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Optimal investment strategies for an insurer and a reinsurer with a jump diffusion risk process under the CEV model

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ABSTRACT

In this paper, we consider the optimal investment problem for both an insurer and a reinsurer. The insurer's wealth process is described by a jump diffusion risk model and the insurer can purchase proportional reinsurance from the reinsurer. Both the insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset whose price process follows the constant elasticity of variance (CEV) model. Moreover, the correlation between risk model and the risky asset's price is considered. The objective is maximizing the expected utility of the insurer's and the reinsurer's terminal wealth. Applying stochastic control theory, we establish the corresponding Hamilton–Jacobi–Bellman (HJB) equations and derive optimal investment–reinsurance strategies for exponential utility function. Finally, numerical examples are provided to analyze the effects of parameters on the optimal strategies.

1. Introduction

Nowadays, optimal investment and reinsurance problem for an insurer has been extensively studied in the literature. For example, Browne [1] investigated the problem for maximizing the utility of terminal wealth and minimizing the probability of ruin for an insurer with the diffusion risk model. A similar problem to Browne [1] for a jump diffusion risk model was studied by Yang and Zhang [2]. For compound Poisson risk model, Hipp and Plum [3] assumed the insurer can invest in a risky asset and obtained the optimal investment strategy for ruin probability minimization. Later Liu and Yang [4] generalized the research of Hipp and Plum [3] by allowing the insurer to invest in a risky-asset and a risk-free asset. Wang et al. [5] obtained the optimal investment strategies for an insurer under different criteria by martingale approach. With regard to reinsurance, Schmidli [6] studied the optimal reinsurance policy to minimize the ruin probability. Promislow and Young [7] obtained the optimal reinsurance–investment strategy for an insurer to minimize the ruin probability. Bai and Guo [8] studied the optimal investment–reinsurance problem with multiple risky assets for utility maximization and ruin probability minimization. Cao and Wan [9] obtained the optimal proportional reinsurance and investment strategies for an insurer to maximize the expected exponential and power utilities of terminal wealth. Liang and Bayraktar [10] considered the optimal reinsurance and investment problem for an insurer whose claim process is governed by an unobservable Markov-modulated compound Poisson process. For a jump diffusion risk model, Irgens and Paulsen [11] obtained the proportional reinsurance and investment strategies of utility maximization for three different utility functions; Huang et al. [12] studied the optimal control problem for an insurer with constrained control variables.

In the above-mentioned literatures, they generally assume that the risky asset's prices are driven by geometric Brownian motions (GBMs), which implies the volatilities of the risky asset's prices are constant and deterministic. But empirical

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analysis has shown that the volatility is not constant, see [13], and the references therein. In this paper, we assume that the price process of the risky asset follows the constant elasticity of variance (CEV) model, which is a natural extension of geometric Brownian motion (GBM) and more practical. Moreover, the CEV model can explain the implied volatility skew and it is analytically tractable in comparison with other stochastic volatility models. The CEV model was proposed by Cox and Ross [14]. Beckers [15], Davydov and Linetsky [16], Jones [17] studied the option pricing problems under the CEV model. Xiao et al. [18] applied the CEV model to investigate the investment problem for pension plan and derived the optimal strategy for logarithm utility function by using Legendre transform and dual theory. Gao [19,20] studied the investment problem for pension plan and obtained the optimal solutions for CRRA and CARA utility functions. Nowadays, the CEV model has also been commonly used in optimal reinsurance and investment problems. Gu et al. [21] considered optimal investment and proportional reinsurance problem for utility maximization. Liang et al. [22] and Lin and Li [23] assumed the price of risky asset followed the CEV model and studied the proportional reinsurance problem under the jump diffusion risk model.

However, most of the researches mentioned previously only consider the optimal strategy for an insurer. But in practice, the optimal reinsurance strategy for an insurer may not be optimal for a reinsurer. Thus, it is necessary to take the management of the reinsurer into account. Currently, some researchers began to study optimal investment–reinsurance problem for both the insurer and the reinsurer. For example, Li et al. [24,25] studied the optimal investment problem for an insurer and a reinsurer for utility maximization. Zhao et al. [26], Li et al. [27] investigated the time-consistent reinsurance–investment strategy for an insurer and a reinsurer under a mean–variance framework. But most of them describe the basic risk process by a Brownian motion with drift. In this paper, we describe the insurer's wealth process by jump diffusion risk model, which is a compound Poisson process perturbed by a Brownian motion. Both the insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset whose price process follows the CEV model. Moreover, we consider the correlation between risk model and the risky asset's price. The objective is to maximize the expected utility of the insurer's and the reinsurer's terminal wealth. By solving the corresponding Hamilton–Jacobi–Bellman (HJB) equations via Legendre transform and dual theory, closed-form solutions to the problems of expected exponential utility maximization are derived under some given assumptions. Furthermore, numerical examples are presented to analyze the effects of parameters on the optimal strategies.

This paper is organized as follows. In Section 2, we introduce the formulation of the model. Section 3 and Section 4 derive the optimal investment–reinsurance strategies to maximize the utility of the insurer's and reinsurer's terminal wealth. In Section 5, numerical examples are carried out to analyze the effects of parameters on the optimal strategies. Finally, we give conclusions in Section 6.

2. Model formulation

In this paper, we consider a filtered complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, P)$ satisfying the usual condition, where $\mathscr{F}_t$ is the information of the market available up to time t, $[0, T]$ is a fixed and finite time horizon. All stochastic processes introduced below are assumed to be well-defined and adapted to $\{\mathscr{F}_t\}_{t\in[0,T]}$.

2.1. Wealth process of the insurer

Without reinsurance and investment, the wealth process of the insurer is described by the jump diffusion risk model

$$\mathrm{d}X(t) = c\mathrm{d}t - \mathrm{d}\left(\sum_{i=1}^{N(t)} Z_i\right) + \beta \mathrm{d}\overline{W}(t), \tag{2.1}$$

where c is premium rate of the insurer, $\sum_{i=1}^{N(t)} Z_i$ is a compound Poisson process representing the cumulative amount of claims in time interval $[0, t]$. $\{N(t), t \geq 0\}$ is a homogeneous Poisson process with intensity $\lambda > 0$ and the claim sizes $\{Z_i (i \geq 1)\}$ are independent and identically distributed ($i.i.d.$) positive random variables with common distribution $F(z)$ and independent of $N(t)$. Denote the mean value $E[Z_i] = \mu_z$, and moment generating function $M_Z(r) = E[e^{rZ_i}]$. We assume that $E[Ze^{rZ}] = M_Z'(r)$ exists for $0 < r < \zeta$ and that $\lim_{r\to\zeta} E[Ze^{rZ}] = \infty$ for some $0 < \zeta \leq +\infty$. $\beta \geq 0$ is a constant, and $\{\overline{W}(t)\}_{t\geq 0}$ is a standard Brownian motion. Suppose the premium is calculated according to the expected value principle, i.e., $c = (1+\eta)\lambda\mu_z$, where $\eta > 0$ is the positive safety loading of the insurer. The diffusion term $\beta \mathrm{d}\overline{W}(t)$ represents the uncertainty related to the insurer's wealth process at time t.

The insurer is allowed to purchase proportional reinsurance from the reinsurer to hedge insurance risk. Let $q_1(t) \in [0, 1]$ be the reinsurance proportion, that is, when the ith claim Z_i occurs, the insurer pays only $q_1(t)Z_i$ while the reinsurer pays $(1 - q_1(t))Z_i$. The reinsurance premium is calculated according to the expected value principle, i.e., $\delta(q_1) = (1+\theta)(1 - q_1(t))\lambda\mu_z$, where $\theta > \eta$ is the safety loading of the reinsurer. Moreover, the insurer is allowed to invest in a risk-free asset and a risky asset. The price process of the risk-free asset $S_0(t)$ is given by

$$\mathrm{d}S_0(t) = rS_0(t)\mathrm{d}t, \qquad S_0(0) = s_0, \tag{2.2}$$

where $r > 0$ is the risk-free interest rate. The price process of the risky asset $S(t)$ is described by the CEV model:

$$\mathrm{d}S(t) = r_1 S(t)\mathrm{d}t + \sigma (S(t))^{k+1}\mathrm{d}W(t), \qquad S(0) = s, \tag{2.3}$$

where $r_1 > r$ is the appreciation rate of the risky asset $S(t)$, $\{W(t)\}$ is a standard Brownian motion defined on $(\Omega, \mathscr{F}, P)$. k is the elasticity parameter and satisfies the general condition $k \leq 0$ according to Gao [19]. $\sigma(S(t))^k$ stands for the instantaneous volatility of risky asset. The correlation coefficient of $\overline{W}(t)$ and $W(t)$ are denoted by ρ, i.e., $E[\overline{W}(t)W(t)] = \rho t$.

Let $\pi_1(t)$ be the money amount invested in the risky asset at time t by the insurer, then $X(t) - \pi_1(t)$ is the money amount invested in the risk-free asset. An investment–reinsurance strategy is described by a pair process $u_1(t) = (\pi_1(t), q_1(t))$. Given a strategy $u_1(t)$, the insurer's wealth process $X(t)$ follows the following dynamic:

$$\begin{cases} \mathrm{d}X(t) &= [rX(t) + (r_1 - r)\pi_1(t) + (\eta - \theta + (1+\theta)q_1(t))\lambda\mu_z]\mathrm{d}t + \beta \mathrm{d}\overline{W}(t) \\ &\quad + \pi_1(t)\sigma(S(t))^k \mathrm{d}W(t) - q_1(t)\mathrm{d}\left(\sum_{i=1}^{N(t)} Z_i\right), \\ X(0) &= x_0. \end{cases} \tag{2.4}$$

An investment–reinsurance strategy $\{u_1(v) := (\pi_1(v), q_1(v)), v \in [t, T]\}$ is said to be admissible if it is $\{\mathscr{F}_t\}_{t\in[0,T]}$-progressively measurable and satisfies $(\pi_1(t), q_1(t)) \in \Pi_1$, where $\Pi_1 = \{(\pi_1(t), q_1(t)) : 0 \leq q_1(t) \leq 1, E[\int_0^T (\pi_1(t))^2 \mathrm{d}t] < \infty\}$.

Suppose that the insurer has a utility function $U_1(\cdot)$ which is strictly concave and continuously differentiable on $(-\infty, +\infty)$. The insurer aims to maximize the expected utility of terminal wealth, i.e.,

$$\max_{(\pi_1, q_1)\in\Pi_1} E[U_1(X(T))]. \tag{2.5}$$

2.2. Wealth process of the reinsurer

In the presence of the proportional reinsurance contract, the wealth process of the reinsurer $Y(t)$ is given by:

$$\mathrm{d}Y(t) = (1+\theta)(1 - q_2(t))\lambda\mu_z \mathrm{d}t - (1 - q_2(t))\mathrm{d}\left(\sum_{i=1}^{N(t)} Z_i\right), \tag{2.6}$$

where $q_2(t)$ is the reinsurance strategy chosen by the reinsurer. In reality, the reinsurer will accept the optimal retention level chosen by the insurer when the reinsurance strategy of the reinsurer is smaller than that of the insurer. While in the opposite case, in order to prevent large losses, the reinsurer may not accept the optimal retention level chosen by the insurer.

Let $\pi_2(t)$ represent the money amount invested in the risky asset at time t by the reinsurer, then $Y(t) - \pi_2(t)$ is the money amount invested in the risk-free asset. The wealth process $Y(t)$ is given by:

$$\begin{cases} \mathrm{d}Y(t) &= [rY(t) + \pi_2(t)(r_1 - r) + (1+\theta)(1 - q_2(t))\lambda\mu_z]\mathrm{d}t + \pi_2(t)\sigma(S(t))^k \mathrm{d}W(t) \\ &\quad - (1 - q_2(t))\mathrm{d}\left(\sum_{i=1}^{N(t)} Z_i\right), \\ Y(0) &= y_0. \end{cases} \tag{2.7}$$

$\{u_2(v) := (\pi_2(v), q_2(v)), v \in [t, T]\}$ is said to be admissible if it is $\{\mathscr{F}_t\}_{t\in[0,T]}$-progressively measurable and satisfies $(\pi_2(t), q_2(t)) \in \Pi_2$, where $\Pi_2 = \{(\pi_2(t), q_2(t)) : 0 \leq q_2(t) \leq 1, E[\int_0^T (\pi_2(t))^2 \mathrm{d}t] < \infty\}$.

The objective of the reinsurer is assumed to maximize the expected utility of terminal wealth $Y(T)$, i.e.,

$$\max_{(\pi_2, q_2)\in\Pi_2} E[U_2(Y(T))]. \tag{2.8}$$

3. Optimal strategy for the insurer

In this section, we first provide the general framework for the optimization problem (2.5) by using the classical tools of stochastic optimal control, and then try to find optimal strategy for exponential utility function via Legendre transform and dual theory.

3.1. General framework

Suppose the insurer has an exponential utility function

$$u_1(x) = -\frac{1}{\gamma_1} e^{-\gamma_1 x}, \qquad \gamma_1 > 0. \tag{3.1}$$

By using stochastic optimal control, we define the value function as

$$V(t, s, x) = \sup_{(\pi_1, q_1)\in\Pi_1} E[u_1(X(T)) | X(t) = x, S(t) = s], \qquad 0 \leq t < T \tag{3.2}$$

with $V(T, s, x) = u_1(x)$.

According to Fleming and Soner [28], if $V(t,s,x) \in C^{1,2,2}([0,T]\times\mathscr{R}\times\mathscr{R})$, then V satisfies the following Hamilton–Jacobi–Bellman (HJB) equation:

$$\sup_{(\pi_1,q_1)\in\Pi_1} \mathscr{A}^{\pi_1,q_1}V(t,s,x)=0, \qquad 0\le t<T \tag{3.3}$$

with the boundary condition $V(T,s,x)=u_1(x)$. Here $\mathscr{A}^{\pi_1,q_1}$ is an operator and

$$\begin{aligned}\mathscr{A}^{\pi_1,q_1}V(t,s,x) \;=\; & V_t+r_1sV_s+[rx+\pi_1(r_1-r)+(\eta-\theta+(1+\theta)q_1)\lambda\mu_z]V_x\\ &+\frac{1}{2}(\beta^2+\pi_1^2\sigma^2s^{2k}+2\beta\rho\pi_1\sigma s^k)V_{xx}+(\rho\beta\sigma s^{k+1}+\pi_1\sigma^2s^{2k+1})V_{sx}\\ &+\frac{1}{2}\sigma^2s^{2k+2}V_{ss}+\lambda E(V(t,s,x-q_1z)-V(t,s,x)).\end{aligned} \tag{3.4}$$

The first order maximizing condition for the optimal investment strategy is

$$\pi_1^*=-\frac{(r_1-r)V_x+\beta\rho\sigma s^kV_{xx}+\sigma^2s^{2k+1}V_{sx}}{\sigma^2s^{2k}V_{xx}}. \tag{3.5}$$

Putting (3.4) and (3.5) into HJB equation (3.3), after simplification, we have

$$\begin{aligned}&V_t+r_1sV_s+[rx+(\eta-\theta)\lambda\mu_z]V_x+\frac{1}{2}\beta^2V_{xx}+\rho\beta\sigma s^{k+1}V_{sx}+\frac{1}{2}\sigma^2s^{2k+2}V_{ss}\\ &-\frac{((r_1-r)V_x+\beta\rho\sigma s^kV_{xx}+\sigma^2s^{2k+1}V_{sx})^2}{2\sigma^2s^{2k}V_{xx}}+\sup_{q_1\in[0,1]}[(1+\theta)q_1\lambda\mu_zV_x\\ &+\lambda E(V(t,s,x-q_1z)-V(t,s,x))]=0.\end{aligned} \tag{3.6}$$

3.2. Optimal results

To solve Eq. (3.6), we try to conjecture a solution in the following form

$$V(t,s,x)=-\frac{1}{\gamma_1}exp[-\gamma_1xe^{r(T-t)}+h(t,s)], \tag{3.7}$$

with $h(T,s)=0$. Then we have

$$\begin{aligned}&V_t=(\gamma_1xre^{r(T-t)}+h_t)V, \qquad V_s=h_sV,\\ &V_{ss}=(h_s^2+h_{ss})V, \quad V_x=(-\gamma_1e^{r(T-t)})V,\\ &V_{xx}=(\gamma_1^2e^{2r(T-t)})V, \quad V_{xs}=(-\gamma_1e^{r(T-t)}h_s)V,\\ &E[V(t,s,x-q_1z)-V(t,s,x)]=V(M_Z(\gamma_1q_1e^{r(T-t)})-1).\end{aligned}$$

Substituting these into (3.6) yields

$$\begin{aligned}&V_t+r_1sV_s+[rx+(\eta-\theta)\lambda\mu_z]V_x+\frac{1}{2}\beta^2V_{xx}+\rho\beta\sigma s^{k+1}V_{sx}+\frac{1}{2}\sigma^2s^{2k+2}V_{ss}\\ &-\frac{((r_1-r)V_x+\beta\rho\sigma s^kV_{xx}+\sigma^2s^{2k+1}V_{sx})^2}{2\sigma^2s^{2k}V_{xx}}+\sup_{q_1\in[0,1]}f(q_1,t)=0,\end{aligned} \tag{3.8}$$

where $f(q_1,t)=(1+\theta)q_1\lambda\mu_z(-\gamma_1e^{r(T-t)})V+\lambda V(M_Z(\gamma_1q_1e^{r(T-t)})-1)$.

Differentiating $f(q_1,t)$ with respect to q_1, we get

$$\begin{aligned}\frac{\partial f(q_1,t)}{\partial q_1} &= -(1+\theta)\lambda\mu_z\gamma_1e^{r(T-t)}V+\lambda\gamma_1e^{r(T-t)}E[Ze^{\gamma_1q_1e^{r(T-t)}Z}]V,\\ \frac{\partial^2 f(q_1,t)}{\partial q_1^2} &= \lambda\gamma_1^2e^{2r(T-t)}E[Z^2e^{\gamma_1q_1e^{r(T-t)}Z}]V<0.\end{aligned}$$

Thus, $f(q_1,t)$ is concave in q_1, and its maximizer $q_1(t)$ satisfies the equation

$$(1+\theta)\mu_z=M_Z'(n), \tag{3.9}$$

where $n:=\gamma_1q_1e^{r(T-t)}$. According to Liang et al. [22], we have the following result.

Lemma 3.1. *Eq. (3.9) has a unique positive root ξ.*

From Lemma 3.1, we get $q_1(t) = \frac{\xi}{\gamma_1} e^{-r(T-t)} \geq 0$. Here ξ is a constant, and it only depends on the safety loading θ and the claim sizes distribution $F(z)$.

Since the optimal reinsurance proportion $q_1^*(t) \in [0, 1]$, if $0 \leq q_1(t) \leq 1$, the optimal reinsurance proportion $q_1^*(t)$ coincides with $q_1(t)$; if $q_1(t) \leq 0$, we let $q_1^*(t)$ be 0; and if $q_1(t) \geq 1$, we set $q_1^*(t) = 1$. Therefore, we discuss the optimal reinsurance strategy in the following three cases.

Case 1. $\xi \leq \gamma_1$.

In this case, $\frac{\xi}{\gamma_1} \leq 1$, and thus, $q_1(t) \leq 1$ for any $t \in [0, T]$, then the optimal reinsurance strategy is

$$q_1^*(t) = q_1(t), \quad 0 \leq t \leq T. \tag{3.10}$$

Case 2. $\gamma_1 < \xi < \gamma_1 e^{rT}$.

Let $t_0 = T + \frac{1}{r} ln \frac{\gamma_1}{\xi}$, then $\frac{\xi}{\gamma_1} > 1$, and hence, $q_1(t) < 1$ for $t \in [0, t_0)$, $q_1(t) \geq 1$ for $t \in [t_0, T]$, thus the optimal reinsurance strategy is

$$q_1^*(t) = \begin{cases} \dfrac{\xi}{\gamma_1} e^{-r(T-t)}, & 0 \leq t < t_0, \\ 1, & t_0 \leq t \leq T. \end{cases} \tag{3.11}$$

Case 3. $\xi \geq \gamma_1 e^{rT}$.

In this case, $q_1(t) \geq e^{rt} \geq 1$ for any $t \in [0, T]$, then the optimal reinsurance strategy is

$$q_1^*(t) \equiv 1, \quad 0 \leq t \leq T. \tag{3.12}$$

Substituting $q_1^*(t)$ into $f(q_1, t)$, we can get the value of $f(q_1^*, t)$ as

$$f(q_1^*, t) = \begin{cases} V[-(1+\theta)\xi\lambda\mu_z + \lambda(M_Z(\xi) - 1)], & q_1^*(t) = q_1(t), \\ V[-(1+\theta)\lambda\mu_z n + \lambda(M_Z(n) - 1)], & q_1^*(t) = 1. \end{cases} \tag{3.13}$$

where $n = \gamma e^{r(T-t)}$.

Replacing the supremum in (3.8) by $f(q_1^*, t)$ yields the following equation:

$$\begin{aligned} &V_t + r_1 s V_s + [rx + (\eta - \theta)\lambda\mu_z]V_x + \frac{1}{2}\beta^2 V_{xx} + \rho\beta\sigma s^{k+1} V_{sx} + \frac{1}{2}\sigma^2 s^{2k+2} V_{ss} \\ &\quad - \frac{((r_1 - r)V_x + \beta\rho\sigma s^k V_{xx} + \sigma^2 s^{2k+1} V_{sx})^2}{2\sigma^2 s^{2k} V_{xx}} + V f_1(q_1^*, t) = 0, \end{aligned} \tag{3.14}$$

where $f_1(q_1^*, t) = \frac{f(q_1^*, t)}{V}$.

Here, the stochastic control problem has been transformed into a non-linear second order PDE, it is difficult to solve it directly. Therefore, we transform the problem into a dual one and get a linear PDE by applying Legendre transform and dual theory.

Definition 3.2. Let $g : \mathscr{R}^n \to \mathscr{R}$ be a convex function; for $\omega > 0$, define the Legendre transform

$$L(\omega) = \max_x \{g(x) - \omega x\}. \tag{3.15}$$

The function $L(\omega)$ is called the Legendre dual of function $g(x)$ (cf. [18]).

If $g(x)$ is strictly convex, the maximum in (3.15) will be attained at just one point, which we denote by $\bar{x}$ and then

$$L(\omega) = g(\bar{x}) - \omega\bar{x}. \tag{3.16}$$

Following Xiao et al. [18] and Gao [19], we define a Legendre transform

$$\hat{V}(t, s, \omega) := \sup_{x>0} \{V(t, s, \omega) - \omega x | 0 < x < \infty\}, \tag{3.17}$$

where $0 < t < T$, and $\omega > 0$ denotes the dual variable to x. The value of x where this optimum is attained is denoted by $p(t, s, \omega)$. Therefore,

$$p(t, s, \omega) := \inf_{x>0} \{x | V(t, s, x) \geq \omega x + \hat{V}(t, s, \omega)\}, \tag{3.18}$$

where $0 < t < T$. The function $\hat{V}$ is related to p by:

$$p = -\hat{V}_\omega. \tag{3.19}$$

At the terminal time, we denote

$$\hat{U}(x) = \sup_{x>0}\{U(x) - \omega x | 0 < x < \infty\},$$
$$P(x) = \inf_{x>0}\{x | U(x) \geq \omega x + \hat{U}(x)\}.$$

As a result,

$$P(x) = (U')^{-1}(x). \tag{3.20}$$

Since $V(T, s, x) = U(x)$, we have

$$P(T, s, x) = \inf_{x>0}\{x | U(x) \geq \omega x + \hat{V}(T, s, \omega)\},$$
$$\hat{V}(T, s, \omega) = \sup_{x>0}\{U(x) - \omega x\}.$$

Therefore,

$$p(T, s, \omega) = (U')^{-1}(x). \tag{3.21}$$

According to (3.16) and (3.17), we derive

$$V_x = \omega, \tag{3.22}$$

and thus

$$\hat{V}(t, s, \omega) = V(t, s, p) - \omega p, \qquad p(t, s, \omega) = x. \tag{3.23}$$

Differentiating (3.22) and (3.23) with respect to t, s and x, we obtain the following derivatives of V and $\hat{V}$

$$V_t = \hat{V}_t, \quad V_s = \hat{V}_s, \quad V_{ss} = \hat{V}_{ss} - \frac{\hat{V}_{s\omega}^2}{\hat{V}_{\omega\omega}}, \quad V_{xx} = -\frac{1}{\hat{V}_{\omega\omega}}, \quad V_{xs} = -\frac{\hat{V}_{s\omega}}{\hat{V}_{\omega\omega}}. \tag{3.24}$$

Substituting (3.22) and (3.24) into (3.14), and let $\rho^2 = 1$, differentiating $\hat{V}$ with respect to ω, we derive:

$$\begin{aligned} & p_t + rsp_s - ((\eta - \theta)\lambda\mu_z + rp) + \frac{1}{2}\sigma^2 s^{2k+2} p_{ss} + \frac{\omega^2(r_1 - r)^2}{2\sigma^2 s^{2k}} p_{\omega\omega} \\ & + \left[\frac{(r_1 - r)^2}{\sigma^2 s^{2k}} - r - f_1(q_1^*, t)\right]\omega p_\omega - (r_1 - r)s\omega p_{s\omega} + \frac{(r_1 - r)\beta\rho}{\sigma s^k} = 0. \end{aligned} \tag{3.25}$$

From (3.5), (3.19), (3.22), (3.23) and (3.24), the optimal investment strategy (3.5) can be rewritten as

$$\pi_1^*(t) = \frac{p_s\sigma^2 s^{2k+1} - \omega p_\omega(r_1 - r) - \beta\rho\sigma s^k}{\sigma^2 s^{2k}}. \tag{3.26}$$

Here, the non-linear second-order PDE (3.14) has been transformed into a linear PDE (3.25). The problem now is to solve (3.25) for the dual p and replace it in (3.26) so as to obtain the optimal strategy.

From the exponential utility given by (3.1) and the dual (3.21), the boundary condition is

$$p(T, s, \omega) = -\frac{1}{\gamma_1}\ln\omega.$$

Thus, we try to conjecture a solution to (3.25) in the following way:

$$p(t, s, \omega) = -\frac{1}{\gamma_1}[b(t)(\ln\omega + m(t, s))] + a(t), \tag{3.27}$$

with $b(T) = 1$, $m(T.s) = 0$ and $a(T) = 0$. Then we have

$$p_t(t, s, \omega) = -\frac{1}{\gamma_1}[b'(t)(\ln\omega + m(t, s)) + b(t)m_t] + a'(t), \quad p_s = -\frac{1}{\gamma_1}b(t)m_s,$$
$$p_\omega = -\frac{b(t)}{\gamma_1\omega}, \quad p_{\omega\omega} = \frac{b(t)}{\gamma_1\omega^2}, \quad p_{ss} = -\frac{b(t)}{\gamma_1}m_{ss}, \quad p_{s\omega} = 0.$$

Substituting the above derivatives into (3.25), we derive:

$$\begin{aligned} & \frac{1}{\gamma_1}\ln\omega[b'(t) - rb(t)] + [ra(t) - a'(t) + (\eta - \theta)\lambda\mu_z] + \frac{b(t)}{\gamma_1}\Bigg[m_t + rsm_s - rm \\ & + \frac{1}{2}\sigma^2 s^{2k+2} m_{ss} + \frac{(r_1 - r)^2}{2\sigma^2 s^{2k}} - r - f_1(q_1^*, t) + \frac{b'(t)}{b(t)}m(t, s) - \frac{\gamma_1}{b(t)}\frac{(r_1 - r)\beta\rho}{\sigma s^k}\Bigg] = 0. \end{aligned} \tag{3.28}$$

Again we can split (3.28) equation into three equations:

$$b'(t) - rb(t) = 0. \tag{3.29}$$

$$a'(t) - ra(t) - (\eta - \theta)\lambda\mu_z = 0. \tag{3.30}$$

$$m_t + rsm_s - rm + \frac{1}{2}\sigma^2 s^{2k+2} m_{ss} + \frac{(r_1 - r)^2}{2\sigma^2 s^{2k}} - r - f_1(q_1^*, t) + \frac{b'(t)}{b(t)} m(t, s) - \frac{\gamma_1}{b(t)} \frac{(r_1 - r)\beta\rho}{\sigma s^k} = 0. \tag{3.31}$$

Taking the boundary conditions $b(T) = 1$ and $a(T) = 0$ into account, we obtain the solutions to (3.29) and (3.30):

$$b(t) = e^{r(t-T)}. \tag{3.32}$$

$$a(t) = (\theta - \eta)\lambda\mu_z \left[\frac{1 - e^{r(t-T)}}{r}\right]. \tag{3.33}$$

It is difficult to solve Eq. (3.31), so we use a power transformation and a variable change technique to transform it into a linear one.

Let

$$m(t, s) = G(t, l), \quad l = s^{-2k} \tag{3.34}$$

with the boundary condition $G(T, l) = 0$.

So that

$$m_t = G_t, \qquad m_s = -2ks^{-2k-1} G_l, \qquad m_{ss} = 2k(2k+1)s^{-2k-2} G_l + 4k^2 s^{-4k-2} G_{ll}.$$

Putting the above derivatives into (3.31), we get the following linear PDE:

$$G_t + [\sigma^2 k(2k+1) - 2rkl]G_l + 2k^2\sigma^2 l G_{ll} + \frac{(r_1 - r)^2}{2\sigma^2} l - f_1(q_1^*, t) - r - \frac{\gamma_1 (r_1 - r)\beta\rho}{\sigma} e^{r(T-t)}\sqrt{l} = 0. \tag{3.35}$$

In the following, we derive the solution to (3.35) for special cases.

We try to conjecture a solution to (3.35) in the following way:

$$G(t, l) = d(t) + i(t)\sqrt{l} + j(t)l, \tag{3.36}$$

with $d(T) = 0$, $i(T) = 0$ and $j(T) = 0$. Then

$$G_t = d_t + i_t\sqrt{l} + j_t l, \qquad G_l = \frac{1}{2} l^{-\frac{1}{2}} i(t) + j(t), \qquad G_{ll} = -\frac{1}{4} l^{-\frac{3}{2}} i(t).$$

Substituting these derivatives in (3.35) leads to

$$\begin{aligned} &[d_t + \sigma^2 k(2k+1)j(t) - r - f_1(q_1^*, t)] + \sqrt{l}\left[i_t - rki(t) - \frac{\gamma_1 (r_1 - r)\beta\rho}{\sigma} e^{r(T-t)}\right] \\ &+ l\left[j_t - 2rkj(t) + \frac{(r_1 - r)^2}{2\sigma^2}\right] + \frac{i(t)\sigma^2}{2\sqrt{l}}[k(2k+1) - k^2] = 0. \end{aligned} \tag{3.37}$$

Again we can split (3.37) into four equations:

$$d_t + \sigma^2 k(2k+1)j(t) - r - f_1(q_1^*, t) = 0. \tag{3.38}$$

$$i_t - rki(t) - \frac{\gamma_1 (r_1 - r)\beta\rho}{\sigma} e^{r(T-t)} = 0. \tag{3.39}$$

$$j_t - 2rkj(t) + \frac{(r_1 - r)^2}{2\sigma^2} = 0. \tag{3.40}$$

$$k(2k+1) - k^2 = 0. \tag{3.41}$$

From (3.41), we have $k = -1$ or $k = 0$.

Next, we try to solve (3.37) for the above two cases.

Case 1. $k = -1$.

Suppose that $k = -1$, the price process of the risky asset satisfies the special CEV model, namely

$$\mathrm{d}S(t) = r_1 S(t)\mathrm{d}t + \sigma \mathrm{d}W(t), \qquad S(0) = s, \tag{3.42}$$

and Eq. (3.37) reduces to

$$[d_t + \sigma^2 j(t) - r - f_1(q_1^*, t)] + \sqrt{l}\left[i_t + ri(t) - \frac{\gamma_1(r_1 - r)\beta\rho}{\sigma}e^{r(T-t)}\right] + l\left[j_t + 2rj(t) + \frac{(r_1 - r)^2}{2\sigma^2}\right] = 0. \tag{3.43}$$

Decomposing (3.43) into three equations, we have

$$d_t + \sigma^2 j(t) - r - f_1(q_1^*, t) = 0. \tag{3.44}$$

$$i_t + ri(t) - \frac{\gamma_1(r_1 - r)\beta\rho}{\sigma}e^{r(T-t)} = 0. \tag{3.45}$$

$$j_t + 2rj(t) + \frac{(r_1 - r)^2}{2\sigma^2} = 0. \tag{3.46}$$

Taking the boundary condition $d(T) = 0$, $i(T) = 0$ and $j(T) = 0$ into account, we obtain the solutions to (3.44)–(3.46)

$$j(t) = \frac{(r_1 - r)^2}{2\sigma^2}\left[\frac{e^{2r(T-t)} - 1}{2r}\right]. \tag{3.47}$$

$$i(t) = -\frac{\gamma_1(r_1 - r)\beta\rho}{\sigma}(T - t)e^{r(T-t)}. \tag{3.48}$$

$$d(t) = -\frac{(r_1 - r)^2[1 - e^{2r(T-t)}]}{8r^2} + \left(\frac{(r_1 - r)^2}{4r} + r\right)(t - T) - \int_t^T f_1(q_1^*, s)\mathrm{d}s, \tag{3.49}$$

where

$$\int_t^T f_1(q_1^*, s)\mathrm{d}s = \begin{cases} [-(1+\theta)\xi\lambda\mu_z + \lambda(M_Z(\xi) - 1)](T - t), & q_1^*(t) = q_1(t), \\ \frac{1}{r}(1+\theta)\lambda\mu_z\gamma_1(1 - e^{r(T-t)}) - \lambda(T - t) + \lambda\int_0^{T-t} M_Z(\gamma_1 e^{ru})\mathrm{d}u, & q_1^*(t) = 1. \end{cases}$$

According to (3.26), (3.27), (3.32), (3.34), (3.36), (3.47) and (3.48), we have

$$\begin{aligned} \pi_1^*(t) &= \frac{p_s\sigma^2 s^{-1} - \omega p_\omega(r_1 - r) - \beta\rho\sigma s^{-1}}{\sigma^2 s^{-2}} \\ &= \frac{-\frac{\sigma^2 s^{-1}}{\gamma_1}b(t)m_s + \frac{1}{\gamma_1}b(t)(r_1 - r) - \beta\rho\sigma s^{-1}}{\sigma^2 s^{-2}} \\ &= \frac{-\frac{2\sigma^2}{\gamma_1}b(t)\left[\frac{1}{2}s^{-1}i(t) + j(t)\right] + \frac{1}{\gamma_1}b(t)(r_1 - r) - \beta\rho\sigma s^{-1}}{\sigma^2 s^{-2}} \\ &= \frac{(r_1 - r)s^2 e^{r(t-T)}}{\gamma_1\sigma^2}\left[1 + \frac{r_1 - r}{2r}(1 - e^{2r(T-t)})\right] + \frac{s\beta\rho(r_1 - r)(T - t)}{\sigma} - \frac{s\beta\rho}{\sigma}. \end{aligned}$$

Case 2. $k = 0$.

When $k = 0$, then the price process of the risky assets are governed by geometric Brownian motion (GBM). Similar to the case of $k = -1$, the solutions to (3.38)–(3.40) are

$$j(t) = \frac{(r_1 - r)^2}{2\sigma^2}(T - t). \tag{3.50}$$

$$i(t) = \frac{\gamma_1\beta\rho(r_1 - r)}{\sigma r}(1 - e^{r(T-t)}). \tag{3.51}$$

$$d(t) = r(T - t) + \int_t^T f_1(q_1^*, s)\mathrm{d}s, \tag{3.52}$$

where

$$\int_t^T f_1(q_1^*, s)\mathrm{d}s = \begin{cases} [-(1+\theta)\xi\lambda\mu_z + \lambda(M_Z(\xi) - 1)](T - t), & q_1^*(t) = q_1(t), \\ \frac{1}{r}(1+\theta)\lambda\mu_z\gamma_1(1 - e^{r(T-t)}) - \lambda(T - t) + \lambda\int_0^{T-t} M_Z(\gamma_1 e^{ru})\mathrm{d}u, & q_1^*(t) = 1. \end{cases}$$

According to (3.26), (3.27), (3.32), (3.34), (3.36), (3.50) and (3.51),we have

$$\begin{aligned}\pi_1^*(t) &= \frac{p_s\sigma^2 s - \omega p_\omega(r_1 - r) - \beta\rho\sigma}{\sigma^2}\\ &= \frac{-\omega p_\omega(r_1 - r) - \beta\rho\sigma}{\sigma^2}\\ &= \frac{(r_1 - r)e^{r(t-T)}}{\gamma_1\sigma^2} - \frac{\beta\rho}{\sigma}.\end{aligned}$$

The following theorem summarizes the above derivation.

Theorem 3.3. *Assume that ξ is the unique positive root to Eq. (3.9), let $t_0 = T + \frac{1}{r}\ln\frac{\gamma_1}{\xi}$. Then the optimal reinsurance strategy for the optimal investment–reinsurance problem (2.5) is given as follows:*

(1) *If*

$$\xi \le \gamma_1,$$

then the optimal reinsurance strategy is given by

$$q_1^*(t) = q_1(t), \qquad 0 \le t \le T. \tag{3.53}$$

(2) *If*

$$\gamma_1 < \xi < \gamma_1 e^{rT},$$

then the optimal reinsurance strategy is given by

$$q_1^*(t) = \begin{cases} \dfrac{\xi}{\gamma_1}e^{-r(T-t)}, & 0 \le t < t_0,\\ 1, & t_0 \le t \le T.\end{cases} \tag{3.54}$$

(3) *If*

$$\xi \ge \gamma_1 e^{rT},$$

then the optimal reinsurance strategy is given by

$$q_1^*(t) \equiv 1, \qquad 0 \le t \le T. \tag{3.55}$$

Furthermore, we have the following cases for the optimal investment strategy:

(i) *If the elasticity parameter $k = -1$, the optimal investment strategy under the exponential utility is*

$$\pi_1^*(t) = \frac{(r_1 - r)s^2 e^{r(t-T)}}{\gamma_1\sigma^2}\left[1 + \frac{r_1 - r}{2r}(1 - e^{2r(T-t)})\right] + \frac{s\beta\rho(r_1 - r)(T - t)}{\sigma} - \frac{s\beta\rho}{\sigma}. \tag{3.56}$$

(ii) *If the elasticity parameter $k = 0$, the CEV model reduces to the GBM model and the optimal investment strategy under the exponential utility is*

$$\pi_1^*(t) = \frac{(r_1 - r)e^{r(t-T)}}{\gamma_1\sigma^2} - \frac{\beta\rho}{\sigma}. \tag{3.57}$$

Remark 3.4. For the case of $k = 0$, note that $\frac{\partial\pi_1(t)}{\partial\beta} = -\frac{\rho}{\sigma}$. Thus, the optimal investment strategy is a decreasing function with β when $\rho > 0$. It is because when $\rho > 0$, the correlation between the risk model and the risky asset's price is positive, as β increases, the underwriting risk becomes larger, thus the insurer will put less money in the risky asset to reduce the financial risk. On the contrary, in the case that $\rho < 0$, the correlation between the risk model and the risky asset's price is negative, the optimal investment strategy increases with β. As β becomes larger, the insurer will get more profits from risky asset. Therefore, the insurer would like to put more money in the risky asset to gain more profits.

Remark 3.5. If $\beta = 0$, the jump diffusion risk process (2.1) reduces to the classical Cramér–Lundberg (C–L) model

$$\mathrm{d}X(t) = c\mathrm{d}t - \mathrm{d}\left(\sum_{i=1}^{N(t)} Z_i\right). \tag{3.58}$$

In this case, the risk model is independent with the risky asset's price and the optimal investment strategy is

$$\pi_1^*(t) = \frac{(r_1 - r)e^{r(t-T)}}{\gamma_1 \sigma^2 s^{2k}} \left[1 + \frac{(r_1 - r)}{2r}(1 - e^{2rk(t-T)})\right]. \tag{3.59}$$

Proof. For $\beta = 0$, (3.35) reduces to

$$G_t + [\sigma^2 k(2k+1) - 2rkl]G_l + 2k^2\sigma^2 l G_{ll} + \frac{(r_1 - r)^2}{2\sigma^2} l - f_1(q_1^*, t) - r = 0. \tag{3.60}$$

We try to conjecture a solution of (3.60) with the following structure:

$$G(t, l) = e(t) + g(t)l, \tag{3.61}$$

with $e(T) = 0$ and $g(T) = 0$. Then

$$G_t = e_t + g_t l, \quad G_l = g(t), \qquad G_{ll} = 0.$$

Putting these derivatives in (3.60) yields

$$[e_t + \sigma^2 k(2k+1)g(t) - f_1(q_1^*, t) - r] + l\left[g_t - 2rkg(t) + \frac{(r_1 - r)^2}{2\sigma^2}\right] = 0. \tag{3.62}$$

In order to eliminate the dependence on l, we decompose (3.62) into two equations:

$$e_t + \sigma^2 k(2k+1)g(t) - f_1(q_1^*, t) - r = 0. \tag{3.63}$$

$$g_t - 2rkg(t) + \frac{(r_1 - r)^2}{2\sigma^2} = 0. \tag{3.64}$$

In terms of the boundary conditions $e(T) = 0$ and $g(T) = 0$, the solutions to (3.63) and (3.64) are

$$e(t) = \left[\frac{(2k+1)(r_1 - r)^2}{4r} - r\right](T - t) - \frac{(2k+1)(r_1 - r)^2}{8kr^2}(1 - e^{2kr(t-T)}) - \int_t^T f_1(q_1^*, s)\mathrm{d}s, \tag{3.65}$$

where

$$\int_t^T f_1(q_1^*, s)\mathrm{d}s = \begin{cases} [-(1+\theta)\xi\lambda\mu_z + \lambda(M_Z(\xi) - 1)](T - t), & q_1^*(t) = q_1(t), \\ \frac{1}{r}(1+\theta)\lambda\mu_z\gamma_1(1 - e^{r(T-t)}) - \lambda(T - t) + \lambda\int_0^{T-t} M_Z(\gamma_1 e^{ru})\mathrm{d}u, & q_1^*(t) = 1. \end{cases}$$

$$g(t) = \frac{(r_1 - r)^2}{2\sigma^2}\left[\frac{1 - e^{2rk(t-T)}}{2rk}\right]. \tag{3.66}$$

From (3.26), (3.27), (3.32), (3.34), (3.61), (3.66), we have

$$\begin{aligned} \pi_1^*(t) &= \frac{p_s\sigma^2 s^{2k+1} - \omega p_\omega(r_1 - r)}{\sigma^2 s^{2k}} \\ &= \frac{-\frac{b(t)}{\gamma_1} m_s \sigma^2 s^{2k+1} + \frac{b(t)}{\gamma_!}(r_1 - r)}{\sigma^2 s^{2k}} \\ &= \frac{b(t)}{\gamma_1\sigma^2 s^{2k}}[(r_1 - r) + 2k\sigma^2 G_l] \\ &= \frac{(r_1 - r)e^{r(t-T)}}{\gamma_1\sigma^2 s^{2k}}\left[1 + \frac{(r_1 - r)}{2r}(1 - e^{2rk(t-T)})\right]. \end{aligned}$$

Remark 3.6. For the case of $\beta = 0$, if $k = 0$, the CEV model reduces to the GBM model, and the optimal investment strategy is

$$\pi_1^*(t) = \frac{(r_1 - r)e^{r(t-T)}}{\gamma_1\sigma^2}. \tag{3.67}$$

Compared to (3.67), the optimal investment strategy under the CEV model can be decomposed into two parts. One is

$$M(t) = \frac{(r_1 - r)e^{r(t-T)}}{\gamma_1\sigma^2 s^{2k}}, \tag{3.68}$$

which is similar to the optimal investment strategy under the GBM model, but the volatility is stochastic. Hence, we call (3.68) as a modified GBM strategy. The second part

$$N(t) = 1 + \frac{(r_1 - r)}{2r}(1 - e^{2rk(t-T)}) \tag{3.69}$$

is called a modification factor, which reflects the insurer's decision to hedge the volatility risk.

The following corollary shows the properties of the modification factor.

Corollary 3.7. *The modification factor $N(t)$ is a monotone increasing function with respect to time t and satisfies*

$$1 + \frac{(r_1 - r)}{2r}(1 - e^{-2rkT}) \le N(t) \le 1. \quad 0 \le t \le T \tag{3.70}$$

Proof. Note that $r_1 > r > 0$ and $k < 0$, then

$$N'(t) = -k(r_1 - r)e^{2rk(t-T)} > 0,$$

thus the modification factor $N(t)$ is a monotone increasing function with respect to time t.

Since $N(0) = 1 + \frac{(r_1-r)}{2r}(1 - e^{-2rkT})$ and $N(T) = 1$, we have

$$1 + \frac{(r_1 - r)}{2r}(1 - e^{-2rkT}) \le N(t) \le 1, \quad 0 \le t \le T.$$

Corollary 3.7 shows that the insurer will invest less in the risky asset at initial time, and steadily increase the amount as time goes on under the CEV model.

4. Optimal strategy for the reinsurer

In this section, we derive the optimal strategy for the reinsurer to maximize the reinsurer's expected exponential utility of terminal wealth.

The utility function of the reinsurer is given by exponential utility

$$u_2(y) = -\frac{1}{\gamma_2}e^{-\gamma_2 y}, \quad \gamma_2 > 0. \tag{4.1}$$

We define the value function of the reinsurer as

$$\psi(t, s, y) = \sup_{(\pi_2, q_2) \in \Pi_2} E[u_2(Y(T))|Y(t) = y, S(t) = s], \quad 0 \le t < T \tag{4.2}$$

with $\psi(T, s, y) = u_2(y)$.

The corresponding HJB equation is

$$\begin{aligned} &\psi_t + r_1 s\psi_s + [ry + (1+\theta)\lambda\mu_z]\psi_y + \frac{1}{2}\sigma^2 s^{2k+2}\psi_{ss} + \sup_{\pi_2}\left\{\pi_2(r_1 - r)\psi_y + \frac{1}{2}\pi_2^2\sigma^2 s^{2k}\psi_{yy} + \sigma^2 s^{2k+1}\pi_2\psi_{sy}\right\} \\ &\quad + \sup_{q_2}\{-(1+\theta)\lambda\mu_z q_2\psi_y + \lambda E(\psi(t, s, y - (1-q_2)z) - \psi(t, s, y))\} = 0, \quad 0 \le t < T \end{aligned} \tag{4.3}$$

with the boundary condition $\psi(T, s, y) = u_2(y)$, where ψ_t, ψ_s, ψ_y, ψ_{ss}, ψ_{yy}, ψ_{sy} denote partial derivatives of first and second orders with respect to time t, risky asset's price s and wealth y.

According to the exponential utility function described by (4.1), we construct the solution to (4.3) with the following form:

$$\psi(t, s, y) = -\frac{1}{\gamma_2}exp[-\gamma_2 y e^{r(T-t)} + h(t, s)], \tag{4.4}$$

with $h(T, s) = 0$. Then we have

$$\begin{aligned} &\psi_t = (\gamma_2 yre^{r(T-t)} + h_t)\psi, \quad \psi_s = h_s\psi, \\ &\psi_{ss} = (h_s^2 + h_{ss})\psi, \quad \psi_y = (-\gamma_2 e^{r(T-t)})\psi, \\ &\psi_{yy} = (\gamma_2^2 e^{2r(T-t)})\psi, \quad \psi_{ys} = (-\gamma_2 e^{r(T-t)}h_s)\psi, \\ &E[\psi(t, s, y - (1-q_2)z) - \psi(t, s, y)] = \psi(M_Z(\gamma_2(1-q_2)e^{r(T-t)}) - 1). \end{aligned}$$

Plugging these back into HJB equation (4.3) yields

$$h_t - (1+\theta)\lambda\mu_z\gamma_2 e^{r(T-t)} + r_1 sh_s + \frac{1}{2}\sigma^2 s^{2k+2}(h_s^2 + h_{ss}) - \lambda + \inf_{\pi_2}\{g_1(\pi_2, t)\} + \inf_{q_2}\{g_2(q_2, t)\} = 0, \tag{4.5}$$

where

$$g_1(\pi_2,t)=-(r_1-r)\pi_2\gamma_2e^{r(T-t)}+\frac{1}{2}\pi_2^2\sigma^2s^{2k}\gamma_2^2e^{2r(T-t)}-\sigma^2s^{2k+1}\pi_2\gamma_2e^{r(T-t)}h_s,$$

and

$$g_2(q_2,t)=(1+\theta)\lambda\mu_z\gamma_2e^{r(T-t)}q_2+\lambda M_Z(\gamma_2(1-q_2)e^{r(T-t)}).$$

Differentiating $g_1(\pi_2,t)$ with respect to π_2 yields the minimizer

$$\pi_2^*(t)=\frac{1}{\gamma_2}e^{r(t-T)}\left[\frac{r_1-r}{\sigma^2s^{2k}}+sh_s\right], \tag{4.6}$$

and the corresponding minimum value is

$$g_1(\pi_2^*,t)=-\frac{1}{2}\left(\frac{r_1-r}{\sigma s^k}+\sigma s^{k+1}h_s\right)^2.$$

Similarly, differentiating $g_2(q_2,t)$ with respect to q_2, we get

$$\frac{\partial g_2(q_2,t)}{\partial q_2}=(1+\theta)\lambda\mu_z\gamma_2e^{r(T-t)}-\lambda\gamma_2e^{r(T-t)}E[Ze^{\gamma_2(1-q_2)e^{r(T-t)}Z}].$$
$$\frac{\partial^2 g_2(q_2,t)}{\partial q_2^2}=\lambda\gamma_2^2e^{2r(T-t)}E[Z^2e^{\gamma_2(1-q_2)e^{r(T-t)}}]>0.$$

Thus, $g_2(q_2,t)$ is a convex function with respect to q_2, and its minimizer $q_2(t)$ satisfies

$$(1+\theta)\mu_z=M'(n), \tag{4.7}$$

where $n:=\gamma_2(1-q_2)e^{r(T-t)}$.

From Lemma 3.1, we get that ξ is the unique positive root of Eq. (4.7).

Therefore, we get $q_2(t)=1-\frac{\xi}{\gamma_2}e^{-r(T-t)}\le 1$. In the following, we analyze the optimal problem under three cases.

Case 1.$\xi\ge\gamma_2$.

In this case, $1-\frac{\xi}{\gamma_2}\le 0$, and thus, $q_2(t)\le 0$ for any $t\in[0,T]$, then the optimal reinsurance strategy is

$$q_2^*(t)\equiv 0,\qquad 0\le t\le T. \tag{4.8}$$

Case 2.$e^{-rT}\gamma_2<\xi<\gamma_2$.

Let $t_1=T+\frac{1}{r}ln\frac{\gamma_2}{\xi}$, then $q_2(t)>0$ for $t\in[0,t_1)$, $q_2(t)\le 0$ for $t\in[t_1,T]$, thus the optimal reinsurance strategy is

$$q_2^*(t)=\begin{cases}1-\dfrac{\xi}{\gamma_2}e^{-r(T-t)}, & 0\le t<t_1,\\ 0, & t_1\le t\le T.\end{cases} \tag{4.9}$$

Case 3.$\xi\le\gamma_2e^{-rT}$.

In this case, $q_2(t)\ge 0$ for any $t\in[0,T]$, then the optimal reinsurance strategy is

$$q_2^*(t)\equiv q_2(t),\qquad 0\le t\le T. \tag{4.10}$$

The value of $g_2(q_2,t)$ at $q_2^*(t)$ is

$$g_2(q_2^*,t)=\begin{cases}(1+\theta)\lambda\mu_z\gamma_2\left[e^{r(T-t)}-\dfrac{\xi}{\gamma_2}\right]+\lambda M_Z(\xi), & q_2^*(t)=q_2(t),\\ \lambda M_Z(\gamma_2e^{r(T-t)}), & q_2^*(t)=0.\end{cases} \tag{4.11}$$

Substituting $g_1(\pi_2^*,t)$ and $g_2(q_2^*,t)$ into (4.5) yields

$$h_t-(1+\theta)\lambda\mu_z\gamma_2e^{r(T-t)}+rsh_s+\frac{1}{2}\sigma^2s^{2k+2}h_{ss}-\lambda-\frac{(r_1-r)^2}{2\sigma^2s^{2k}}+g_2(q_2^*,t)=0. \tag{4.12}$$

Through the similar method, let

$$h(t,s)=A(t,l),\qquad l=s^{-2k} \tag{4.13}$$

with the boundary condition $A(T,l)=0$.

Thus,

$$h_t=A_t,\quad h_s=-2ks^{-2k-1}A_l,\quad h_{ss}=2k(2k+1)s^{-2k-2}A_l+4k^2s^{-4k-2}A_{ll}.$$

With the use of this transformation, (4.12) becomes

$$A_t - 2klrA_l + \sigma^2 k(2k+1)A_l + 2\sigma^2 k^2 lA_{ll} - \frac{(r_1 - r)^2}{2\sigma^2} l - (1+\theta)\lambda\mu_z\gamma_2 e^{r(T-t)} + g_2(q_2^*, t) - \lambda = 0. \tag{4.14}$$

To solve (4.14), we try a solution with the following form:

$$A(t, l) = I(t)l + J(t), \tag{4.15}$$

with the boundary condition $I(T) = 0$ and $J(T) = 0$.

Then (4.14) is transformed into

$$\left[I_t - 2krI(t) - \frac{(r_1 - r)^2}{2\sigma^2}\right] l + J_t + \sigma^2 k(2k+1)I(t) - (1+\theta)\lambda\mu_z\gamma_2 e^{r(T-t)} + g_2(q_2^*, t) - \lambda = 0. \tag{4.16}$$

In order to eliminate the dependence on l, we decompose (4.16) into

$$I_t - 2krI(t) - \frac{(r_1 - r)^2}{2\sigma^2} = 0, \tag{4.17}$$

and

$$J_t + \sigma^2 k(2k+1)I(t) - (1+\theta)\lambda\mu_z\gamma_2 e^{r(T-t)} + g_2(q_2^*, t) - \lambda = 0. \tag{4.18}$$

Taking the boundary conditions into account, we get

$$I(t) = \frac{(r_1 - r)^2}{4rk\sigma^2}[e^{-2rk(T-t)} - 1], \tag{4.19}$$

$$J(t) = B(t) + \int_t^T g_2(q_2^*, s)ds, \tag{4.20}$$

where

$$B(t) = \frac{(r_1 - r)^2(2k+1)}{8r^2 k}[1 - e^{-2kr(T-t)}] - \frac{1}{r}(1+\theta)\lambda\mu_z\gamma_2(e^{r(T-t)} - 1) - \left(\frac{(r_1 - r)^2(2k+1)}{4r} + \lambda\right)(T-t),$$

$$\int_t^T g_2(q_2^*, s)ds = \begin{cases} C_1(t), & q_2^*(t) = q_2(t), \\ C_2(t), & q_2^*(t) = 0 \end{cases}$$

where

$$\begin{cases} C_1(t) = \dfrac{(1+\theta)\lambda\mu_z\gamma_2}{r}(1 - e^{r(T-t)}) - [(1+\theta)\lambda\mu_z\xi - \lambda M_Z(\xi)](T-t), \\ C_2(t) = \lambda \displaystyle\int_0^{T-t} M_Z(\gamma_2 e^{ru})du. \end{cases} \tag{4.21}$$

The following theorem summarizes the above derivation and gives the optimal investment–reinsurance strategy for the reinsurer.

Theorem 4.1. *The optimal investment strategy for problem (2.8) under the exponential utility function is given by*

$$\pi_2^*(t) = \frac{(r_1 - r)e^{r(t-T)}}{\gamma_2\sigma^2 s^{2k}}\left[1 - \frac{r_1 - r}{2r}(e^{-2rk(T-t)} - 1)\right]. \tag{4.22}$$

Furthermore, define

$$\begin{cases} h_1(t, s) = I(t)s^{-2k} + B(t) + C_1(t), \\ h_2(t, s) = I(t)s^{-2k} + B(t) + C_2(t) \end{cases}$$

the optimal reinsurance strategies and the corresponding value function of the reinsurer are as follows:

(1) If $\xi \geq \gamma_2$, the optimal reinsurance strategy is

$$q_2^*(t) \equiv 0, \quad 0 \leq t \leq T \tag{4.23}$$

and the value function is

$$\psi(t, s, y) = -\frac{1}{\gamma_2} exp[-\gamma_2 y e^{r(T-t)} + h_2(t, s)]. \tag{4.24}$$

(2) *If $e^{-rT}\gamma_2 < \xi < \gamma_2$, the optimal reinsurance strategy is*

$$q_2^*(t) = \begin{cases} 1 - \dfrac{\xi}{\gamma_2} e^{-r(T-t)}, & 0 \le t < t_1, \\ 0, & t_1 \le t \le T, \end{cases} \tag{4.25}$$

where $t_1 = T + \frac{1}{r} ln \frac{\gamma_2}{\xi}$, and the value function is

$$\psi(t, s, y) = \begin{cases} -\dfrac{1}{\gamma_2} exp[-\gamma_2 y e^{r(T-t)} + h_1(t, s) + \tau], & 0 \le t < t_1, \\ -\dfrac{1}{\gamma_2} exp[-\gamma_2 y e^{r(T-t)} + h_2(t, s)], & t_1 \le t \le T, \end{cases} \tag{4.26}$$

where $\tau = h_2(t_1, s) - h_1(t_1, s)$.

(3) *If $\xi \le \gamma_2 e^{-rT}$, the optimal reinsurance strategy is*

$$q_2^*(t) \equiv q_2(t), \quad 0 \le t \le T. \tag{4.27}$$

and the value function is

$$\psi(t, s, y) = -\frac{1}{\gamma_2} exp[-\gamma_2 y e^{r(T-t)} + h_1(t, s)]. \tag{4.28}$$

Proof. According to (4.6), (4.13), (4.15) and (4.19), we have

$$\begin{aligned} \pi_2^*(t) &= \frac{1}{\gamma_2} e^{r(t-T)} \left[\frac{r_1 - r}{\sigma^2 s^{2k}} + s h_s \right] \\ &= \frac{1}{\gamma_2} e^{r(t-T)} \left[\frac{r_1 - r}{\sigma^2 s^{2k}} - 2ks^{-2k} A_l \right] \\ &= \frac{1}{\gamma_2} e^{r(t-T)} \left[\frac{r_1 - r}{\sigma^2 s^{2k}} - 2ks^{-2k} I(t) \right] \\ &= \frac{(r_1 - r) e^{r(t-T)}}{\gamma_2 \sigma^2 s^{2k}} \left[1 - \frac{r_1 - r}{2r} (e^{-2rk(T-t)} - 1) \right]. \end{aligned}$$

Moreover, because $h_2(t_1, s) = h_1(t_1, s) + \tau$, $\psi(t, s, y)$ is a continuous function on $[0, T] \times R \times R$. Furthermore,

$$h_1'(t_1, s) = I'(t_1) s^{-2k} + B'(t_1) - \lambda M_Z(\xi) = h_2'(t_1, s),$$

where $h_i'(t_1, s)$ is the first derivative of $h_i(t, s)$ with respect to t at t_1. Thus, $\psi \in C^{1,2}$ is a classical solution of the HJB equation (4.3).

Remark 4.2. If $k = 0$, the CEV model reduces to the GBM model, and the optimal investment strategy is

$$\pi_2^*(t) = \frac{(r_1 - r) e^{r(t-T)}}{\gamma_2 \sigma^2}. \tag{4.29}$$

Similarly, the optimal investment strategy $\pi_2^*(t)$ can be decomposed into two parts. One part has an analogical form of the optimal strategy under the GBM model, i.e.,

$$\hat{M}(t) = \frac{(r_1 - r) e^{r(t-T)}}{\gamma_2 \sigma^2 s^{2k}}. \tag{4.30}$$

The other is the modification factor

$$\hat{N}(t) = 1 - \frac{r_1 - r}{2r} (e^{-2rk(T-t)} - 1). \tag{4.31}$$

The modification factor $\hat{N}(t)$ is a monotone increasing function with respect to time t and satisfies

$$1 + \frac{(r_1 - r)}{2r} (1 - e^{-2rkT}) \le \hat{N}(t) \le 1, \quad 0 \le t \le T. \tag{4.32}$$

Therefore, the reinsurer will invest less wealth in the risky asset at initial time, and steadily increase money amount as time goes on.

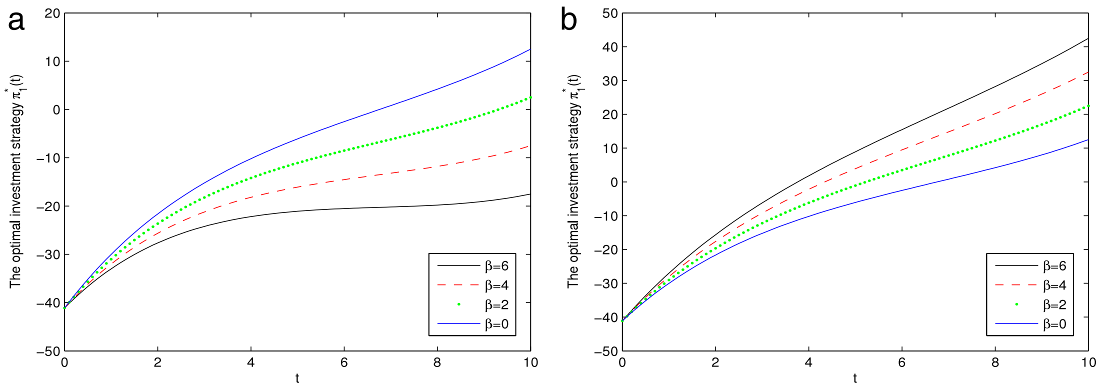

Fig. 5.1. (a) The effect of β on $\pi_1^*(t)$ when $\rho = 1$. (b) The effect of β on $\pi_1^*(t)$ when $\rho = -1$.

5. Numerical analysis

In this section, we provide some numerical simulations to illustrate our results. Here we assume that the claim size Z_i are independent and exponentially distributed with parameter $\frac{1}{\mu_z}$. Throughout the numerical analysis, the basic parameters are given by: $r = 0.3$, $r_1 = 0.4$, $k = -1$, $T = 10$, $t = 5$, $s = 5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.2$, $\sigma = 1$, $\rho = \pm 1$, $\eta = 1$, $\theta = 1$, $\lambda = 0.5$, $\mu_z = 1$.

5.1. Sensitivity analysis of insurer's investment strategy

For the case of $k = -1$, the optimal investment strategy is

$$\pi_1^*(t) = \frac{(r_1 - r)s^2 e^{r(t-T)}}{\gamma_1 \sigma^2}\left[1 + \frac{r_1 - r}{2r}(1 - e^{2r(T-t)})\right] + \frac{s\beta\rho(r_1 - r)(T - t)}{\sigma} - \frac{s\beta\rho}{\sigma}. \tag{5.1}$$

The effects of β on the optimal investment strategy are shown in Fig. 5.1. From Fig. 5.1(a), we can see that the optimal investment strategy $\pi_1^*(t)$ decreases with β in the case of $\rho = 1$. When $\rho > 0$, the correlation between risk model and risky asset's price is positive. As β increases, the underwriting risk becomes larger, in order to reduce overall risk, the insurer will put less money in the risky asset. From Fig. 5.1(b), $\pi_1^*(t)$ is an increasing function of β when $\rho = -1$. In the case of $\rho < 0$, the correlation between risk model and risky asset's price is negative. As β becomes bigger, the insurer will get more profits from risky asset. Therefore, the insurer would like to put more money in the risky asset to gain more profits.

5.2. Sensitivity analysis of reinsurance strategies

Example 5.1. In this example, sensitivity analysis of the reinsurer's strategy for jump diffusion risk model are shown in Fig. 5.2.

Fig. 5.2(a) shows the effects of the safety loading parameter θ on the optimal reinsurance strategies. From Fig. 5.2(a), we find that θ exerts a positive effect on $q_1^*(t)$. As θ becomes larger, the reinsurance premium will be more expensive. Thus the insurer prefers to retain a greater share of each claim and purchase less reinsurance. $q_2^*(t)$ is a decreasing function of θ. When the reinsurance premium becomes more expensive, the reinsurer prefers to take more share of reinsurance to gain more profits.

Fig. 5.2(b) shows the effects of risk aversion coefficient on the reinsurance strategies. The insurer's risk aversion coefficient γ_1 has a negative effect on the reinsurance proportion $q_1^*(t)$. The insurer is risk averse and it will purchase more reinsurance to hedge risk as the risk aversion coefficient becomes higher. The reinsurance strategy of the reinsurer $q_2^*(t)$ increases with γ_2. As γ_2 increases, the reinsurer is more conservative and it will accept lower reinsurance proportion.

In reality, the reinsurance proportion is decided by both the insurer and the reinsurer. If the optimal retention level chosen by the insurer is larger than that of the reinsurer, the reinsurer will accept the strategy. But in the opposite case, the reinsurer may not accept the optimal retention level chosen by the insurer. We discuss the optimal reinsurance strategy in the following three cases.

Case 1. $\xi = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}e^{r(T-t)}$.

In this case, $q_1^*(t) = q_2^*(t)$, both the insurer and reinsurer obtain the maximum value of terminal wealth, thus, the reinsurer will accept the optimal reinsurance strategy chosen by the insurer.

Case 2. $\xi > \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}e^{r(T-t)}$.

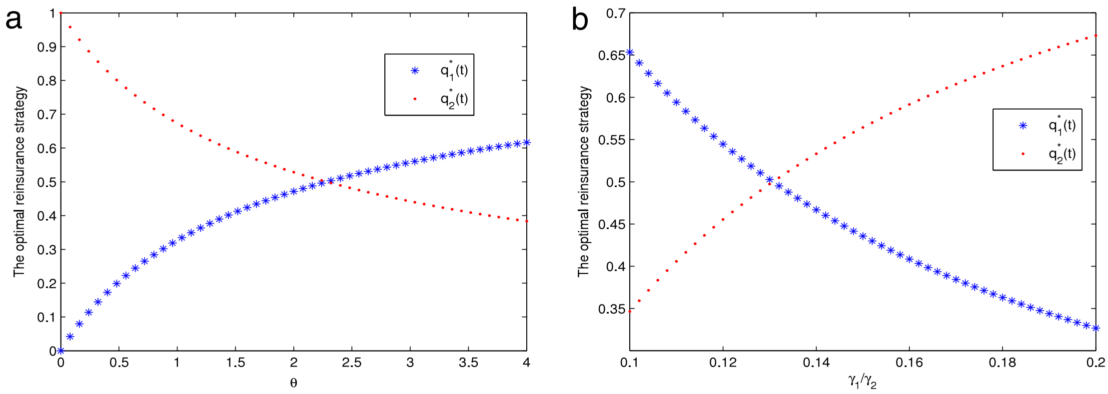

Fig. 5.2. (a) The effect of θ on the optimal reinsurance strategies. (b) The effect of risk aversion on the optimal reinsurance strategies.

In this case, $q_1^*(t) > q_2^*(t)$, the insurer's retention proportion is larger than that chosen by the reinsurer, thus, the reinsurer has enough wealth to accept the optimal reinsurance strategy chosen by the insurer.

Case 3. $\xi < \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2} e^{r(T-t)}$.

In this case, $q_1^*(t) < q_2^*(t)$, the reinsurer may not accept the optimal reinsurance strategy chosen by the insurer.

Example 5.2. In this example, sensitivity analysis of the reinsurer's strategy for diffusion approximation risk model are shown in Fig. 5.3.

Let $C(t) = \sum_{i=1}^{N(t)} Z_i$, according to Grandell [29], the claim process $C(t)$ can be approximated by the following diffusion risk model

$$d\hat{C}(t) = \alpha_1 dt - \alpha_2 d\hat{W}(t), \tag{5.2}$$

where $\alpha_1 = \lambda E[Z_i]$, $\alpha_2^2 = \lambda E[Z_i^2]$. By using the similar methods in Sections 3 and 4, we obtain the optimal reinsurance strategies for the diffusion model.

Let $\hat{t}_0 = T - \frac{1}{r} \ln \frac{\lambda\mu_z\theta}{\alpha_2^2\gamma_1}$, the optimal reinsurance strategy of the insurer with the diffusion risk model are as follows:

(1)if $\frac{\lambda\mu_z\theta}{\alpha_2^2\gamma_1} \le 1$, then $\hat{t}_0 \ge T$, thus the optimal reinsurance strategy is given by

$$\hat{q}_1^*(t) = \frac{\lambda\mu_z\theta}{\alpha_2^2\gamma_1} e^{-r(T-t)}, \quad 0 \le t \le T; \tag{5.3}$$

(2) if $1 < \frac{\lambda\mu_z\theta}{\alpha_2^2\gamma_1} < e^{rT}$, the optimal reinsurance strategy is

$$\hat{q}_1^*(t) = \begin{cases} \dfrac{\lambda\mu_z\theta}{\alpha_2^2\gamma_1} e^{-r(T-t)}, & 0 \le t < \hat{t}_0, \\ 1, & \hat{t}_0 \le t \le T; \end{cases} \tag{5.4}$$

(3) if $\frac{\lambda\mu_z\theta}{\alpha_2^2\gamma_1} \ge e^{rT}$, the optimal reinsurance strategy is

$$\hat{q}_1^*(t) \equiv 1, \quad 0 \le t \le T. \tag{5.5}$$

Let $\hat{t}_1 = T - \frac{1}{r} \ln \frac{\lambda\mu_z\theta}{\alpha_2^2\gamma_2}$, the optimal reinsurance strategy of the reinsurer with the diffusion risk model are as follows:

(1) if $\frac{\lambda\mu_z\theta}{\alpha_2^2\gamma_2} \le 1$, then $\hat{t}_1 \ge T$, thus the optimal reinsurance strategy is given by

$$\hat{q}_2^*(t) = 1 - \frac{\lambda\mu_z\theta}{\alpha_2^2\gamma_2} e^{-r(T-t)}, \quad 0 \le t \le T; \tag{5.6}$$

(2) if $1 < \frac{\lambda\mu_z\theta}{\alpha_2^2\gamma_2} < e^{rT}$, the optimal reinsurance strategy is

$$\hat{q}_2^*(t) = \begin{cases} 1 - \dfrac{\lambda\mu_z\theta}{\alpha_2^2\gamma_2} e^{-r(T-t)}, & 0 \le t < \hat{t}_1, \\ 0, & \hat{t}_1 \le t \le T; \end{cases} \tag{5.7}$$

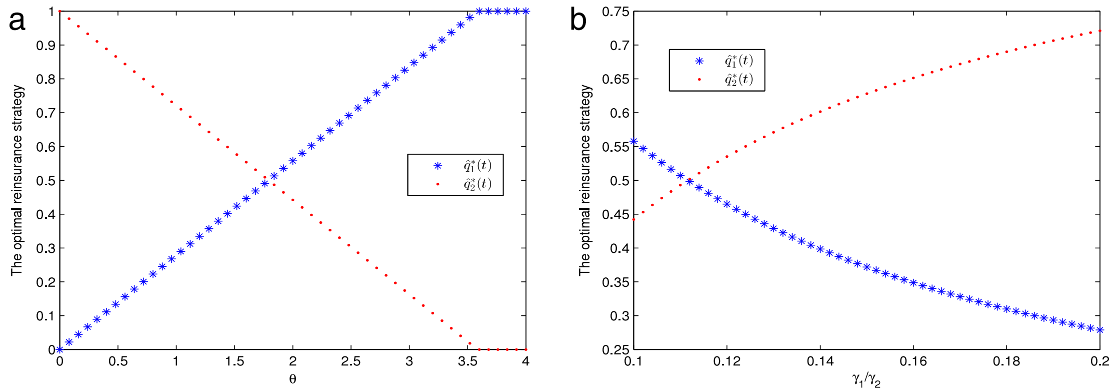

Fig. 5.3. (a) The effect of θ on the optimal reinsurance strategies. (b) The effect of risk aversion on the optimal reinsurance strategies.

(3) if $\frac{\lambda\mu_z\theta}{\alpha_2^2\gamma_2} \geq e^{rT}$, the optimal reinsurance strategy is given by

$$\hat{q}_2^*(t) \equiv 0, \qquad 0 \leq t \leq T. \tag{5.8}$$

Fig. 5.3 illustrates the effects of the safety loading parameter θ and the risk aversion coefficient on the optimal reinsurance strategies. $\hat{q}_1^*(t)$ increases with θ and decreases with γ_1, while $\hat{q}_2^*(t)$ decreases with θ and increases with γ_2. The effects are similar to those under the jump diffusion risk model. Thus, it is reasonable to approximate the compound Poisson process by the Brownian motion with drift. But the reinsurance strategy is more sensitive with θ when the claim process is described by the diffusion risk model. Moreover, when $\theta \geq \frac{\alpha_2^2\gamma_1\gamma_2}{\lambda\mu_z(\gamma_1+\gamma_2)}e^{r_0(T-t)}$, then $\hat{q}_1^*(t) \geq \hat{q}_2^*(t)$, thus the reinsurer will accept the optimal retention level chosen by the insurer.

6. Conclusion

In this paper, we study the optimal investment–reinsurance problem for both an insurer and a reinsurer. We consider an insurer with a jump diffusion risk model, and she/he can purchase proportional reinsurance from the reinsurer. Both the insurer and the reinsurer are allowed to invest in a risky asset and a risk-free asset. Moreover, we adopt the constant elasticity of variance (CEV) model to describe risky asset's price process. By applying stochastic control approach, we establish the corresponding Hamilton–Jacobi–Bellman (HJB) equations. For exponential utility maximization, we obtain closed-form reinsurance–investment strategies for insurer and reinsurer, respectively. Finally, a numerical simulation is presented to analyze the properties of the optimal strategies. Some interesting results are found: (1) The insurer's reinsurance strategy is different from the reinsurer's strategy. (2) In comparison with the GBM model, the optimal investment strategies under the CEV model contain an extra modification factor, which reflects the insurer's and reinsurer's decisions to hedge the volatility risk.

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References

[1] S. Browne, Optimal investment policies for a firm with a random risk process: exponential utility and minimizing the probability of ruin, Math. Oper. Res. 20 (1995) 937–958.
[2] H.L. Yang, L.H. Zhang, Optimal investment for insurer with jump-diffusion risk process, Insurance Math. Econom. 37 (2005) 615–634.
[3] C. Hipp, M. Plum, Optimal investment for insurers, Insurance Math. Econom. 27 (2000) 215–228.
[4] C. Liu, H. Yang, Optimal investment for an insurer to minimize its probability of ruin, N. Am. Actuar. J. 8 (2004) 11–31.
[5] Z. Wang, J. Xia, L. Zhang, Optimal investment for an insurer: The martingale approach, Insurance Math. Econom. 40 (2007) 322–334.
[6] H. Schmidli, Optimal proportional reinsurance policies in a dynamic setting, Scand. Actuar. J. 1 (2001) 55–68.
[7] D.S. Promislow, V.R. Young, Minimizing the probability of ruin when claims follow Brownian motion with drift, N. Am. Actuar. J. 9 (2005) 109–128.
[8] L. Bai, J. Guo, Optimal proportional reinsurance and investment with multiple risky assets and no-shorting constraint, Insurance Math. Econom. 42 (2008) 968–975.
[9] Y. Cao, N. Wan, Optimal proportional reinsurance and investment based on Hamilton–Jacobi-Bellman equation, Insurance Math. Econom. 45 (2009) 157–162.
[10] Z. Liang, E. Bayraktar, Optimal reinsurance and investment with unobservable claim size and intensity, Insurance Math. Econom. 55 (2014) 156–166.

[11] C. Irgens, J. Paulsen, Optimal control of risk exposure, reinsurance and investments for insurance portfolios, Insurance Math. Econom. 35 (2004) 21–51.
[12] Y. Huang, X. Yang, J. Zhou, Optimal investment and proportional reinsurance for a jump-diffusion risk model with constrained control variables, J. Comput. Appl. Math. 296 (2016) 443–461.
[13] D.G. Hobson, L.C.G. Rogers, Complete models with stochastic volatility, Math. Finance 8 (1998) 27–48.
[14] J.C. Cox, S.A. Ross, The valuation of options for alternative stochastic processes, J. Financ. Econ. 4 (1976) 145–166.
[15] S. Beckers, The constant elasticity of variance model and its implications for option pricing, J. Financ. 35 (1980) 661–673.
[16] D. Davydov, V. Linetsky, The valuation and hedging of barrier and lookback option under the CEV process, Manage. Sci. 47 (2001) 949–965.
[17] C. Jones, The dynamics of the stochastic volatility: evidence from underlying and options markets, J. Econ. 116 (2003) 181–224.
[18] J. Xiao, Z. Hong, C. Qin, The constant elasticity of variance (CEV) model and the Legendre transform-dual solution for annuity contracts, Insurance Math. Econom. 40 (2007) 302–310.
[19] J. Gao, Optimal portfolio for DC pension plans under a CEV model, Insurance Math. Econom. 44 (2009) 479–490.
[20] J. Gao, Optimal investment strategy for annuity contracts under the constant elasticity of variance (CEV) model, Insurance Math. Econom. 45 (2009) 9–18.
[21] M. Gu, Y. Yang, S. Li, J. Zhang, Constant elasticity of variance model for proportional reinsurance and investment strategies, Insurance Math. Econom. 46 (2010) 580–587.
[22] Z. Liang, K. Yuen, K. Cheung, Optimal reinsurance-investment problem in a constant elasticity of variance stock market for jump-diffusion risk model, Appl. Stoch. Models Bus. 28 (2012) 585–597.
[23] X. Lin, Y. Li, Optimal reinsurance and investment for a jump diffusion risk process under the CEV model, N. Am. Actuar. J. 15 (2012) 417–431.
[24] D. Li, X. Rong, H. Zhao, Optimal reinsurance-investment problem for maximizing the product of the insurers and the reinsurers utilities under a CEV model, J. Comput. Appl. Math. 255 (2014) 671–683.
[25] D. Li, X. Rong, H. Zhao, Optimal investment problem for an insurer and a reinsurer, J. Syst. Sci. Complex. 28 (2015) 1326–1343.
[26] H. Zhao, C. Weng, Y. Zeng, Time-consistent investment-reinsurance strategies towards joint interests of the insurer and the reinsurer under CEV models, Sci. China Math. 60 (2017) 317–344.
[27] D. Li, X. Rong, H. Zhao, Time-consistent reinsurance-investment strategy for an insurer and a reinsurer with mean–variance criterion underthe CEV model, J. Comput. Appl. Math. 283 (2015) 142–162.
[28] W. Fleming, H. Soner, Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, NY, 1993.
[29] J. Grandell, Aspects of Risk Theory, Springer-Verlag, NY, 1991.