K > 0$ such that

$$\|\chi(t)\|_\alpha \leq KM\|\chi(0)\|_\alpha e^{-\gamma t}, \quad t \geq 0,$$

which completes the proof the theorem.

Let $y(\tau;t)$, $\tau \leq t$ be the solution of the equation

$$\begin{cases} y_\tau + A_u y = \Pi_u\big[f\big(y+\xi(\tau,y),p(\tau)\big)\big], & \tau \leq t; \\ \qquad y = x_u(t), & \tau = t. \end{cases} \tag{4.23}$$

We first estimate

$$
\begin{aligned}
&\|y(\tau;t)-x_u(\tau)\|_\alpha \le \int_\tau^t Me^{\beta(\tau-r)}\big\|f\big(y(r;t)+\xi(r,y(r;t)),p(r)\big)-f\big(x(r),p(r)\big)\big\|dr\\
&\le ML_f\int_\tau^t e^{\beta(\tau-r)}\big(\|\xi(r,y(r;t))-x_s(r)\|_\alpha+\|y(r;t)-x_u(r)\|_\alpha\big)dr\\
&\le ML_f\int_\tau^t e^{\beta(\tau-r)}\big[\|\xi(r,y(r;t))-\xi(r,x_u(r))\|_\alpha+\\
&\qquad\qquad\qquad +\|\xi(r,x_u(r))-x_s(r)\|_\alpha+\|y(r;t)-x_u(r)\|_\alpha\big]dr\\
&\le ML_f(L_\xi+1)\int_\tau^t e^{\beta(\tau-r)}\|y(r;t)-x_u(r)\|_\alpha dr+ML_f\int_\tau^t e^{\beta(\tau-r)}\|\chi(r)\|_\alpha dr\\
&:= a\int_\tau^t e^{\beta(\tau-r)}\|\chi(r)\|_\alpha dr+b\int_\tau^t e^{\beta(\tau-r)}\|y(r;t)-x_u(r)\|_\alpha dr,
\end{aligned}
$$

where $a := ML_f$, $b := ML_f(L_\xi+1)$. By Gronwall's inequality, we have

$$
\begin{aligned}
&e^{-\beta\tau}\|y(\tau;t)-x_u(\tau)\|_\alpha\\
&\le a\int_\tau^t e^{-\beta r}\|\chi(r)\|_\alpha dr+b\int_\tau^t e^{b(r-\tau)}\left(a\int_r^t e^{-\beta\mu}\|\chi(\mu)\|_\alpha d\mu\right)dr,\ 0\le\tau\le t.
\end{aligned}
$$

Integrating by parts, we find

$$
\begin{aligned}
&b\int_\tau^t e^{b(r-\tau)}\left(a\int_r^t e^{-\beta\mu}\|\chi(\mu)\|_\alpha d\mu\right)dr\\
&\quad= a\int_\tau^t\left(\int_r^t e^{-\beta\mu}\|\chi(\mu)\|_\alpha d\mu\right)d\left(e^{b(r-\tau)}\right)\\
&\quad= ae^{b(r-\tau)}\int_r^t e^{-\beta\mu}\|\chi(\mu)\|_\alpha d\mu\Big|_\tau^t+a\int_\tau^t e^{b(r-\tau)}e^{-\beta r}\|\chi(r)\|_\alpha dr\\
&\quad= -a\int_\tau^t e^{-\beta r}\|\chi(r)\|_\alpha dr+a\int_\tau^t e^{b(r-\tau)}e^{-\beta r}\|\chi(r)\|_\alpha dr.
\end{aligned}
$$

So

$$
\|y(\tau;t)-x_u(\tau)\|_\alpha\le a\int_\tau^t e^{-(\beta-b)(r-\tau)}\|\chi(r)\|_\alpha dr,\quad 0\le\tau\le t. \tag{4.24}
$$

Let $\tau\le t_0\le t$. We then calculate

$$
\begin{aligned}
&y(\tau;t)-y(\tau;t_0)=[y(\tau;t)-x_u(\tau)]-[y(\tau;t_0)-x_u(\tau)]\\
&=\int_t^\tau e^{-A_u(\tau-r)}\Pi_u\big[f\big(y(r;t)+\xi(r,y(r;t)),p(r)\big)-f\big(x(r),p(r)\big)\big]dr\\
&\quad-\int_{t_0}^\tau e^{-A_u(\tau-r)}\Pi_u\big[f\big(y(r;t_0)+\xi(r,y(r;t_0)),p(r)\big)-f\big(x(r),p(r)\big)\big]dr\\
&=\int_{t_0}^\tau e^{-A_u(\tau-r)}\Pi_u\big[f\big(y(r;t)+\xi(r,y(r;t)),p(r)\big)-\\
&\qquad\qquad\qquad -f\big(y(r;t_0)+\xi(r,y(r;t_0),p(r))\big)\big]dr\\
&\quad+\int_t^{t_0} e^{-A_u(\tau-r)}\Pi_u\big[f\big(y(r;t)+\xi(r,y(r;t)),p(r)\big)-f\big(x(r),p(r)\big)\big]dr\\
&:= I+J.
\end{aligned}
$$

Now

$$\|I\|_\alpha \leq b\int_\tau^{t_0} e^{\beta(\tau-r)}\|y(r;t)-y(r;t_0)\|_\alpha dr, \tag{4.25}$$

$$\begin{aligned}
\|J\|_\alpha &\leq b\int_{t_0}^{t} e^{\beta(\tau-r)}\|y(r;t)-x_u(r)\|_\alpha dr + a\int_{t_0}^{t} e^{\beta(\tau-r)}\|\chi(r)\|_\alpha dr\\
&\leq \quad (\text{by } (4.24)) \leq ab\int_{t_0}^{t} e^{\beta(\tau-r)}\int_r^t e^{(b-\beta)(\mu-r)}\|\chi(\mu)\|_\alpha d\mu dr\\
&\quad + a\int_{t_0}^{t} e^{\beta(\tau-r)}\|\chi(r)\|_\alpha dr\\
&= ae^{\beta\tau}\int_{t_0}^{t} be^{-br}\left(\int_r^t e^{(b-\beta)\mu}\|\chi(\mu)\|_\alpha d\mu\right)dr + a\int_{t_0}^{t} e^{\beta(\tau-r)}\|\chi(r)\|_\alpha dr\\
&= ae^{\beta\tau}\int_{t_0}^{t} \left(\int_r^t e^{(b-\beta)\mu}\|\chi(\mu)\|_\alpha d\mu\right)d\left(-e^{-br}\right) + a\int_{t_0}^{t} e^{\beta(\tau-r)}\|\chi(r)\|_\alpha dr\\
&= ae^{\beta\tau}\int_{t_0}^{t} e^{b(r-t_0)}e^{-\beta r}\|\chi(r)\|_\alpha dr \text{ (by integration-by-parts formula).}
\end{aligned} \tag{4.26}$$

Inequalities (4.25) and (4.26) imply

$$\begin{aligned}
e^{-\beta\tau}\|y(\tau;t)-y(\tau;t_0)\|_\alpha \leq b\int_\tau^{t_0} e^{-\beta r}\|y(r;t)-y(r;t_0)\|_\alpha dr\\
+ a\int_{t_0}^{t} e^{b(r-t_0)}e^{-\beta r}\|\chi(r)\|_\alpha dr.
\end{aligned}$$

By Gronwall's inequality, we have

$$\|y(\tau;t)-y(\tau;t_0)\|_\alpha \leq a\int_{t_0}^{t} e^{-(\beta-b)(r-\tau)}\|\chi(r)\|_\alpha dr, \;\; \tau\leq t_0\leq t. \tag{4.27}$$

From (4.16) and (4.23), one knows that $\xi(t,x_u(t))$, $t\geq 0$ satisfies

$$\xi(t,x_u(t)) = \int_{-\infty}^{t} e^{-A_s(t-\tau)}\Pi_s\big[f\big(y(\tau;t)+\xi(\tau,y(\tau;t)),p(\tau)\big)\big]d\tau.$$

Then for $t\geq 0$,

$$\begin{aligned}
\chi(t)-e^{-A_s t}\chi(0) = \int_0^t e^{-A_s(t-\tau)}\Pi_s\big[f\big(x(\tau),p(\tau)\big) - f\big(y(\tau;t)+\\
+\xi(\tau,y(\tau;t)),p(\tau)\big)\big]d\tau\\
+\int_{-\infty}^{0} e^{-A_s(t-\tau)}\Pi_s\big[f\big(y(\tau;0)+\xi(\tau,y(\tau;0)),p(\tau)\big)-\\
- f\big(y(\tau;t)+\xi(\tau,y(\tau;t)),p(\tau)\big)\big]d\tau,
\end{aligned}$$

and thus

$$\begin{aligned}
\|\chi(t)\|_\alpha &\leq Me^{-\beta t}\|\chi(0)\|_\alpha\\
&+\int_0^t (t-\tau)^{-\alpha}e^{-\beta(t-\tau)}\big(b\|x_u(\tau)-y(\tau;t)\|_\alpha + a\|\chi(\tau)\|_\alpha\big)d\tau\\
&+\int_{-\infty}^{0}(t-\tau)^{-\alpha}e^{-\beta(t-\tau)}b\|y(\tau;0)-y(\tau;t)\|_\alpha d\tau.
\end{aligned}$$

In light of (4.24) and (4.27), we deduce

$$
\begin{aligned}
\|\chi(t)\|_\alpha \le\ & Me^{-\beta t}\|\chi(0)\|_\alpha + a\int_0^t (t-\tau)^{-\alpha}e^{-\beta(t-\tau)}\|\chi(\tau)\|_\alpha d\tau \\
& + ab\int_0^t (t-\tau)^{-\alpha}e^{-\beta(t-\tau)}\int_\tau^t e^{-(\beta-b)(r-\tau)}\|\chi(r)\|_\alpha dr d\tau \\
& + ab\int_{-\infty}^0 (t-\tau)^{-\alpha}e^{-\beta(t-\tau)}\int_0^t e^{-(\beta-b)(r-\tau)}\|\chi(r)\|_\alpha dr d\tau \\
:=\ & Me^{-\beta t}\|\chi(0)\|_\alpha + a\int_0^t (t-\tau)^{-\alpha}e^{-\beta(t-\tau)}\|\chi(\tau)\|_\alpha d\tau + U + V.
\end{aligned}
$$

We first calculate

$$
\begin{aligned}
U &\le ab\int_0^t \left((t-r)^{-\alpha}e^{-\frac{\beta(t-r)}{2}}\right)e^{-\frac{\beta(t-\tau)}{2}}\int_\tau^t \|\chi(r)\|_\alpha dr d\tau \\
&\le ab\int_0^t e^{-\frac{\beta(t-\tau)}{2}}d\tau\int_0^t (t-r)^{-\alpha}e^{-\frac{\beta(t-r)}{2}}\|\chi(r)\|_\alpha dr \\
&\le \frac{2ab}{\beta}\int_0^t (t-r)^{-\alpha}e^{-\frac{\beta(t-r)}{2}}\|\chi(r)\|_\alpha dr,
\end{aligned}
$$

where we have used the condition (4.21), i.e., $\beta > b$. Similarly,

$$
\begin{aligned}
V &\le ab\int_{-\infty}^0 \left((t-r)^{-\alpha}e^{-\frac{\beta(t-r)}{2}}\right)e^{-\frac{\beta(t-\tau)}{2}}\int_0^t \|\chi(r)\|_\alpha dr d\tau \\
&\le ab\int_{-\infty}^0 e^{-\frac{\beta(t-\tau)}{2}}d\tau\int_0^t (t-r)^{-\alpha}e^{-\frac{\beta(t-r)}{2}}\|\chi(r)\|_\alpha dr \\
&\le \frac{2ab}{\beta}\int_0^t (t-r)^{-\alpha}e^{-\frac{\beta(t-r)}{2}}\|\chi(r)\|_\alpha dr.
\end{aligned}
$$

Consequently,

$$
e^{\frac{\beta}{2}t}\|\chi(t)\|_\alpha \le M\|\chi(0)\|_\alpha + C\int_0^t (t-\tau)^{-\alpha}\big(e^{\frac{\beta}{2}\tau}\|\chi(\tau)\|_\alpha\big)d\tau, \tag{4.28}
$$

where

$$
C := 4ab/\beta + a.
$$

Applying Gronwall inequality (see e.g. [19, Lemma 7.1.1]) to (4.28), we obtain that there is a constant $K > 0$ (depending only on α) such that

$$
\|\chi(t)\|_\alpha \le KM\|\chi(0)\|_\alpha e^{-\gamma t},
$$

where

$$
\gamma = \frac{\beta}{2} - \left(C\int_0^\infty \tau^{-\alpha}e^{-\tau}d\tau\right)^{1/(1-\alpha)}.
$$

By condition (4.22), $\gamma > 0$. The proof is complete. □

5. **Application to stochastic partial differential equations with additive white noise.** In the section, we apply the main results obtained in Section 4 to stochastic partial equation with additive white noise.

Let X be a Polish space with norm $\|\cdot\|$. We first recall some basic concepts in random dynamical systems (RDSs).

5.1. **RDSs.** An RDS consists of a "metric dynamical system" and a "cocycle semiflow" that is in some sense driven by the metric dynamical system.

Definition 5.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\vartheta := \{\vartheta_t\}_{t\in\mathbb{R}}$ be a family of measure preserving transformations on Ω such that $(t, \omega) \to \vartheta_t\omega$ is measurable, $\vartheta_0 = id_\Omega$, and $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$ for all $t, s \in \mathbb{R}$. Then the quadruple $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is called a *metric dynamical system.*

Definition 5.2. A *cocycle semiflow* on X is a mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X,$$

which is $\big(\mathcal{B}(\mathbb{R}\otimes\mathcal{F}\otimes\mathcal{B}(X)), \mathcal{F}\big)$-measurable such that $(t, x) \to \phi(t, \omega, x)$ is continuous for all $\omega \in \Omega$ and the family $\phi(t, \omega) := \phi(t, \omega, \cdot) : X \to X$ satisfies

- $\phi(0, \omega) = id_X$,
- $\phi(t + s, \omega) = \phi(t, \vartheta_s\omega) \circ \phi(s, \omega)$ for all $s, t \in \mathbb{R}^+$ and all $\omega \in \Omega$ (cocycle property).

For a comprehensive exposition on RDS see [1, 12, 13].

Definition 5.3. A random variable Ξ is called a *random equilibrium* of ϕ if for each $\omega \in \Omega$, $\gamma(t) := \Xi(\vartheta_t\omega)$, $t \in \mathbb{R}$ is a full solution of ϕ.

Definition 5.4. Let Ξ be a random equilibrium of ϕ. For each $\omega \in \Omega$, the *ω-section unstable (resp. stable) manifold* of Ξ is defined to be the set

$$W^u(\Xi, \omega) = \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there is a } \omega\text{-solution } \gamma(t) \text{ on } (-\infty, 0] \text{ with } \gamma(0) = x \\ \text{such that } \lim_{t\to-\infty} d(\gamma(t), \Xi(\vartheta_t\omega)) = 0 \end{array} \right\}.$$

$$\left(resp.\, W^s(\Xi, \omega) = \left\{ x \in X \;\middle|\; \begin{array}{l} \text{there is a } \omega\text{-solution } \gamma(t) \text{ on } [0, \infty) \text{ with } \gamma(0) = x \\ \text{such that } \lim_{t\to+\infty} d(\gamma(t), \Xi(\vartheta_t\omega)) = 0 \end{array} \right\}. \right)$$

Definition 5.5. A random variable Λ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in X is called tempered if for each $\omega \in \Omega$, the mapping $t \to \|\Lambda(\vartheta_t\omega)\|$ is subexponentially growing, namely, $t \to \|\Lambda(\vartheta_t\omega)\|$ is *μ-subexponentially growing for any $\mu > 0$.*

5.2. **Stochastic hyperbolic equilibrium and synchronising behavior.** In what follows we suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is the classic Wiener space, i.e., $\Omega = \{\omega : \omega(\cdot) \in C(\mathbb{R}, X),\ \omega(0) = 0\}$ endowed with the open compact topology, $\mathcal{F}$ is the associated Borel-σ-algebra, $\mathbb{P}$ is the Wiener measure and the σ-fields $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$, called a filtration, given by

$$\mathcal{F}_t := \sigma\{\omega(\tau) - \omega(s) : s, \tau \le t\}.$$

Remark 4. Denote

$$\omega^-(t) = \omega(-t), \quad t \in \mathbb{R}.$$

Since $\omega \in \Omega$ if and only if $\omega^- \in \Omega$, we may assume without loss of generality that each $\omega \in \Omega$ is an even function, i.e.,

$$\omega(-t) = \omega(t), \quad t \in \mathbb{R}.$$

We can define a measurable dynamical system $\vartheta := \{\vartheta_t\}_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ by $\vartheta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t)$. It is well-known that $\mathbb{P}$ is invariant and ergodic under ϑ. Then $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a metric dynamical system. A Wiener process $\{W(t)\}_{t\in\mathbb{R}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is given by

$$W(t, \omega) = \omega(t), \quad W(t, \vartheta_s\omega) = \omega(t + s) - \omega(s) = W(t + s, \omega) - W(s, \omega).$$

We now consider stochastic partial differential equations (SPDEs) with additive noise on X which have the form:

$$du + Audt = f(u)dt + dW(t), \tag{5.1}$$

where A is a hyperbolic sectorial operator in X, $f : X^\alpha \to H$ is a Lipshitz continuous mapping with Lipschitz constant L_f, W is a X valued Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance operator Q.

Let $B : X^\alpha \to X$ be a Lipschitz continuous Hilbert-Schmidt operator. When the noise term dW in (5.1) is replaced by $B(u)dW$, we will obtain a more general SPDE. The existence theory for such general equations is formulated as in Da Prato and Zabczyk [14]. Specifically, for any initial data $u_0 \in X^\alpha$, there exists a unique mild solution given by

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)} f(u(s))d\tau + \int_0^t e^{-A(t-s)} B(u(s))dW(s), \quad t \geq 0.$$

The above mild solution is only defined *almost surely* where the exceptional set may depend on the initial data u_0. Such a dependence contradicts the definition of an RDS. In fact, it is still an open problem to interpret general SPDEs as RDSs. However, for the special additive noise in (5.1), using a *perfection procedure*, the equation (5.1) indeed generates an RDS (see [1,17]). The key idea is to transform this SPDE into a random evolutionary equation with random coefficients. For this purpose, we first need to construct a stationary solution of linear Langevin SPDE in X^α:

$$dz + Azdt = dW(t). \tag{5.2}$$

The following existence result of stationary solution for equation (5.2) is inspired by the works of Brune and Schmalfuss [2, Lemma 4.2].

Lemma 5.6. *Suppose the Wiener process $W(t)$, $t \in \mathbb{R}$ has a covariance operator Q such that Tr$(Q(-A_u)^{2\alpha-1+\varepsilon}) < \infty$ and Tr$(QA_s^{2\alpha-1+\varepsilon}) < \infty$ for some $\varepsilon > 0$. Then the equation (5.2) possesses a stationary solution $\mathbb{R} \times \Omega \ni (t, \omega) \to Z(\vartheta_t\omega) \in X^\alpha$, which is $\{\mathcal{F}_{|t|}\}_{t\in\mathbb{R}}$-adapted and is given by a tempered and $\mathcal{F}_0$-measurable random variable $Z \in X^\alpha$. Moreover, $t \to Z(\vartheta_t\omega)$ is continuous.*

Proof. Split (5.2) into two linear equations

$$dz_s + A_s z_s dt = \Pi_s dW(t), \tag{5.3}$$

$$dz_u + A_u z_u dt = \Pi_u dW(t). \tag{5.4}$$

Since the real part of the spectrum $\operatorname{Re}\sigma(A_s) > 0$ and $\operatorname{Tr}(QA_s^{2\alpha-1+\varepsilon}) < \infty$, from [2, Lemma 4.2], there is a tempered and $\mathcal{F}_0$-measurable random variable $Z_s \in X_s^\alpha$ such that the stochastic process $Z_s(\vartheta_t\omega) \in X_s^\alpha$ is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$-adapted and continuous (in t) stationary solution for (5.3), where

$$Z_s(\vartheta_t\omega) = \int_{-\infty}^t e^{-A_s(t-\tau)} \Pi_s dW(\tau, \omega), \quad t \in \mathbb{R}.$$

Meanwhile, we use a coordinate transform $y(t) = z_u(-t)$ for (5.4), then y solves

$$dy - A_u y dt = \Pi_u dW(-t), \tag{5.5}$$

where $-A_u$ is a bounded linear operator on X^α with $\operatorname{Re}\sigma(-A_u) > 0$. By Remark 4, we can rewrite (5.5) as

$$dy - A_u y dt = \Pi_u dW(t).$$

Since $\mathrm{Tr}(Q(-A_u)^{2\alpha-1+\varepsilon}) < \infty$, repeating the argument above, there is a tempered and $\mathcal{F}_0$-measurable random variable $Y_u \in X_u^\alpha$ such that the stochastic process $Y_u(\vartheta_t\omega) \in X_u^\alpha$ is an $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$-adapted and continuous (in t) stationary solution for (5.5), where

$$Y_u(\vartheta_t\omega) = \int_{-\infty}^{t} e^{A_u(t-\tau)}\Pi_u dW(\tau,\omega), \quad t \in \mathbb{R}.$$

Then

$$Z_u(\vartheta_t\omega) = Y_u(\vartheta_{-t}\omega) = \int_{-\infty}^{-t} e^{-A_u(t+\tau)}\Pi_u dW(\tau,\omega), \quad t \in \mathbb{R}$$

is an $\{\mathcal{F}_{-t}\}_{t\in\mathbb{R}}$-adapted stationary solution for (5.4).

Consequently,

$$Z(\vartheta_t\omega) := Z_u(\vartheta_t\omega) + Z_s(\vartheta_t\omega) \in X^\alpha, \quad t \in \mathbb{R}$$

is the desired stationary solution for (5.2), and $Z : \Omega \to X^\alpha$ is tempered. □

Let $v(t) = u(t) - Z(\vartheta_t\omega)$. Then $v(t)$ satisfies the following random evolution equation

$$v_t + Av = f\big(v + Z(\vartheta_t\omega)\big). \tag{5.6}$$

Let $x(t, 0; v_0, \omega)$ denote the unique globally defined solution of (5.6) for the initial value $v_0 \in X^\alpha$. Define

$$\phi(t,\omega)v_0 := x(t, 0; v_0, \omega), \qquad \omega \in \Omega, v_0 \in X^\alpha.$$

Suppose ϕ is an RDS on X^α driven by the base flow ϑ on Ω.

We summarise our conclusions in the following theorem.

Theorem 5.7. *Suppose the assumptions of Lemma 5.6 hold and the Lipschitz constant L_f of f is sufficiently small such that*

$$ML_f \int_0^\infty \big(1 + \tau^{-\alpha}\big)e^{-\beta\tau} d\tau < 1.$$

Then

(a) *the RDS ϕ has a unique tempered stochastic hyperbolic equilibrium $\Xi : \Omega \to X^\alpha$, with its the ω-section unstable and stable manifolds being represented as*

$$W^u(\Xi,\omega) = \{y + \xi_\omega(y) : y \in X_u^\alpha\}$$
$$W^s(\Xi,\omega) = \{\zeta_\omega(y) + y : y \in X_s^\alpha\}$$

respectively, where

$$\xi_\omega : X_u^\alpha \to X_s^\alpha \quad \text{and} \quad \zeta_\omega : X_s^\alpha \to X_u^\alpha$$

are the graphs for the two manifolds.

Moreover,

$$\lim_{t\to-\infty} \|\gamma^u(t) - \Xi(\vartheta_t\omega)\|_\alpha = 0, \quad \gamma^u(0) \in W^u(\Xi,\omega);$$
$$\lim_{t\to\infty} \|\gamma^s(t) - \Xi(\vartheta_t\omega)\|_\alpha = 0, \quad \gamma^s(0) \in W^s(\Xi,\omega)$$

exponentially fast;

(b) *the ω-section unstable manifold $W^u(\Xi,\omega)$ completely characterizes the dynamics of ϕ, i.e., for every $x_0 \in X^\alpha$,*

$$\lim_{t\to\infty} \|\phi(t,\omega)x_0 - W^u(\Xi,\omega)\|_\alpha = 0. \tag{5.7}$$

Proof. For each $\omega \in \Omega$, let $h(t) = Z(\vartheta_t \omega)$ and $v(t) = u(t) - h(t)$. Then $v(t)$ satisfies the following evolution equation

$$v_t + Av = f\big(v + h(t)\big). \tag{5.8}$$

We learn from Lemma 5.6 that $h : \mathbb{R} \to X^\alpha$ is subexponentially growing.

Let

$$\mathcal{H} := \mathcal{H}[h] = \overline{\{h(\tau + \cdot); \tau \in \mathbb{R}\}}^{C(\mathbb{R}, X^\alpha)}$$

be the hull of h, and

$$\theta_t : \mathcal{H} \to \mathcal{H},\ t \in \mathbb{R}, \quad \text{as } \theta_t p(\cdot) = p(t + \cdot)$$

be the shift operator on $\mathcal{H}$. Now we embed (5.8) into the following cocycle system

$$v_t + Av = f\big(v + p(t)\big), \quad p \in \mathcal{H}. \tag{5.9}$$

The unique solution of (5.9) will generate a cocycle semiflow φ on X^α driven by the base flow θ on $\mathcal{H}$, namely,

$$\varphi(t, p)x_0 := x(t, 0; x_0, p), \qquad x_0 \in X^\alpha.$$

Employing the same techniques used in the Theorem 4.2, we know that φ has a unique subexponential hyperbolic equilibrium $\Gamma : \mathcal{H} \to X^\alpha$ such that its p-section unstable and stable manifolds being represented as

$$W^u(\Gamma, p) = \{y + \xi_p(y) : y \in X_u^\alpha\},$$
$$W^s(\Gamma, p) = \{\zeta_p(y) + y : y \in X_s^\alpha\}.$$

Moreover, $\Gamma : \mathcal{H} \to X^\alpha$ is continuous;

$$\lim_{p \to q} \delta_{X^\alpha}\big(W^u(\Gamma, p),\, W^u(\Gamma, q)\big) = 0, \qquad \forall\, q \in \mathcal{H}, \tag{5.10}$$

$$\lim_{p \to q} \delta_{X^\alpha}\big(W^s(\Gamma, p),\, W^s(\Gamma, q)\big) = 0, \qquad \forall\, q \in \mathcal{H}; \tag{5.11}$$

and

$$\lim_{t \to -\infty} \|\gamma^u(t) - \Gamma(\theta_t p)\|_\alpha = 0, \quad \gamma^u(0) \in W^u(\Gamma, p), \tag{5.12}$$
$$\lim_{t \to +\infty} \|\gamma^s(t) - \Gamma(\theta_t p)\|_\alpha = 0, \quad \gamma^s(0) \in W^s(\Gamma, p)$$

exponentially fast. Furthermore, by Theorem 4.3, we know that

$$\lim_{t \to \infty} \|\varphi(t, p)x_0 - W^u(\Gamma, \theta_t p)\|_\alpha = 0, \qquad \forall x_0 \in X^\alpha. \tag{5.13}$$

Let $p = h = Z(\vartheta.\omega)$ and define a random variable $\Xi : \Omega \to X^\alpha$ by

$$\Xi(\omega) = \Gamma\big(h\big) = \Gamma\big(Z(\vartheta.\omega)\big). \tag{5.14}$$

By the uniqueness of the solution of (5.8), we have for any $s \geq 0$,

$$\begin{aligned} \phi(s, \omega)\Xi(\omega) &= \varphi\big(s, Z(\vartheta.\omega)\big)\Gamma\big(Z(\vartheta.\omega)\big) \\ &= (\text{by the invariance of } \Gamma \text{ under } \varphi) = \Gamma\big(\theta_s Z(\vartheta.\omega)\big) \\ &= (\text{by the definition of } \theta) = \Gamma\big(Z(\vartheta_{s+}.\omega)\big) \\ &= \Gamma\big(Z(\vartheta.\vartheta_s\omega)\big) = (\text{by } (5.14)) = \Xi(\vartheta_s \omega), \end{aligned} \tag{5.15}$$

that is, Ξ is invariant under the RDS ϕ. Notice that the full solution

$$\Xi(\vartheta_t \omega) = \Gamma\big(\theta_t Z(\vartheta.\omega)\big), \quad t \in \mathbb{R}$$

is subexponentially growing. Accordingly, Ξ is a tempered equilibrium of ϕ.

Now let us demonstrate the hyperbolicity of Ξ. Let

$$W^u(\Xi, \omega) := W^u\big(\Gamma, Z(\vartheta.\omega)\big) = \{y + \xi_\omega(y) : y \in X_u^\alpha\}$$

and

$$W^s(\Xi,\omega) := W^s\big(\Gamma, Z(\vartheta.\omega)\big) = \{\zeta_\omega(y) + y : y \in X_s^\alpha\},$$

where $\xi_\omega(y) := \xi_{Z(\vartheta.\omega)}(y)$ and $\zeta_\omega(y) := \zeta_{Z(\vartheta.\omega)}(y)$. We can see that $W^u(\Xi,\omega)$ and $W^s(\Xi,\omega)$ are invariant under ϕ by a similar argument used in (5.15). For each $\gamma^u(0) \in W^u(\Xi,\omega) = W^u\big(\Gamma, Z(\vartheta.\omega)\big)$, by (5.14) and (5.12),

$$\lim_{t\to-\infty} \|\gamma^u(t) - \Xi(\vartheta_t\omega)\|_\alpha = \lim_{t\to-\infty} \|\gamma^u(t) - \Gamma\big(\theta_t Z(\vartheta.\omega)\big)\|_\alpha = 0 \tag{5.16}$$

exponentially fast. Similarly, we can verify

$$\lim_{t\to\infty} \|\gamma^s(t) - \Xi(\vartheta_t\omega)\|_\alpha = 0, \qquad \gamma^s(0) \in W^s(\Xi,\omega) \tag{5.17}$$

exponentially fast. Equalities (5.16) and (5.17) indicate that $W^u(\Xi,\omega)$ and $W^s(\Xi,\omega)$ are ω-section unstable and stable manifolds of Ξ respectively. Consequently, $\Xi : \Omega \to X^\alpha$ is hyperbolic.

To complete the proof of the theorem, we show that $W^u(\Xi,\omega)$ completely characterizes the dynamics of ϕ. Indeed, by (5.13), one has for every $x_0 \in X^\alpha$,

$$\lim_{t\to\infty} \|\phi(t,\omega)x_0 - W^u(\Xi,\omega)\|_\alpha = \lim_{t\to\infty} \|\varphi\big(t, Z(\vartheta.\omega)\big)x_0 - W^u\big(\Gamma, Z(\vartheta.\omega)\big)\|_\alpha = 0.$$

The proof is complete. □

Remark 5. The continuity of $\Gamma : \mathcal{H} \to X^\alpha$, (5.10), (5.11) and (5.7) manifest that for each $\omega \in \Omega$, the original system (5.1) exhibits a *global synchronising behavior* (in the sense of Remark 2) with $h(t) = Z(\vartheta_t\omega)$, $t \in \mathbb{R}$.

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