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Hypercyclic composition operators on the little Bloch space and the Besov spaces

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Abstract

Let $S(\mathbb{D})$ be the collection of all holomorphic self-maps on $\mathbb{D}$ of the complex plane $\mathbb{C}$, and C_φ the composition operator induced by $\varphi \in S(\mathbb{D})$. We obtain that there are no hypercyclic composition operators on the little Bloch space $\mathcal{B}_0$ and the Besov space B_p.

Keywords: Hypercyclic; Composition operator; Bloch space; Besov space

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $S(\mathbb{D})$ be the collection of all holomorphic self-maps on $\mathbb{D}$. We denote $dA(z) = dxdy$ the Lebesgue area measure on $\mathbb{C}$. For the composition operator C_φ induced by $\varphi \in S(\mathbb{D})$ is defined as

$$C_\varphi f(z) = f \circ \varphi(z), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

The one-to-one holomorphic functions that map $\mathbb{D}$ onto itself, called the *Möbius* transformations, and denoted by $\mathcal{M}$ (also $Aut(\mathbb{D})$), have the form $\lambda\varphi_a$, where $|\lambda| = 1$ and φ_a is the basic

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conformal automorphism defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad z \in \mathbb{D},$$

for $a \in \mathbb{D}$. The following identities are easily verified:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2}$$

and

$$(1-|z|^2)|\varphi_a'(z)| = 1 - |\varphi_a(z)|^2. \tag{1.1}$$

A linear space X of analytic functions on the open unit disk $\mathbb{D}$ is said to be *Möbius-invariant*, if $f \circ S \in X$ for all $f \in X$ and all $S \in \mathcal{M}$ and X has a seminorm $\| \ \|_X$ such that $\|f \circ S\|_X = \|f\|_X$ for each $f \in X$ and each $S \in \mathcal{M}$.

The well-known *Möbius-invariant* function space — the Besov spaces B_p $(1 < p < \infty)$ are defined as follows

$$B_p = \{f \in H(\mathbb{D}) : \ \|f\|_{B_p}^p = \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty\},$$

that is, $f \in B_p$ if and only if the function $(1-|z|^2)f' \in L^p(\mathbb{D}, d\lambda)$, where

$$d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}.$$

Although the measure λ is not a finite measure on $\mathbb{D}$, it is a *Möbius-invariant*. Indeed, by (1.1)

$$d\lambda(\varphi_a(z)) = \frac{|\varphi_a'(z)|^2}{(1-|\varphi_a(z)|^2)^2} dA(z) = \frac{dA(z)}{(1-|z|^2)^2} = d\lambda(z).$$

Hence we have the following change-of-variable formula

$$\int_{\mathbb{D}} f \circ \varphi_a(z) d\lambda(z) = \int_{\mathbb{D}} f(u) d\lambda(u),$$

for every positive measurable function f on $\mathbb{D}$, from which it is easily seen that

$$\|f \circ \varphi_a\|_{B_p} = \|f\|_{B_p}, \tag{1.2}$$

and the above identity also holds for $S = \lambda\varphi_a \in \mathcal{M}$ with $|\lambda| = 1$. Thus

$$\text{if } f \in B_p \text{ then } f \circ S \in B_p \text{ for all } S \in \mathcal{M}.$$

That is, B_p $(1 < p < \infty)$ are *Möbius-invariant* spaces.

For $p = 1$, the Besov space B_1 consists of the analytic functions f on $\mathbb{D}$ that admit the representation

$$f(z) = \sum_{n=1}^{\infty} a_n \varphi_{\lambda_n}(z), \quad z \in \mathbb{D},$$

where $\{a_n\} \in l^1$ and $\lambda_n \in \mathbb{D}$ for $n \in \mathbb{N}$. The norm in B_1 is defined as

$$\|f\|_{B_1} = \inf \left\{ \sum_{n=1}^{\infty} |a_n| : \ f(z) = \sum_{n=1}^{\infty} a_n \varphi_{\lambda_n}(z), z \in \mathbb{D} \right\}.$$

It is evident that B_1 is the *Möbius* invariant subset of the bounded analytic functions space H^∞. On the other hand, B_1 has the following definition,

$$B_1 = \{f \in H(\mathbb{D}) : \ \|f\|_{B_1} = \int_{\mathbb{D}} |f''(z)| dA(z) < \infty\},$$

even though the above semi-norm is not *Möbius-invariant*, the Besov space B_1 is the minimal *Möbius-invariant* space (see, e.g. [1,2]).

It is well-known that B_p $(1 < p < \infty)$ are Banach spaces endowed with the norm denoted by $\|f\|_p$,

$$\|f\|_p^p = |f(0)|^p + \|f\|_{B_p}^p.$$

Another *Möbius-invariant* space of analytic functions on $\mathbb{D}$ is the Bloch space $\mathcal{B}$,

$$\mathcal{B} = \{f \in H(\mathbb{D}) : \ \|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(z)| < \infty\}.$$

By (1.1) it follows that

$$\begin{aligned} \|f \circ \varphi_a\|_{\mathcal{B}} &= \sup_{z\in\mathbb{D}} (1 - |z|^2)|(f \circ \varphi_a)'(z)| \\ &= \sup_{z\in\mathbb{D}} (1 - |z|^2)|f'(\varphi_a(z))\varphi_a'(z)| \\ &= \sup_{z\in\mathbb{D}} (1 - |\varphi_a(z)|^2)|f'(\varphi_a(z))| \\ &= \|f\|_{\mathcal{B}}, \end{aligned} \tag{1.3}$$

for all $a \in \mathbb{D}$, and the above identities also hold for all $S \in \mathcal{M}$. That is,

$$\text{if } f \in \mathcal{B} \text{ then } f \circ S \in \mathcal{B} \text{ for all } S \in \mathcal{M}.$$

Hence $\mathcal{B}$ is a *Möbius-invariant* space.

The little Bloch space $\mathcal{B}_0$ consists of all $f \in \mathcal{B}$ such that

$$\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0.$$

Replacing "$\sup_{z\in\mathbb{D}}$" by "$\lim_{|z|\to 1}$" in (1.3), we get that $f \circ \varphi_a \in \mathcal{B}_0$ for every $f \in \mathcal{B}_0$ and $a \in \mathbb{D}$. Similarly, $\mathcal{B}_0$ is also a *Möbius-invariant* space. Both the Bloch space $\mathcal{B}$ and the little Bloch space $\mathcal{B}_0$ are Banach spaces under the norm

$$\|f\|_{Bloch} = |f(0)| + \|f\|_{\mathcal{B}}.$$

The above *Möbius-invariant* spaces have the relationship $B_1 \subset B_p \subset B_q \subset \mathcal{B}$ for each $1 < p < q < \infty$ (see, e.g. [19, Lemma 1.1]). Moreover, B_1 is a subset of the little Bloch space $\mathcal{B}_0$ (see [21]) and the Bloch space $\mathcal{B}$ is maximal among all *Möbius-invariant* Banach spaces of analytic functions on $\mathbb{D}$ (see, e.g. [17]). The Besov space B_2 is often referred to as the Dirichlet space $\mathcal{D}$, which is a Hilbert space with inner product

$$\langle f, g\rangle = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} dA(z)/\pi.$$

The problem of boundedness and compactness of C_φ has been studied in many function spaces, we refer the readers to the books [6,19,22,23]. In the recent time, the papers [4,5,7, 8,10,16] play important parts in the theory of the hypercyclicity of composition operators C_φ acting on analytic function spaces.

In the following, we introduce some definitions in dynamic systems. Let $L(X)$ denote the space of all linear continuous operators on a *separable infinite dimensional Banach space* X. For a positive integer n, the nth iterate of $T \in L(X)$ denoted by T^n, is the function obtained by composing T with itself n times.

A continuous linear operator $T \in L(X)$ is called *hypercyclic* provided there is some $f \in X$ such that the orbit

$$Orb(T, f) = \{T^n f : n = 0, 1, \ldots\}$$

is dense in X. Such a vector f is said to be a *hypercyclic* vector for T. Therefore, if a Banach space X admits a *hypercyclic* operator, X must be separable and infinite dimensional.

Since the polynomials are dense in the little Bloch space $\mathcal{B}_0$ (see, e.g. [22, Proposition 3.10]) and the polynomials are dense in B_p $(1 \le p < \infty)$ (see, e.g. [22, Proposition 6.2]), thus the little Bloch space $\mathcal{B}_0$ and B_p $(1 \le p < \infty)$ are *separable infinite dimensional Banach spaces*. This is why we investigate the composition operators on $\mathcal{B}_0$ and B_p $(1 \le p < \infty)$. For motivation, examples and background about linear dynamics we refer the reader to the books [3] by Bayart and Matheron, [7] by Grosse-Erdmann and Manguillot, and articles by Godefroy and Shapiro [9]. The papers [12–14] investigate other aspects of the hypercyclic property.

This paper is inspired by the result [7, Theorem 1.8]: "**No linear fractional composition operator is *hypercyclic* on the Dirichlet space** $\mathcal{D}$". We refer the readers to the paper [20], which contains the proof. Now in this paper, we want to characterize the hypercyclicity of composition operator C_φ acting on $\mathcal{B}_0$ and B_p $(1 \le p < \infty)$. We will show that "**No linear fractional composition operator is *hypercyclic* on the little Bloch space** $\mathcal{B}_0$, B_1 **and** B_p $(2 \le p < \infty)$". Since the Besov space B_2 is the Dirichlet space $\mathcal{D}$, hence we generalize the above result to some extend. The paper is organized as follows: some lemmas are listed in Section 2 and the main results are given in Sections 3 and 4.

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

2. Auxiliary results

A linear fractional transformation is a mapping of the form

$$\varphi(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$. We will write $LFT(\mathbb{D})$ to refer to the set of all such maps, which are self-maps of the unit disk $\mathbb{D}$. Those maps that take $\mathbb{D}$ onto itself are precisely the members of $Aut(\mathbb{D})$, so that $Aut(\mathbb{D}) \subset LFT(\mathbb{D}) \subset S(\mathbb{D})$.

We classify those maps according to their fixed point behavior, see [18, p. 5]:

(a) *Parabolic* members of $LFT(\mathbb{D})$ have their fixed point on $\partial\mathbb{D}$.

(b) *Hyperbolic* members of $LFT(\mathbb{D})$ must have an attractive fixed point in $\overline{\mathbb{D}}$, with the other fixed point outside $\mathbb{D}$, and lying on $\partial\mathbb{D}$ if and only if the map is an automorphism of $\mathbb{D}$.

(c) *Loxodromic* and *elliptic* members of $\varphi \in LFT(\mathbb{D})$ have a fixed point in $\mathbb{D}$ and a fixed point outside $\overline{\mathbb{D}}$. The elliptic ones are precisely the automorphisms in $LFT(\mathbb{D})$ with this fixed point configuration.

The following two lemmas are well-known, so we omit the details.

Lemma 2.1 (*[22, p. 82 (3.5)]*)**.** *For each $f \in \mathcal{B}$, we have*

$$|f(z)| \le \|f\|_{Bloch} \log \frac{2}{1-|z|^2}.$$

Lemma 2.2 (*[21, Theorem 8]*). *For every $f \in B_p$ with $1 < p < \infty$, we have*

$$|f(z)| \leq C\|f\|_{B_p}\left(\log\frac{2}{1-|z|^2}\right)^{1-1/p}.$$

Remark 2.3. From Lemmas 2.1 and 2.2, we obtain that the norm convergence in $\mathcal{B}_0$ (respectively, B_p $(1 < p < \infty)$) implies pointwise convergence.

Lemma 2.4. *Let $\varphi \in S(\mathbb{D})$ with an interior fixed point on $\mathbb{D}$. Suppose that C_φ is bounded on $\mathcal{B}_0$ (respectively, B_p $(1 < p < \infty)$). Then the operator C_φ is not hypercyclic on $\mathcal{B}_0$ (respectively, B_p $(1 < p < \infty)$).*

Proof. We prove for the little Bloch space $\mathcal{B}_0$. Let $a \in \mathbb{D}$ be the fixed point of φ. Suppose that $f \in \mathcal{B}_0$ is hypercyclic for C_φ and for each $g \in \mathcal{B}_0$ there exists a sequence $\{n_k\}$ such that $C_\varphi^{n_k} f$ tends to g in $\mathcal{B}_0$, that is,

$$\|C_\varphi^{n_k} f - g\|_{Bloch} \to 0 \quad \text{as} \quad k \to \infty.$$

By Remark 2.3 and $f(\varphi^{n_k}(a)) = f(a)$ for every $k \in \mathbb{N}$, it follows that

$$g(a) = \lim_{k\to\infty}(C_\varphi^{n_k} f)(a) = \lim_{k\to\infty}(C_{\varphi^{n_k}} f)(a) = \lim_{k\to\infty} f(\varphi^{n_k}(a)) = f(a),$$

that is not the case for every $g \in \mathcal{B}_0$. Thus the operator C_φ is not hypercyclic on $\mathcal{B}_0$. The proof for the Besov spaces B_p is similar, so we omit the details. This ends the proof. □

Remark 2.5. In the following, we need only consider φ is *parabolic* or *hyperbolic* case.

The following lemma is a necessary condition of the hypercyclic composition operator C_φ on $\mathcal{B}_0$ and B_p.

Lemma 2.6. *Suppose that $\varphi \in S(\mathbb{D})$ and the bounded composition operator C_φ is hypercyclic on $\mathcal{B}_0$ (respectively, B_p $(1 < p < \infty)$), then the compositional symbol φ is univalent.*

Proof. We prove for the space $\mathcal{B}_0$. Suppose that $\varphi(z_1) = \varphi(z_2)$ for some $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$. We pick $g \in \mathcal{B}_0$ such that $g(z_1) \neq g(z_2)$, and let $f \in \mathcal{B}_0$ be a hypercyclic vector for C_φ. By Lemma 2.1, for each $n \in \mathbb{N}$,

$$|C_\varphi^n f - g|(z) \leq \|C_\varphi^n f - g\|_{Bloch}\log\frac{2}{1-|z|^2}.$$

So for $\epsilon := |g(z_1) - g(z_2)| > 0$, we choose $n \in \mathbb{N}$ large enough so that

$$|C_\varphi^n f - g|(z) < \epsilon/4 \quad \text{for} \quad z = z_1,\ z_2. \tag{2.1}$$

On the other hand, since $\varphi(z_1) = \varphi(z_2)$, it follows that

$$\begin{aligned}
\epsilon &= |g(z_1) - g(z_2)| \leq |g(z_1) - C_\varphi^n(f)(z_1)| + |C_\varphi^n(f)(z_1) - g(z_2)| \\
&= |g(z_1) - C_\varphi^n(f)(z_1)| + |f(\varphi^n(z_1)) - g(z_2)| \\
&= |g(z_1) - C_\varphi^n(f)(z_1)| + |f(\varphi^n(z_2)) - g(z_2)| \\
&= |g(z_1) - C_\varphi^n(f)(z_1)| + |C_\varphi^n(f)(z_2) - g(z_2)| \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2},
\end{aligned}$$

we get a contraction. So φ must be univalent. The proof for the B_p $(1 < p < \infty)$ is similar. This completes the proof. $\square$

Lemma 2.7 (*[15, Schwarz–Pick Lemma]*). *For $\varphi \in S(\mathbb{D})$, we have*

$$\frac{1-|z|^2}{1-|\varphi(z)|^2}|\varphi'(z)| \leq 1$$

for all $z \in \mathbb{D}$.

Lemma 2.8 (*[11, Theorem 6.7]*). *Let T be an operator on a complex Fréchet space X. If $x \in X$ is such that $\{\lambda T^n x,\ \lambda \in \mathbb{C},\ |\lambda| = 1,\ and\ n \in \mathbb{N}_0\}$ is dense in X, then $orb(x, \lambda T)$ is dense in X for each $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

In particular, for any $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, T and λT have the same hypercyclic vectors, that is,

$$HC(T) = HC(\lambda T). \tag{2.2}$$

3. Hypercyclicity on the little Bloch space $\mathcal{B}_0$

For $\varphi \in S(\mathbb{D})$ and by the Schwarz–Pick lemma (Lemma 2.7), we obtain that

$$\begin{aligned}\|C_\varphi f\|_{Bloch} &= |f(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(\varphi(z))\varphi'(z)| \\ &\leq |f(\varphi(0))| + \sup_{z\in\mathbb{D}}(1-|\varphi(z)|^2)|f'(\varphi(z))| \\ &\leq |f(\varphi(0))| + \|f\|_{\mathcal{B}} < \infty,\end{aligned}$$

then the composition operator C_φ is always a bounded operator from $\mathcal{B}$ into $\mathcal{B}$. Moreover, if $\varphi \in \mathcal{B}_0$, then C_φ maps $\mathcal{B}_0$ into $\mathcal{B}_0$. In this section, we always assume C_φ is bounded on $\mathcal{B}_0$.

Case I. $\varphi \in Aut(\mathbb{D})$.

Theorem 3.1. *Suppose $\varphi \in Aut(\mathbb{D})$ and the composition operator C_φ is bounded on $\mathcal{B}_0$. Then C_φ is not hypercyclic on $\mathcal{B}_0$.*

Proof. Suppose that there is an $f \in \mathcal{B}_0$ such that the set $\{C_\varphi^k f : k \in \mathbb{N} \cup \{0\}\}$ is dense in $\mathcal{B}_0$. By (1.3) it follows that

$$\begin{aligned}\|C_\varphi^k f\|_{Bloch} &= |f(\varphi^k(0))| + \|f \circ \varphi^k\|_{\mathcal{B}} \\ &= |f(\varphi^k(0))| + \|f\|_{\mathcal{B}}.\end{aligned}$$

For $f_1(z) = z \in \mathcal{B}_0$, there exists a subsequence $\{\varphi^{k_j}\}_j$ such that

$$\|C_\varphi^{k_j} f - f_1\|_{Bloch} \to 0 \quad \text{as} \quad j \to \infty, \tag{3.1}$$

from which and Remark 2.3, it is clear that for $z = 0$,

$$f(\varphi^{k_j}(0)) \to 0 \quad \text{as} \quad j \to \infty. \tag{3.2}$$

By (3.1) and (3.2), we have

$$\begin{aligned}\|f\|_{Bloch} &= |f(0)| + \|f\|_{\mathcal{B}} \\ &= |f(0)| + \|C_\varphi^{k_j} f\|_{Bloch} - |f(\varphi^{k_j}(0))|\end{aligned}$$

$$\begin{aligned} &\to |f(0)| + \|f_1\|_{Bloch}, \quad j \to \infty \\ &= |f(0)| + 1. \end{aligned} \tag{3.3}$$

On the other hand, there exists another sequence $\{\varphi^{\bar{k}_j}\}_j$ such that

$$\|C_\varphi^{\bar{k}_j} f - f_1^2\|_{Bloch} \to 0 \text{ as } j \to \infty.$$

Similarly, $f \circ \varphi^{\bar{k}_j}(0) \to 0$ as $j \to \infty$. Besides

$$\begin{aligned} \|f\|_{Bloch} &= |f(0)| + \|f\|_{\mathcal{B}} \\ &= |f(0)| + \|C_\varphi^{\bar{k}_j} f\|_{Bloch} - |f \circ \varphi^{\bar{k}_j}(0)| \\ &\to |f(0)| + \|f_1^2\|_{Bloch}, \quad j \to \infty \\ &= |f(0)| + \frac{2\sqrt{3}}{9}. \end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we get a contraction. Therefore, the composition operator C_φ is not hypercyclic on $\mathcal{B}_0$. This completes the proof. □

Case II. $\varphi \notin Aut(\mathbb{D})$. For this case, we only consider φ with no interior fixed point in $\mathbb{D}$ by Remark 2.5.

Theorem 3.2. *Suppose $\varphi \notin Aut(\mathbb{D})$ and φ has no interior fixed point in $\mathbb{D}$. Further assume that the composition operator C_φ is bounded on $\mathcal{B}_0$, then C_φ is still not hypercyclic on $\mathcal{B}_0$.*

Proof. Suppose that C_φ is hypercyclic on $\mathcal{B}_0$ and $f \in \mathcal{B}_0$ is a hypercyclic vector for C_φ. Hence for every $g \in \mathcal{B}_0$, there exists $\{\varphi^k\}_k$ satisfying

$$\|C_\varphi^k f - g\|_{Bloch} \to 0 \text{ as } k \to \infty.$$

In particular, we choose $g(z) = nz$ for a fixed $n \in \mathbb{N}$, then there exists a subsequence $\{\varphi^{k_j}\}_j$ such that

$$\|C_\varphi^{k_j} f - nz\|_{Bloch} \to 0 \text{ as } j \to \infty.$$

From which and Remark 2.3, it follows that

$$f(\varphi^{k_j}(0)) \to 0 \text{ as } j \to \infty. \tag{3.5}$$

By Schwarz–Pick Lemma (Lemma 2.7) and (3.5), we get

$$\begin{aligned} &\|C_\varphi^{k_j} f\|_{Bloch} = |f(\varphi^{k_j}(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)|(f\circ\varphi^{k_j})'(z)| \\ &= |f(\varphi^{k_j}(0))| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(\varphi^{k_j}(z))\big(\varphi^{k_j}\big)'(z)| \\ &= |f(\varphi^{k_j}(0))| + \sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|\varphi^{k_j}(z)|^2} |\big(\varphi^{k_j}\big)'(z)|(1-|\varphi^{k_j}(z)|^2)|f'(\varphi^{k_j}(z))| \\ &\le |f(\varphi^{k_j}(0))| + \sup_{z\in\mathbb{D}}(1-|\varphi^{k_j}(z)|^2)|f'(\varphi^{k_j}(z))| \\ &\le |f(\varphi^{k_j}(0))| + \|f\|_{\mathcal{B}} \\ &\to \|f\|_{\mathcal{B}} = \|f\|_{Bloch} - |f(0)| < \infty, \quad j \to \infty. \end{aligned} \tag{3.6}$$

At the same time, since

$$\|nz\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1-|z|^2)n = n \to \infty \ \text{ as } \ n \to \infty.$$

Thus

$$\|C_\varphi^{k_j} f\|_{Bloch} \to \infty, \quad j \to \infty. \tag{3.7}$$

Combining (3.6) and (3.7), we get a contraction. Thus C_φ is not hypercyclic on $\mathcal{B}_0$. This completes the proof. □

From Lemma 2.8, we obtain that $HC(C_\varphi) = HC(\lambda C_\varphi)$ for $|\lambda| = 1$, hence the following corollary holds.

Corollary 3.3. *For every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $\varphi \in S(\mathbb{D})$. Suppose that the composition operator C_φ is bounded in $\mathcal{B}_0$, then the operator λC_φ is not hypercyclic on $\mathcal{B}_0$.*

Since the Besov space $B_1 \subset \mathcal{B}_0$, we obtain the following corollary.

Corollary 3.4. *For every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and $\varphi \in S(\mathbb{D})$. Suppose that the composition operator C_φ is bounded in B_1, then the operator λC_φ is not hypercyclic on B_1.*

4. Hypercyclicity on B_p

This section is similar to Section 3, we include the brief proof for the convenience of the readers.

Case I. $\varphi \in Aut(\mathbb{D})$.

Theorem 4.1. *Suppose $\varphi \in Aut(\mathbb{D})$ and the composition operator C_φ is bounded on B_p ($1 < p < \infty$), then C_φ is not hypercyclic on B_p.*

Proof. Suppose that there is an $f \in B_p$ such that $\{C_\varphi^k f : k \in \mathbb{N} \cup \{0\}\}$ is dense in B_p. By (1.2) it follows that

$$\begin{aligned}\|C_\varphi^k f\|_p^p &= |f(\varphi^k(0))|^p + \|f \circ \varphi^k\|_{B_p}^p \\ &= |f(\varphi^k(0))|^p + \|f\|_{B_p}^p.\end{aligned}$$

For $f_1(z) = z \in B_p$, there exists a subsequence $\{\varphi^{k_j}\}$ such that

$$\|C_\varphi^{k_j} f - f_1\|_p \to 0 \ \text{ as } \ j \to \infty,$$

from which and Remark 2.3, it is clear that for $z = 0$,

$$f(\varphi^{k_j}(0)) \to 0 \ \text{ as } \ j \to \infty.$$

By the above three formulas, we have

$$\begin{aligned}\|f\|_p^p &= |f(0)|^p + \|f\|_{B_p}^p \\ &= |f(0)|^p + \|C_\varphi^{k_j} f\|_p^p - |f(\varphi^{k_j}(0))|^p \\ &\to |f(0)|^p + \|f_1\|_p^p, \quad j \to \infty \\ &= |f(0)|^p + \frac{1}{2(p-1)}.\end{aligned} \tag{4.1}$$

On the other hand, there exists another sequence $\{\varphi^{\hat{k}_j}\}$ such that

$$\|C_{\varphi}^{\hat{k}_j} f - f_1^2\|_p \to 0 \;\text{ as }\; j \to \infty.$$

Similarly, $f \circ \varphi^{\hat{k}_j}(0) \to 0$ as $j \to \infty$. Using the $Beta$ function

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad \text{for } \; p > 0, \; q > 0,$$

it follows that

$$\begin{aligned} \|f\|_p^p &= |f(0)|^p + \|f\|_{B_p}^p \\ &= |f(0)|^p + \|C_{\varphi}^{\hat{k}_j} f\|_p^p - |f \circ \varphi^{\hat{k}_j}(0)| \\ &\to |f(0)|^p + \|f_1^2\|_p^p, \quad j \to \infty \\ &= |f(0)|^p + 2^{p-1}B(\frac{p}{2}+1, p-1). \end{aligned} \tag{4.2}$$

From (4.1) and (4.2), we get a contraction. Thus the composition operator C_φ is not hypercyclic on B_p $(1 < p < \infty)$. This completes the proof. □

Case II. $\varphi \notin Aut(\mathbb{D})$, we only consider the case $p - 2 \geq 0$.

Theorem 4.2. *Suppose $\varphi \notin Aut(\mathbb{D})$ and φ has no interior fixed point in $\mathbb{D}$. Further assume that the composition operator C_φ is bounded on B_p $(2 \leq p < \infty)$, then C_φ is still not hypercyclic on B_p.*

Proof. Suppose that C_φ is hypercyclic on B_p and $f \in B_p$ is a hypercyclic vector for C_φ. Then for each $g \in B_p$, there exists $\{\varphi^k\}_k$ satisfying

$$\|C_{\varphi}^{k} f - g\|_p \to 0 \;\text{ as }\; k \to \infty.$$

In particular, we choose $g(z) = nz^2$ for a fixed $n \in \mathbb{N}$, then there exists a subsequence $\{\varphi^{k_j}\}_j$ such that

$$\|C_{\varphi}^{k_j} f - nz^2\|_p \to 0 \;\text{ as }\; j \to \infty.$$

From which and Remark 2.3, it follows that

$$f(\varphi^{k_j}(0)) \to 0 \;\text{ as }\; j \to \infty. \tag{4.3}$$

For $p \geq 2$, by Schwarz–Pick lemma (Lemma 2.7) and Lemma 2.6, we get that

$$\begin{aligned} &\|C_{\varphi}^{k_j} f\|_p^p = |f(\varphi^{k_j}(0))|^p + \|f \circ \varphi^{k_j}\|_{B_p}^p \\ &= |f(\varphi^{k_j}(0))|^p + \int_{\mathbb{D}} |(f \circ \varphi^{k_j})'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= |f(\varphi^{k_j}(0))|^p + \int_{\mathbb{D}} |f'(\varphi^{k_j}(z))(\varphi^{k_j})'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= |f(\varphi^{k_j}(0))|^p + \int_{\mathbb{D}} \frac{(1-|\varphi^{k_j}(z)|^2)^p |f'(\varphi^{k_j}(z))|^p}{(1-|\varphi^{k_j}(z)|^2)^p} |(\varphi^{k_j})'(z)|^p (1-|z|^2)^{p-2} dA(z) \\ &= |f(\varphi^{k_j}(0))|^p \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{D}}\left(\frac{(1-|z|^2)|(\varphi^{k_j})'(z)|}{1-|\varphi^{k_j}(z)|^2}\right)^{p-2}\frac{(1-|\varphi^{k_j}(z)|^2)^p|f'(\varphi^{k_j}(z))|^p|(\varphi^{k_j})'(z)|^2}{(1-|\varphi^{k_j}(z)|^2)^2}dA(z)\\
&\le|f(\varphi^{k_j}(0))|^p+\int_{\mathbb{D}}\frac{(1-|\varphi^{k_j}(z)|^2)^p|f'(\varphi^{k_j}(z))|^p|(\varphi^{k_j})'(z)|^2}{(1-|\varphi^{k_j}(z)|^2)^2}dA(z)\ \ (\varphi \text{ is univalent})\\
&=|f(\varphi^{k_j}(0))|^p+\int_{\varphi^{k_j}(\mathbb{D})}|f'(w)|^p(1-|w|^2)^{p-2}dA(w)\\
&\le|f(\varphi^{k_j}(0))|^p+\int_{\mathbb{D}}|f'(w)|^p(1-|w|^2)^{p-2}dA(w)\\
&=|f(\varphi^{k_j}(0))|^p+\|f\|_{B_p}^p\\
&\to\|f\|_{B_p}^p=\|f\|_p^p-|f(0)|^p<\infty,\quad j\to\infty.
\end{aligned}\tag{4.4}
$$

At the same time, since

$$
\begin{aligned}
\|nz^2\|_p^p&=0+\|nz^2\|_{B_p}^p=\int_{\mathbb{D}}|(nz^2)'|^p(1-|z|^2)^{p-2}dA(z)\\
&=\int_{\mathbb{D}}(2n)^p|z|^p(1-|z|^2)^{p-2}dA(z)\\
&=\int_0^{2\pi}\frac{d\theta}{2\pi}\int_0^1(2n)^pr^p(1-r^2)^{p-2}dr\\
&=\int_0^1 2^{p-1}n^p(r^2)^{p/2+1-1}(1-r^2)^{p-1-1}d(r^2)\\
&=n^p2^{p-1}B\left(\frac{p}{2}+1,p-1\right)\to\infty\ \text{ as }\ n\to\infty.
\end{aligned}
$$

Thus

$$
\|C_\varphi^{k_j}f\|_p^p\to\infty\ \text{ as }\ j\to\infty.\tag{4.5}
$$

From the above, comparing (4.4) with (4.5), we get a contraction. Thus C_φ is not hypercyclic on B_p $(2\le p<\infty)$. This completes the proof. □

Corollary 4.3. *For any $\lambda\in\mathbb{C}$ with $|\lambda|=1$ and $\varphi\in S(\mathbb{D})$. Suppose that the composition operator C_φ is bounded in B_p $(2\le p<\infty)$, then the operator λC_φ is not hypercyclic on B_p.*

Open question: Is the composition operator C_φ hypercyclic on the space B_p $(1<p<2)$ with $\varphi\notin Aut(\mathbb{D})$?

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References

[1] J. Arazy, S.D. Fisher, Some aspects of the minimal, Möbius-invariant space of analytic functions on the unit disc, in: Interpolation Spaces and Allied Topics in Analysis (Lund, 1983), in: Lecture Notes in Math., vol. 1070, Springer, Berlin, 1984, pp. 24–44.

[2] J. Arazy, S.D. Fisher, The uniqueness of the Dirichlet space among Möbius-invariant Hilbert spaces, Illinois J. Math. 29 (1985) 449–462.

[3] F. Bayart, E. Matheron, Dynamics of Linear Operators, Camberidge University Press, 2009.
[4] P.S. Bourdon, J.H. Shapiro, Cyclic composition operators on H^2, Proc. Sympos. Pure Math. 51 (Part 2) (1990) 43–53.
[5] P.S. Bourdon, J.H. Shapiro, Cyclic phenomena for composition operators, Mem. Amer. Math. Soc. 596 (1997) 1–150.
[6] C.C. Cowen, B.D. MacCluer, Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton, FL, 1995.
[7] E.A. Gallardo-Gutiérrez, A. Montes-Rodríguez, The role of the spectrum in cyclic behavior of composition operators, Mem. Amer. Math. Soc. (2004).
[8] R.M. Gethner, J.H. Shapiro, Universal vectors for operators on spaces of holomorphic functions, Proc. Amer. Math. Soc. 100 (1987) 281–288.
[9] G. Godefroy, J.H. Shapiro, Operators with dense invariant cyclic vector manifolds, J. Funct. Anal. 98 (1991) 229–269.
[10] K.G. Grosse-Erdmann, Recent developments in hypercyclicity, Rev. R. Acad. Cien. Serie A. Mat. 97 (2003) 273–286.
[11] K.G. Grosse-Erdmann, A.P. Manguillot, Linear Chaos, Springer, New York, 2011.
[12] Y.X. Liang, Z.H. Zhou, Hereditarily hypercyclicity and supercyclicity of different weighted shifts, J. Korean Math. Soc. 51 (1) (2014) 125–135.
[13] Y.X. Liang, Z.H. Zhou, Hypercyclic behaviour of multiples of composition operators on the weighted Banach space, Bull. Belg. Math. Soc. Simon Stevin 21 (2014) 1–17.
[14] Y.X. Liang, Z.H. Zhou, Supercyclic tuples of the adjoint weighted composition operators on Hilbert spaces, Bull. Iranian Math. Soc. 41 (1) (2015) 121–139.
[15] B.D. Maccluer, K. Stroethoff, R.H. Zhao, Generalized Schwarz–Pick estimates, Proc. Amer. Math. Soc. 131 (2002) 593–599.
[16] A. Miralles, E. Wolf, Hypercyclic composition operators on $H_{\nu,0}^{\infty}$ spaces, Math. Nachr. 286 (2013) 34–41.
[17] L.A. Rubel, R. Timoney, An extremal property of the Bloch space, Proc. Amer. Math. Soc. 75 (1979) 45–49.
[18] J.H. Shapiro, Composition Operators and Classical Function Theory, Springer-Verlag, New York, 1993.
[19] M. Tjani, Compact composition operators on some Möbius invariant Banach spaces, doctor of philosophy, 1996.
[20] X.H. Wang, G.F. Cao, Cyclic composition operators on Dirichlet space, Acta Math. Sci. Ser. A Chin. Ed. 23A (2003) 91–95 (in Chinese).
[21] K. Zhu, Analytic Besov spaces, J. Math. Anal. Appl. 157 (1991) 318–336.
[22] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, in: Graduate Texts in Mathematics, vol. 226, Springer, New York, 2005.
[23] N. Zorboska, Composition operators on weighted Dirichlet spaces, Proc. Amer. Math. Soc. 126 (1998) 2013–2023.