Weighted differentiation composition operator from logarithmic Bloch spaces to Bloch-type spaces

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In this paper, we give a new characterization for the boundedness of the weighted differentiation composition operator from logarithmic Bloch spaces to Bloch-type spaces and calculate its essential norm in terms of the \(n\)-th power of the induced analytic self-map on the unit disk. From which a sufficient and necessary condition of compactness of the operator follows immediately.

1 Introduction

Denote \(H(D)\) the space of all holomorphic functions on the unit disk \(D\) and \(S(D)\) the set of all self-maps on \(D\). Throughout this paper, \(\log\) denotes the natural logarithm function. Given a bounded, continuous and strictly positive function \(\mu\) on \(D\), we define the \(\mu\)-Bloch space \(B_\mu = B_\mu(D)\), consisting of all \(f \in H(D)\) such that
\[
\|f\|_\mu = \sup_{z \in D} \mu(z) |f'(z)| < \infty.
\]
The space \(B_\mu\) is a Banach space under the norm
\[
\|f\|_{B_\mu} = |f(0)| + \|f\|_\mu.
\]

For \(\alpha > 0\) and \(\mu : D \to (0, 1]\) is defined by \(\mu(z) = (1 - |z|^2)^\alpha\), then in this case, \(B_\mu\) is denoted by \(B_\alpha\), the so-called \(\alpha\)-Bloch space on \(D\). when \(\alpha = 1\), \(B_\alpha\) is the classical Bloch space \(B\). Moreover, let \(\mu = v_{\log} : D \to (0, \infty)\) be given by
\[
v_{\log}(z) = (1 - |z|) \log \left( \frac{3}{1 - |z|} \right),
\]
then we obtain the log-Bloch space and denote \(B_{v_{\log}}(D)\) by \(B_{\log}\). It is well-known that the log-Bloch space \(B_{\log}\) is a Banach space endowed with the norm
\[
\|f\|_{B_{\log}} = |f(0)| + \|f\|_{log},
\]
where
\[
\|f\|_{\log} = \sup_{z \in D} (1 - |z|) \log \left( \frac{3}{1 - |z|} \right) |f'(z)|.
\]

We refer the readers to the book [22] by K. H. Zhu, which is excellent source for the development of the theory of function spaces.

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For $\varphi \in S(\mathbb{D})$, the composition operator $C_\varphi$ is defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in H(\mathbb{D}).$$

It is an interesting topic to provide function-theoretic characterizations of when $\varphi$ induces a bounded or compact composition operator on various holomorphic function spaces. For this theory, we refer the readers to the books [4] by Cowen and MacCluer, and [14] by Shapiro.

As we all know the differentiation operator is defined as $Df = f'$ for $f \in H(\mathbb{D})$. For $u \in H(\mathbb{D})$, the weighted composition operator $uC_\varphi$ is given by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

Now let $m \in \mathbb{N}$, the weighted differentiation composition operator, denoted by $D^m_{\varphi,u}$, is defined as follows

$$(D^m_{\varphi,u}f)(z) = u(z)f^{(m)}(\varphi(z)), \quad f \in H(\mathbb{D}).$$

When $m = 0$, then the operator $D^m_{\varphi,u}$ becomes the weighted composition operator $uC_\varphi$. That’s the reason why we call $D^m_{\varphi,u}$ the weighted differentiation composition operator. If $m = 0$ and $u(z) = 1$, then $D^m_{\varphi,u} = C_\varphi$. If $m = 1$ and $u(z) = 1$, then $D^m_{\varphi,u} = C_\varphi D$. If $m = 1$ and $u(z) = \varphi'(z)$, then $D^m_{\varphi,u} = DC_\varphi$. For the operator $D^m_{\varphi,u}$, we refer the readers to the papers [15, 16].

The essential norm of a continuous linear operator $T : X \rightarrow Y$ is the distance from $T$ to the set of all compact operators, that is,

$$\|T\|_e = \inf \{\|T - K\| : K \text{ is compact}\}.$$

Since $\|T\|_e = 0$ if and only if $T$ is compact, so the estimates on $\|T\|_e$ lead to conditions for $T$ to be compact.

Recently there has been a great interest in the new characterizations for the essential norms of composition operator and differentiation operator between Bloch-type spaces on the unit disk. In papers [10] and [11], we respectively studied the new characterizations for the operators $C_\varphi D^m : B_\alpha \rightarrow B_\beta$ and $DC_\varphi : B_\alpha \rightarrow B_\beta$. Concerning the composition operator from the log-Bloch space to $\mu$-Bloch space, we refer the readers to the papers [2, 5]. Moreover, the papers [6–9, 13, 19–21] are also about the new subject, which are helpful for our study.

Based on the above foundations, the goal of this paper is to give the new characterizations for the weighted differentiation operator $D^m_{\varphi,u} : B_{\log}^\alpha \rightarrow B_\mu^\beta$ on the unit disk. In section 2, we list some lemmas. The characterizations for the boundedness and essential norm of $D^m_{\varphi,u} : B_{\log}^\alpha \rightarrow B_\mu^\beta$ are given in section 3 and section 4, respectively.

Throughout the remainder of this paper, the notations $A \asymp B$, $A \preceq B$, $A \succeq B$ mean that there maybe different positive constants $C$ such that $B/C \leq A \leq CB$, $A \leq CB$, $CB \leq A$.

## 2 Some lemmas

Let $L = 1 - \frac{1}{\log 3} \in (0, 1)$ in this paper. In this section, we list some auxiliary facts. We define a sequence $(r_j)_{j \in \mathbb{N}}$ by $r_0 = 0$ and $r_j = 1 - \frac{m-1+L}{m+1+L}$ for each $j \in \mathbb{N}$. The sequence $(r_j)_{j \in \mathbb{N}}$ lies in $[0, 1)$ is strictly increasing and satisfies $r_j \rightarrow 1^-$ as $j \rightarrow \infty$. For $\varphi \in S(\mathbb{D})$ and $j \in \mathbb{N}$, we define the set

$$A^j_\varphi = \{z \in \mathbb{D} : r_j \leq |\varphi(z)| < r_{j+1}\}.$$

It is obvious that $A^j_\varphi \cap A^k_\varphi = \emptyset$ for $j \neq k$.

**Lemma 2.1** Define $A : [1, \infty) \rightarrow (0, 1]$ by

$$A(x) = \left(\frac{x+1}{x+m+L}\right)^x.$$

Then we have that

$$\inf_{x\geq1} A(x) = \lim_{x \rightarrow \infty} A(x) = e^{-(m+1+L)}.$$
Proof.

\[
\lim_{x \to \infty} A(x) = \lim_{x \to \infty} \left( \frac{x + 1}{x + m + L} \right)^x \\
= \lim_{x \to \infty} \left( 1 + \frac{-(m - 1 + L)}{x + m + L} \right)^{-x(m-1+L)} \\
= e^{-(m-1+L)}.
\]

\[
(\log A(x))' = \log \frac{x + 1}{x + m + L} + \frac{x(m + L - 1)}{(x + 1)(x + m + L)} \\
\leq \log \frac{x + 1}{x + m + L} + \frac{m + 1}{x + m + L} \\
= \log \frac{x + 1}{x + m + L} - \frac{x + 1}{x + m + L} + 1.
\]

The above inequality is negative, due to the function \(\log \eta - \eta + 1\) takes values \(-\infty\) and 0 at \(\eta = 0\) and \(\eta = 1\), respectively and is strictly increasing in \(\eta \in (0,1)\). Thus \(\log A\) is strictly decreasing, in turn the function \(A\) is strictly decreasing on \([1,\infty)\). Hence it follows that \(\inf_{x \geq 1} A(x) = \lim_{x \to \infty} A(x) = e^{-(m-1+L)}\). This ends the proof.

\[\square\]

Lemma 2.2 Let \(m, j \in \mathbb{N}\), then the function

\[f_j(x) = x^j(1-x)^m \log \frac{3}{1-x}, \quad x \in (0,1),\]

is decreasing on \([r_j,1)\). Also, we have that

\[\frac{j^m}{\log \frac{j+1}{m+1}} f_j(x) \geq \frac{L^m}{3^m e^{m-1+L}} \quad \text{for all} \quad x \in [r_j, r_{j+1}],\]

\[\lim_{j \to \infty} \log \frac{j^m}{\log \frac{j+1}{m+1}} \min_{x \in [r_j, r_{j+1}]} f_j(x) = \frac{(m-1+L)^m}{e^{m-1+L}}.\]

Proof. It suffices to show that \(f_j\) is decreasing on \([r_j, r_{j+1})\). Since

\[f_j'(x) = x^j(1-x)^{m-1} \left( (j(1-x) - mx) \log \frac{3}{1-x} + x \right),\]

and \(L \in (0,1)\), then we have \(j - (j + m)x < 0\) holds for all \(x \in [r_j,1)\). By the fact \(\log \frac{3}{1-x} \geq \log 3\) for all \(x \in (0,1)\), it follows that

\[f_j'(x) \leq x^{j-1}(1-x)^{m-1} \left( (j - (j + m)x) \log 3 + x \right).\]

Since the function

\[h_j(x) = (j - (j + m)x) \log 3 + x = j \log 3 - ((j + m) \log 3 - 1)x\]

is decreasing for \(x \in [r_j, 1)\) and \(h_j(r_j) = 0\), thus \(f_j'(x) < 0\) for all \(x \in (r_j, 1)\). That is, \(f_j\) is decreasing on \([r_j, 1)\). On the other hand, by the first statement in this lemma and lemma 2.1, we get that for each \(j \in \mathbb{N}\) and all
Moreover, we have that
\[ x \in [r_j, r_{j+1}], \]
\[ \frac{j^m}{\log \frac{j+1}{m+1}} f_j(x) \geq \frac{j^m}{\log \frac{j+1}{m+1}} f_j(r_{j+1}) \]
\[ = \frac{j^m}{\log \frac{j+1}{m+1}} \left( \frac{1 - m - 1 + L}{j + m + L} \right)^j \frac{m - 1 + L}{j + m + L} \cdot \log \frac{3(j + m + L)}{m - 1 + L} \]
\[ = \log \frac{3(j + m + L)}{m + 1 + L} \left( \frac{j + 1}{j + m + L} \right)^j \left( \frac{(m - 1 + L)}{j + m + L} \right)^m \]
\[ \geq 1 \cdot e^{-(m-1+L)} \left( \frac{m - 1 + L}{m + 1 + L} \right)^m \]
\[ \geq \frac{L^m}{3m e^{m-1+L}}. \]

Moreover, we have that
\[ \lim_{j \to \infty} \log \frac{j^m}{\log \frac{j+1}{m+1}} \min_{x \in [r_j, r_{j+1}]} f_j(x) = \lim_{j \to \infty} \log \frac{j^m}{\log \frac{j+1}{m+1}} f_j(r_{j+1}) \]
\[ = \lim_{j \to \infty} \log \frac{3(j + m + L)}{m + 1 + L} \left( \frac{j + 1}{j + m + L} \right)^j \left( \frac{(m - 1 + L)}{j + m + L} \right)^m \]
\[ = (m - 1 + L)^m e^{m-1+L}. \]

This completes the proof. \( \square \)

A positive continuous function \( v \) on \([0, 1)\) is called normal (see, e.g. [18]), if there exist three positive constants \( 0 \leq \delta < 1 \), and \( 0 < a < b < \infty \), such that for \( r \in [\delta, 1) \)
\[ v(r) = (1 - |z|)^m \log \left( \frac{3}{1 - |z|} \right), \]
it is clear that \( v_1 \) is a normal weight. From [17, Lemma 3] we obtain that "Assume that \( v \) is a normal weight, then \( \sup_{z \in \mathbb{D}} v(z) = |f(0)| \) and \( \sup_{z \in \mathbb{D}} v(z)(1 - |z|)|f'(z)| \) for every \( f \in H(\mathbb{D}) \)." Hence the following result holds for the log-Bloch space \( B_{\log} \) on the unit disk:

**Lemma 2.3** For \( f \in H(\mathbb{D}), m \in \mathbb{N}, \)
\[ f \in B_{\log} \Leftrightarrow \|f\|_{\log} = \sup_{z \in \mathbb{D}} (1 - |z|)^m \log \left( \frac{3}{1 - |z|} \right) |f^{(m)}(z)| < \infty. \]

Hence, \( f \in B_{\log} \Leftrightarrow f^{(m)} \in H_{v_1}^{\infty} = \{ f \in H(\mathbb{D}) : \|f\|_{v_1} = \sup_{z \in \mathbb{D}} v_1(z)|f(z)| < \infty \}. \) Moreover, \( v_1 \) is also a essential weight. In fact, a weight \( v : \mathbb{D} \to \mathbb{R}_+ \) is called radial if \( v(z) = v(|z|) \) for all \( z \in \mathbb{D} \). The so-called associated weights are defined by
\[ \tilde{v}(z) = (\sup \{|f(z)| : f \in H_{v_1}^{\infty}, \|f\|_{v_1} \leq 1 \})^{-1}. \]

It is evident that \( \tilde{v} \) is also a weight. A weight \( \nu \) is called essential if there exists a constant \( C > 0 \) such that
\[ v(z) \leq \tilde{v}(z) \leq Cv(z) \quad \text{for each} \quad z \in \mathbb{D}. \]

Besides, the following condition \((L1)\) which was introduced by Lusky in [12] plays an important part in deciding whether a weight is essential or not,
\[ (L1) \inf_{n \in \mathbb{N}} \frac{v(1 - 2^{-n-1})}{v(1 - 2^{-n})} > 0. \]
Radial weights which satisfy $(L1)$ are always essential (see [1]). It is obvious that the weight $v_1$ is radial and satisfies

$$\inf_{n \in \mathbb{N}} \frac{v_1(1 - 2^{n-1})}{v_1(1 - 2^{-n})} = \inf_{n \in \mathbb{N}} \frac{1}{2^m \log 3 \cdot 2^m} > \frac{1}{2} > 0.$$ 

Hence $v_1$ is essential, which will be used to show the following theorem A.

**Lemma 2.4** [2, Lemma 2.3]

$$\|z^j\|_{\log} \lesssim \frac{1}{e} \text{ as } j \to \infty. \quad (3)$$

Besides, since

$$\lim_{j \to \infty} \frac{\log(j+1)}{\log \frac{j+1}{m-1+L}} = 1,$$

thus

$$\frac{\|z^j\|_{\log}}{\log \frac{j+1}{m-1+L}} \lesssim \frac{1}{e} \text{ as } j \to \infty. \quad (4)$$

**Lemma 2.5** For $f_j$ defined in lemma 2.2, the following statements hold:

(a) for all $j \in \mathbb{N}$, there is a unique $x_j \in (0,1)$ such that $f_j(x_j)$ is the absolute maximum of $f_j$.

(b) the sequence $(x_j)_{j \in \mathbb{N}}$ satisfies

$$\lim_{j \to \infty} x_j = 1^-,$$

where "−" denotes that $x_j$ tends to 1 from the left, moreover,

$$\lim_{j \to \infty} j(1-x_j) = m. \quad (6)$$

(c)

$$\lim_{j \to \infty} \frac{j^m}{\log(j+1)} \max_{0 < x < 1} f_j(x) = \lim_{j \to \infty} \frac{j^m}{\log(j+1)} f_j(x_j) = \frac{m^m}{e^m}. \quad (7)$$

**Proof.** It is obvious that

$$f'_j(x) = x^j(1-x)^{m-1} \left( (j(1-x) - mx) \log \frac{3}{1-x} + x \right).$$

We denote

$$g_j(x) = (j(1-x) - mx) \log \frac{3}{1-x} + x.$$ 

Then

$$g'_j(x) = j + 1 + m - \frac{m}{1 - x} - (j + m) \log \frac{3}{1-x} \leq j + 1 + m - m - (j + m) \log 3 = j + 1 - (j + m) \log 3 < 0.$$ 

Thus $g_j$ is strictly decreasing on $(0,1)$, and since

$$\lim_{x \to 0^+} g_j(x) = j \log j > 0 \text{ and } \lim_{x \to 1^-} g_j(x) = -\infty,$$

hence there is a unique $x_j \in (0,1)$ such that $g_j(x_j) = 0$. It is clear that $g_j(x) > 0$ whenever $x \in (0, x_j)$ and that $g_j(x) < 0$ whenever $x \in (x_j, 1)$. Since

$$f'_j(x) = x^{j-1}(1-x)^{m-1} g_j(x),$$

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so the function \( f_j \) is increasing on \((0, x_j)\) and decreasing on \((x_j, 1)\). Therefore, \( f_j \) has a unique absolute maximum at \( x_j \), which implies \((a)\) holds.

Since \( g_j(x_j) = (j(1-x_j) - mx_j) \log \frac{3}{1-x_j} + x_j = 0 \), thus
\[
\left(1 - x_j - \frac{mx_j}{j}\right) \log \frac{3}{1-x_j} = \frac{-x_j}{j}.
\]

Since \( x_j \in (0, 1) \), then letting \( j \to \infty \) in the above equation, it follows that
\[
\lim_{j \to \infty} \left(1 - x_j - \frac{mx_j}{j}\right) \log \frac{3}{1-x_j} = 0.
\]

However, \( \log \frac{3}{1-x_j} \geq \log 3 \), then \( \lim_{j \to \infty} [(1 - x_j) - \frac{mx_j}{j}] = 0 \). Hence \( \lim_{j \to \infty} x_j = 1^- \). That is, \((5)\) is true.

Using \( g_j(x_j) = 0 \) again, we have that
\[
\lim_{j \to \infty} [j(1-x_j) - mx_j] = \lim_{j \to \infty} \frac{-x_j}{\log \frac{3}{1-x_j}} = 0.
\]

Which implies that \( \lim_{j \to \infty} j(1-x_j) = \lim_{j \to \infty} mx_j = m \). That is, \((6)\) is true.

Since
\[
\lim_{j \to \infty} j \log x_j = \lim_{j \to \infty} j(x_j - 1) \frac{\log [1+(x_j - 1)]}{x_j - 1} = -m,
\]
then \( \lim_{j \to \infty} x_j = e^{-m} \). Further by
\[
\lim_{j \to \infty} \frac{3}{1-x_j} \log (j+1) = \lim_{j \to \infty} \frac{3j}{j(1-x_j)} = \lim_{j \to \infty} \left( \frac{\log 3j}{\log (j+1)} - \frac{\log (j(1-x_j))}{\log (j+1)} \right) = 1,
\]
we have that
\[
\lim_{j \to \infty} \frac{j^m}{\log (j+1)} \max_{0 < x < 1} f_j(x) = \lim_{j \to \infty} \frac{j^m}{\log (j+1)} x_j^j (1-x_j)^m \log \frac{3}{1-x_j} = \lim_{j \to \infty} \frac{1}{\log (j+1)} j^m (1-x_j)^m x_j^j = \frac{m^m}{e^m}.
\]

This completes the proof. \( \square \)

In this paper, we use the following notation. Let \( u \in H(\mathbb{D}) \) and \( f \in H(\mathbb{D}) \), define
\[
I_u f(z) = \int_0^z f'(\zeta)u(\zeta)d\zeta, \quad J_u f(z) = \int_0^z f(\zeta)u'(\zeta)d\zeta.
\]

Then it follows that
\[
I_u (\varphi^{j+1})(z) = \int_0^z (\varphi^{j+1})(\zeta)u(\zeta)d\zeta, \quad J_u (\varphi^{j})(z) = \int_0^z \varphi^{j}(\zeta)u'(\zeta)d\zeta. \quad (8)
\]

Besides,
\[
\|I_u (\varphi^{j+1})\|_\mu = (j+1) \sup_{z \in \mathbb{D}} \mu(z) |\varphi(z)|^j |u(z)| |\varphi'(z)|, \quad \|J_u (\varphi^j)\|_\mu = \sup_{z \in \mathbb{D}} \mu(z) |\varphi^j(z)| u'(z).\]

Since \( D_{\varphi^j,u}^m f' = u' f^{(m)}(\varphi + w\varphi'^{(m+1)}(z) \varphi' \), thus \( D_{\varphi^j,u}^m : B_{\log} \to B_{\log} \) is bounded (or compact) if and only if the weighted composition operators \( u'C_{\varphi} : H_{v_2} \to H_{\mu} \) and \( [w\varphi^j]C_{\varphi} : H_{v_2} \to H_{\mu} \) are bounded (or compact), where \( v_2(z) = (1-|z|)^{m+1} \log \left( \frac{3}{1-|z|} \right) \) is also a essential weight. Similar to the paper \([3, \text{Proposition } 3.1]\) we obtain the following theorem A.
By the above equation, there exists a constant $K$ if and only if
\[ u \]

Theorem A Let $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}$ and $\mu$ be a weight on $\mathbb{D}$. Then the operator $D_{\varphi,u}^m : B_\log \to B_\mu$ is bounded if and only if
\[
M_1 := \sup_{z \in \mathbb{D}} \frac{\mu(z)|u'(z)|}{(1 - |\varphi(z)|)^m \log \frac{3}{1 - |\varphi(z)|}} < \infty,
\]
\[
M_2 := \sup_{z \in \mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|)^{m+1} \log \frac{3}{1 - |\varphi(z)|}} < \infty.
\]

Based on the above result, we will give the new criterion for the boundedness of the operator $D_{\varphi,u}^m : B_\log \to B_\mu$.

3 The boundedness

Theorem 3.1 Let $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}$ and $\mu$ be a weight on $\mathbb{D}$. Then $D_{\varphi,u}^m : B_\log \to B_\mu$ is bounded if and only if $u \in B_\mu$, $\sup_{z \in \mathbb{D}} \mu(z)|u(z)\varphi'(z)| < \infty$ and both of the inequalities hold:
\[
\sup_{j \geq 1} \frac{j^m \|J_u \varphi^j\|_\mu}{\|z^j\|_{\log}} < \infty, \quad j \geq 1
\]
\[
\sup_{j \geq 1} \frac{j^m \|J_u \varphi^{j+1}\|_\mu}{\|z^j\|_{\log}} < \infty. \quad (10)
\]

Proof. Necessity. Suppose the operator $D_{\varphi,u}^m : B_\log \to B_\mu$ is bounded. Since the function $f_1(z) = z^m \in B_\log$ and $f_2(z) = z^{m+1} \in B_\log$, then we have that
\[
m! \sup_{z \in \mathbb{D}} \mu(z)|u'(z)| < \infty.
\]
\[
(m + 1)! \sup_{z \in \mathbb{D}} \mu(z)|u(z)\varphi'(z) + u(z)\varphi'(z)| < \infty.
\]

From the above inequalities, we obtain that $u \in B_\mu$ and $\sup_{z \in \mathbb{D}} \mu(z)|u(z)\varphi'(z)| < \infty$. Next we will show (9) and (10) hold. From (7), we know that there exists a constant $K_1 > 0$ such that
\[
\sup_{j \geq 1} \frac{j^m}{\log(j+1)} \max_{0 < x < 1} x^j (1 - x)^m \log \frac{3}{1 - x} < K_1. \quad (11)
\]

Since the operator $D_{\varphi,u}^m : B_\log \to B_\mu$ is bounded, then by theorem A, lemma 2.4 and (11), it follows that
\[
\sup_{j \geq 1} \frac{j^m \|J_u \varphi^j\|_\mu}{\|z^j\|_{\log}} \leq \sup_{j \geq 1} \frac{j^m}{\log(j+1)} \sup_{z \in \mathbb{D}} \mu(z)|\varphi^j(z)u'(z)|
\]
\[
= \sup_{j \geq 1} \frac{j^m}{\log(j+1)} \sup_{z \in \mathbb{D}} \frac{\mu(z)|u'(z)|}{(1 - |\varphi(z)|)^m \log \frac{3}{1 - |\varphi(z)|}} \frac{|\varphi(z)|^j (1 - |\varphi(z)|)^m \log \frac{3}{1 - |\varphi(z)|}}{3 (1 - |\varphi(z)|)} < M_1 K_1.
\]

That is, (9) holds. Similarly, from (7), we obtain that
\[
\lim_{j \to \infty} \frac{j^m+1}{\log(j+1)} \max_{0 < x < 1} x^j (1 - x)^{m+1} \log \frac{3}{1 - x} = \frac{(m + 1)^{m+1}}{e^{m+1}}.
\]

By the above equation, there exists a constant $K_2 > 0$ such that
\[
\sup_{j \geq 1} \frac{j^{m+1}}{\log(j+1)} \max_{0 < x < 1} x^j (1 - x)^{m+1} \log \frac{3}{1 - x} < K_2. \quad (12)
\]
Similarly, by theorem A, lemma 2.4 and (12),
\[
\sup_{j \geq 1} \frac{j^m |D^{j}_{\varphi} u|_{\mathbb{B}}}{|\varphi|_{\log}} \leq \frac{1}{\log(j+1)} \sup_{j \geq 1} \frac{j^m}{\log(j+1)} \sup_{z \in \mathbb{D}} \frac{\mu(z) u' z (\varphi(z))}{1 - |\varphi(z)|} \left| (1 - |\varphi(z)|)^{m+1} \log \frac{3}{1 - |\varphi(z)|} \right| \left| f_j(z) \right| (1 - |\varphi(z)|)^{m+1} \log \frac{3}{1 - |\varphi(z)|} \leq M_2 K_2.
\]
That is, (10) holds.

**Sufficiency.** Firstly, if sup $|\varphi(z)| < 1$, then there is a number $r \in (0, 1)$ such that sup $|\varphi(z)| < r$. For every $f \in \mathbb{B}_{\log}$ with $\|f\|_{\mathbb{B}_{\log}} \leq 1$, by the condition $u \in \mathbb{B}_{\mu}$ and sup $\mu(z) u(z) \varphi(z) < \infty$, it follows that
\[
\frac{|u(0)|}{1 - |\varphi(0)|} m \log \frac{3}{1 - |\varphi(0)|} + \sup_{z \in \mathbb{D}} \frac{\mu(z) u'(z)}{1 - |\varphi(z)|} m \log \frac{3}{1 - |\varphi(z)|} \leq \frac{|u(0)|}{1 - |\varphi(0)|} m \log \frac{3}{1 - |\varphi(0)|} + \sup_{z \in \mathbb{D}} \frac{\mu(z) u(z) \varphi'(z)}{1 - |\varphi(z)|} m \log \frac{3}{1 - |\varphi(z)|} \leq \frac{|u(0)|}{1 - |\varphi(0)|} m \log \frac{3}{1 - |\varphi(0)|} + \sup_{z \in \mathbb{D}} \frac{\mu(z) u(z) \varphi'(z)}{1 - |\varphi(z)|} m \log \frac{3}{1 - |\varphi(z)|} < \infty,
\]
which implies the boundedness of $D^{m}_{\varphi} u : \mathbb{B}_{\log} \rightarrow \mathbb{B}_{\mu}$.

Secondly, if sup $|\varphi(z)| = 1$. For every $f \in \mathbb{B}_{\log}$ with $\|f\|_{\mathbb{B}_{\log}} \leq 1$, we have that
\[
\frac{|u(0)|}{1 - |\varphi(0)|} m \log \frac{3}{1 - |\varphi(0)|} + \sup_{z \in \mathbb{D}} \frac{\mu(z) u'(z)}{1 - |\varphi(z)|} m \log \frac{3}{1 - |\varphi(z)|} \leq \frac{|u(0)|}{1 - |\varphi(0)|} m \log \frac{3}{1 - |\varphi(0)|} + \sup_{z \in \mathbb{D}} \frac{\mu(z) u(z) \varphi'(z)}{1 - |\varphi(z)|} m \log \frac{3}{1 - |\varphi(z)|} \leq \frac{|u(0)|}{1 - |\varphi(0)|} m \log \frac{3}{1 - |\varphi(0)|} + \sup_{z \in \mathbb{D}} \frac{\mu(z) u(z) \varphi'(z)}{1 - |\varphi(z)|} m \log \frac{3}{1 - |\varphi(z)|} \leq \frac{|u(0)|}{1 - |\varphi(0)|} m \log \frac{3}{1 - |\varphi(0)|} + \sup_{z \in \mathbb{D}} \frac{\mu(z) u(z) \varphi'(z)}{1 - |\varphi(z)|} m \log \frac{3}{1 - |\varphi(z)|} \leq \frac{|u(0)|}{1 - |\varphi(0)|} m \log \frac{3}{1 - |\varphi(0)|} + \sup_{z \in \mathbb{D}} \frac{\mu(z) u(z) \varphi'(z)}{1 - |\varphi(z)|} m \log \frac{3}{1 - |\varphi(z)|} \leq \frac{|u(0)|}{1 - |\varphi(0)|} m \log \frac{3}{1 - |\varphi(0)|} + \sup_{z \in \mathbb{D}} \frac{\mu(z) u(z) \varphi'(z)}{1 - |\varphi(z)|} m \log \frac{3}{1 - |\varphi(z)|}\]
\[
= I_1 + I_2 + I_3,
\]
It is obvious that $I_1 < \infty$. We only need to show that $I_2$ and $I_3$ is finite. For any integer $j \geq 1$, let
\[
A_{\varphi}^j = \{ z \in \mathbb{D} : r_j \leq |\varphi(z)| < r_{j+1} \},
\]
where $r_j = 1 - \frac{m-1+L}{j+m-1+L}$. Let $k$ be the smallest positive integer such that $A_{\varphi}^k \neq \emptyset$. Since sup $|\varphi(z)| = 1$, hence the set $A_{\varphi}^k$ is not empty for every integer $j \geq k$, and $\mathbb{D} = \bigcup_{k=1}^{\infty} A_{\varphi}^k$. By (1), for every $j \in \mathbb{N}$,
\[
\frac{j^m}{\log \frac{1}{m+1+L}} \min_{x \in [r_j, r_{j+1}]} f_j(x) \geq \frac{L^m}{3^m m^{-1} + L} = \delta_1.
\]
Hence by (14) and (4), it follows that

\[ I_2 = \sup_{z \in D} \frac{\mu(z)|u(z)|}{(1 - |\varphi(z)|)^m \log \frac{3}{1 - |\varphi(z)|}} \]

\[ = \sup_{k \leq j} \sup_{z \in A_k^j} \frac{j^m \mu(z)|u'(z)||\varphi(z)|}{\log \frac{j^m + j^1}{j^m + j + 1} |\varphi(z)|^j (1 - |\varphi(z)|)^m \log \frac{3}{1 - |\varphi(z)|} \log \frac{j^m + j^1 + 1}{j^m + 1 + 1} } \]

\[ \leq \frac{1}{\delta_1} \sup_{k \leq j} \sup_{z \in A_k^j} \frac{j^m \mu(z)|u'(z)||\varphi(z)|^j}{\log \frac{j^m + j^1 + 1}{j^m + 1 + 1} } \]

\[ \leq \frac{1}{\delta_1} \sup_{1 \leq j} \sup_{z \in D} \frac{j^m \mu(z)|u'(z)||\varphi(z)|^j}{\log \frac{j^m + j^1 + 1}{j^m + 1 + 1} } \]

\[ \leq \frac{1}{\delta_1} \sup_{j \geq 1} \sup_{z \in D} \frac{j^m \mu(z)|u'(z)||\varphi(z)|^j}{\log \frac{j^m + j^1 + 1}{j^m + 1 + 1} } \]

\[ = \frac{1}{\delta_1} \sup_{j \geq 1} \frac{j^m I_u \varphi_j^j \|\mu\|}{\|z\|^j \log} \quad (15) \]

Similarly, \( \frac{j^{m+1}}{\log \frac{j^m + j+1}{j^m + 1 + 1}} \min_{x \in A_k^j} x^j (1 - x)^{m+1} \log \frac{3}{j^m + j + 1} = \delta_2. \) Hence

\[ I_3 = \sup_{z \in D} \frac{\mu(z)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|)^{m+1} \log \frac{3}{1 - |\varphi(z)|} } \]

\[ = \sup_{k \leq j} \sup_{z \in A_k^j} \frac{\mu(z)|u(z)\varphi'(z)||j^{m+1}|\varphi(z)|^j}{\log \frac{j^m + j^1 + 1}{j^m + 1 + 1} |\varphi(z)|^j (1 - |\varphi(z)|)^{m+1} \log \frac{3}{1 - |\varphi(z)|} \log \frac{j^m + j^1 + 1}{j^m + 1 + 1} } \]

\[ \leq \frac{1}{\delta_2} \sup_{k \leq j} \sup_{z \in A_k^j} \frac{\mu(z)|u(z)\varphi'(z)||j^{m+1}|\varphi(z)|^j}{\log \frac{j^m + j^1 + 1}{j^m + 1 + 1} } \]

\[ \leq \frac{1}{\delta_2} \sup_{j \geq 1} \sup_{z \in D} \frac{\mu(z)|u(z)\varphi'(z)||j^{m+1}|\varphi(z)|^j}{\log \frac{j^m + j^1 + 1}{j^m + 1 + 1} } \]

\[ \leq \frac{1}{\delta_2} \sup_{j \geq 1} \sup_{z \in D} \frac{\mu(z)|u(z)\varphi'(z)||j^{m+1}|\varphi(z)|^j}{\|z\|^j \log} \]

\[ = \frac{1}{\delta_2} \sup_{j \geq 1} \frac{j^{m+1} I_u \varphi^{j+1} \|\mu\|}{\|z\|^j \log} \quad (16) \]

Combining (13), (15) and (16) we obtain the boundedness of \( D_{\varphi,u}^m : B_{\log} \to B_\mu \) in this case. Now the proof is complete. \( \square \)

### 4 The essential norm

In this section, we will give an estimate for the essential norm of \( D_{\varphi,u}^m : B_{\log} \to B_\mu \). To simplify the notations, we denote

\[ A := \limsup_{j \to \infty} \frac{j^m I_u \varphi_j \|\mu\|}{\|z\|^j \log} \quad \text{and} \quad B := \limsup_{j \to \infty} \frac{j^m I_u \varphi^{j+1} \|\mu\|}{\|z\|^j \log}. \]

For \( r \in [0, 1] \), we define the linear dilation operator \( K_r : H(D) \to H(D) \) by \( K_r f = f_r \), where \( f_r(z) = f(rz) \). Then we have:
Employing lemma 4.2, it is clear that the operator $D^{m}_{\varphi,u}$ converging to zero uniformly on compact subsets of $D$. In fact, let $f_{j} \in B_{\mu}$ such that $f_{j}$ converges to 0 uniformly on compact subsets of $D$, then $\|D^{m}_{\varphi,u}(f_{j})\|_{B_{\mu}} \to 0$ as $j \to \infty$.

Theorem 4.3 Let $u \in H(D)$, $\varphi \in S(D)$, $m \in \mathbb{N}$ and $\mu$ be a weight on $D$. Suppose that the operator $D^{m}_{\varphi,u} : B_{\log} \to B_{\mu}$ is bounded. Then

$$\|D^{m}_{\varphi,u}\|_{c} \asymp A + B. \quad (17)$$

Proof. Since the operator $D^{m}_{\varphi,u} : B_{\log} \to B_{\mu}$ is bounded, then $A < \infty$ and $B < \infty$. Moreover,

$$L_{1} := \sup_{z \in D} \mu(z) |u'(z)| < \infty, \quad L_{2} := \sup_{z \in D} \mu(z) |u(z)\varphi'(z)| < \infty.$$

Firstly, if $\sup_{z \in D} |\varphi(z)| < 1$, then there is a number $r \in (0, 1)$ such that $\sup_{z \in D} |\varphi(z)| < r$. It’s easy to verify the operator $D^{m}_{\varphi,u} : B_{\log} \to B_{\mu}$ is compact by lemma 4.2. In fact, let $\{f_{k}\}_{k \in \mathbb{N}}$ be a bounded sequence in $B_{\log}$ converging to zero uniformly on compact subsets of $D$ as $k \to \infty$. We denote $L := \sup_{k \in \mathbb{N}} \|f_{k}\|_{B_{\log}} < \infty$. By Cauchy’s integral formula, we obtain that

$$\|D^{m}_{\varphi,u}f_{k}\|_{B_{\mu}} = \|(D^{m}_{\varphi,u}f_{k})(0)\| + \|D^{m}_{\varphi,u}f_{k}\|_{\mu}$$

$$= |u(0)||f^{(m)}_{k}(\varphi(0))| + \sup_{z \in D} \mu(z)|u'(z)f^{(m)}_{k}(\varphi(z)) + u(z)f^{(m+1)}_{k}(\varphi(z))\varphi'(z)|$$

$$\leq |u(0)||f^{(m)}_{k}(\varphi(0))| + \sup_{z \in D} \mu(z)|u'(z)f^{(m)}_{k}(\varphi(z))|$$

$$+ \sup_{z \in D} \mu(z)|u(z)\varphi'(z)||f^{(m+1)}_{k}(\varphi(z))|$$

$$\leq |u(0)||f^{(m)}_{k}(\varphi(0))| + L_{1}\sup_{z \in D} |f^{(m)}_{k}(\varphi(z))| + L_{2}\sup_{z \in D} |f^{(m+1)}_{k}(\varphi(z))|$$

$$\leq |u(0)||f^{(m)}_{k}(\varphi(0))| + L_{1}\sup_{|w| \leq r} |f^{(m)}_{k}(w)| + L_{2}\sup_{|w| \leq r} |f^{(m+1)}_{k}(w)|$$

$$\to 0, \quad \text{as} \quad k \to \infty.$$

Employing lemma 4.2, it is clear that the operator $D^{m}_{\varphi,u} : B_{\log} \to B_{\mu}$ is compact. Thus

$$\|D^{m}_{\varphi,u}\|_{c} = 0. \quad (18)$$
On the other hand, by (3) we obtain that
\[
A = \limsup_{j \to \infty} \frac{j^m |J_u \varphi|^\mu}{\|z\|^\log(j+1)} = \limsup_{j \to \infty} \frac{j^m |J_u \varphi|^\mu}{\log(j+1)}
\]
\[
= \limsup_{j \to \infty} \frac{j^m}{\log(j+1)} \sup_{z \in B} \mu(z) |u'(z) \varphi'(z)|
\]
\[
\leq L_1 \limsup_{j \to \infty} \frac{j^m}{\log(j+1)} |f|^j = 0.
\]
(19)

\[
B = \limsup_{j \to \infty} \frac{j^m |I_u \varphi|^\mu}{\|z\|^\log(j+1)} = \limsup_{j \to \infty} \frac{j^m |I_u \varphi|^\mu}{\log(j+1)}
\]
\[
= \limsup_{j \to \infty} \frac{j^m}{\log(j+1)} \sup_{z \in B} \mu(z) (j+1) |\varphi(z)|^j |\varphi'(z)| |u(z)|
\]
\[
\leq L_2 \limsup_{j \to \infty} \frac{j^m (j+1)}{\log(j+1)} |f|^j = 0.
\]
(20)

From (18)-(20), both sides of (17) are zero. Therefore the essential norm formula is true in this case. In the sequel, we assume that \( \sup \{ |\varphi(z)| \} = 1 \).

The upper bounded. By the boundedness of the operator \( D_{\varphi,u}^m : B_{\log} \to B_\mu \) and lemma 4.2, we know the operator \( D_{\varphi,u}^m K_{r_j} : B_{\log} \to B_\mu \) is compact. Then

\[
\| D_{\varphi,u}^m e \| \leq \limsup_{j \to \infty} \| D_{\varphi,u}^m - D_{\varphi,u}^m K_{r_j} \|
\]
\[
= \limsup_{j \to \infty} \sup_{\|f\|_{B_{\log}} \leq 1} \| (D_{\varphi,u}^m - D_{\varphi,u}^m K_{r_j}) f \|_{B_\mu}
\]
\[
\leq \limsup_{j \to \infty} \sup_{\|f\|_{B_{\log}} \leq 1} \| (D_{\varphi,u}^m - D_{\varphi,u}^m K_{r_j}) f(0) \|
\]
\[
+ \limsup_{j \to \infty} \sup_{\|f\|_{B_{\log}} \leq 1} \sup_{z \in B} \mu(z) \| (D_{\varphi,u}^m - D_{\varphi,u}^m K_{r_j})' f(z) \|
\]
\[
\leq \limsup_{j \to \infty} \sup_{\|f\|_{B_{\log}} \leq 1} |u(0)f^m(\varphi(0)) - r_j^m u(0)f^m(r_j\varphi(0))|
\]
\[
+ \limsup_{j \to \infty} \sup_{\|f\|_{B_{\log}} \leq 1} \sup_{z \in B} \mu(z) |u'(z)||f^m(\varphi(z)) - r_j^m f^m(r_j\varphi(z))|
\]
\[
+ \limsup_{j \to \infty} \sup_{\|f\|_{B_{\log}} \leq 1} \sup_{z \in B} \mu(z) |u(z)\varphi'(z)||f^{m+1}(\varphi(z)) - r_j^{m+1} f^{m+1}(r_j\varphi(z))|
\]
\[
\leq \limsup_{j \to \infty} \sup_{\|f\|_{B_{\log}} \leq 1} |u(0)||f^m(\varphi(0)) - r_j^m f^m(r_j\varphi(0))|
\]
\[
+ J_1 + J_2.
\]
(21)

On the one hand, similar to [2, Lemma 5.3], by \( r_j \to 1 \) as \( j \to \infty \) and Cauchy’s integral formula, it follows that

\[
\limsup_{j \to \infty} \sup_{\|f\|_{B_{\log}} \leq 1} |u(0)||f^m(\varphi(0)) - r_j^m f^m(r_j\varphi(0))| = 0.
\]

On the other hand, for each positive integer \( j \in \mathbb{N} \), denote the set \( A_{\varphi}^j = \{ z \in \mathbb{D} : r_j \leq |\varphi(z)| < r_{j+1} \} \). Since \( \sup \{ |\varphi(z)| \} = 1 \), then there is a \( k \in \mathbb{N} \) such that \( A_{\varphi}^k \neq \emptyset \) and \( A_{\varphi}^j \) is not empty for every integer \( j \geq k \).

Hence \( \mathbb{D} = \bigcup_{j=k}^{\infty} A_{\varphi}^j \). By (2), for an arbitrary \( \epsilon > 0 \), there is \( N_1 \in \mathbb{N} \) such that for any integer \( j \geq N_1 \),

\[
\frac{j^m}{\log^j \frac{j+1}{m+L}} \min_{x \in [r_j, r_{j+1}]} x^j (1-x)^m \log \frac{3}{1-x} > \frac{(m-1+L)^m}{e^{m-1+L}} - \epsilon.
\]
(22)
Further by
\[
\lim_{j \to \infty} \frac{j^{m+1}}{\log \frac{j+1}{m+L}} \min_{x \in [r_j, r_{j+1}]} x^j(1-x)^{m+1} \log \frac{3}{1-x} = \frac{(m + L)^{m+1}}{e^{m+L}},
\]
there is $N_2 \in \mathbb{N}$ such that for any integer $j \geq N_2$,
\[
\frac{j^{m+1}}{\log \frac{j+1}{m+L}} \min_{x \in [r_j, r_{j+1}]} x^j(1-x)^{m+1} \log \frac{3}{1-x} > \frac{(m + L)^{m+1}}{e^{m+L}} - \epsilon. \tag{23}
\]

In the following, denote $N = \max\{N_1, N_2\}$. In fact (22) is used to prove $J_1$ and (23) for $J_2$. We divide $J_1$ into two parts

\[
J_{1,1} = \limsup_{j \to \infty} \sup_{\|f\|_{B_{m,k} \leq 1}} \sup_{1 \leq j \leq N} \sup_{z \in A'_j} \mu(z)|u'(z)|f^{(m)}(\varphi(z)) - r_j^m f^{(m)}(r_j \varphi(z));
\]
\[
J_{1,2} = \limsup_{j \to \infty} \sup_{\|f\|_{B_{m,k} \leq 1}} \sup_{N+1 \leq j \leq N} \sup_{z \in A'_j} \mu(z)|u'(z)||f^{(m)} (\varphi(z)) - r_j^m f^{(m)}(r_j \varphi(z))|.
\]

It is clear that
\[
J_{1,1} \leq L_1 \limsup_{j \to \infty} \sup_{\|f\|_{B_{m,k} \leq 1}} \sup_{r_k \leq |\varphi(z)| \leq r_{N+1}} |f^{(m)}(\varphi(z)) - r_j^m f^{(m)}(r_j \varphi(z))| = 0.
\]

On the other hand,
\[
J_{1,2} \leq \limsup_{j \to \infty} \sup_{\|f\|_{B_{m,k} \leq 1}} \sup_{N+1 \leq j \leq N} \sup_{z \in A'_j} \mu(z)|u'(z)||f^{(m)}(\varphi(z))| + |r_j^m f^{(m)}(r_j \varphi(z))|)
\]
\[
\leq \limsup_{j \to \infty} \sup_{\|f\|_{B_{m,k} \leq 1}} \sup_{N+1 \leq j \leq N} \sup_{z \in A'_j} \mu(z)|u'(z)||f^{(m)}(\varphi(z))| + |f^{(m)}(r_j \varphi(z))|).
\]

Denote the expression $s(\rho)$ for $\rho \in (0, 1)$,
\[
s(\rho) = \sup_{N+1 \leq j \leq N} \sup_{z \in A'_j} \mu(z)|u'(z)||f^{(m)}(\varphi(z))|$
\[
= \sup_{N+1 \leq j \leq N} \sup_{z \in A'_j} \mu(z)|u'(z)||f^{(m)}(\varphi(z))|(1 - |\varphi(z)|)^m \log \frac{3}{1-|\varphi(z)|}
\]
\[
\leq \sup_{N+1 \leq j \leq N} \sup_{z \in A'_j} \mu(z)|u'(z)| |(1 - |\varphi(z)|)^m \log \frac{3}{1-|\varphi(z)|}
\]
\[
\leq \sup_{N+1 \leq j \leq N} \sup_{z \in A'_j} \mu(z)|u'(z)| |(1 - |\varphi(z)|)^m \log \frac{3}{1-|\varphi(z)|}
\]
\[
= \sup_{N+1 \leq j \leq N} \sup_{z \in A'_j} \mu(z)|u'(z)| |(1 - |\varphi(z)|)^m \log \frac{3}{1-|\varphi(z)|}
\]
\[
\leq \sup_{N+1 \leq j \leq N} \sup_{z \in A'_j} \mu(z)|u'(z)| |(1 - |\varphi(z)|)^m \log \frac{3}{1-|\varphi(z)|}
\]
\[
\leq \frac{1}{(m+1)^{m+1}} \sup_{N+1 \leq j} \sup_{|z'| \leq |z|} |f^{(m)}(z')|_{\mu} |(z'|z|^m |\varphi(z')|^m \log \frac{3}{1-|\varphi(z')|}
\]
\[
\leq \frac{1}{(m+1)^{m+1}} \sup_{N+1 \leq j} \sup_{|z'| \leq |z|} |f^{(m)}(z')|_{\mu} |(z'|z|^m |\varphi(z')|^m \log \frac{3}{1-|\varphi(z')|}
\]
In the above chain of relations, the first inequality follows from the definition of the norm \( \| \cdot \|_{\text{log}} \) and lemma 2.3. The second inequality is due to the fact that the function \((1 - x)^{m+1}\log \frac{3}{1-x}\) is decreasing on \([0, 1]\). The third and fourth inequalities follow from (4) and (22), respectively. Since \( \epsilon \) is arbitrary, then

\[
J_{1,2} \leq \limsup_{j \to \infty} \sup_{\|f\|_{\text{log}} \leq 1} (s(1) + s(r_j)) \\
\leq \limsup_{j \to \infty} \frac{j^m \| J_u \varphi_j \|_\mu}{\| z^j \|_{\text{log}}}. 
\]

The above inequalities imply that

\[
J_1 \leq \limsup_{j \to \infty} \frac{j^m \| J_u \varphi_j \|_\mu}{\| z^j \|_{\text{log}}}. 
\tag{24}
\]

Using the similar way by replacing (22) with (23), we can obtain that

\[
J_2 = \limsup_{j \to \infty} \sup_{\|f\|_{\text{log}} \leq 1} \sup_{z \in \mathbb{D}} \mu(z) |a(z)\varphi'(z)||f^{(m+1)}(\varphi(z)) - f_j^{(m+1)}(\varphi(z))| \\
\leq \limsup_{j \to \infty} \frac{j^m \| J_u \varphi_j^{(m+1)} \|_\mu}{\| z^j \|_{\text{log}}}. 
\tag{25}
\]

Combining (21), (24) and (25), it follows the upper bounded.

The lower bounded. The assumption \( \sup_{z \in \mathbb{D}} |\varphi(z)| = 1 \) holds. For every compact operator \( K : B_{\text{log}} \to B_\mu \) and each sequence \( \{f_k\}_{k \in \mathbb{N}} \subset B_{\text{log}} \) with \( \sup_{k \in \mathbb{N}} \|f_k\|_{B_{\text{log}}} < \infty \), and \( f_k \) converging to zero on the compact subsets of \( \mathbb{D} \), it follows that \( \lim_{k \to \infty} \|Kf_k\|_{B_\mu} = 0 \). Thus

\[
\|D_{\varphi,u}^m - K\| \geq \limsup_{k \to \infty} \| (D_{\varphi,u}^m - K)f_k \|_{B_\mu} \geq \limsup_{k \to \infty} \| D_{\varphi,u}^m (f_k) \|_{B_\mu},
\]

that is, \( \|D_{\varphi,u}^m\|_{\infty} \geq \limsup_{k \to \infty} \| D_{\varphi,u}^m (f_k) \|_{B_\mu} \).

Choosing a sequence \( \{a_k\}_{k \in \mathbb{N}} \subset \mathbb{D} \) such that \( |\varphi(a_k)| \to 1 \) as \( k \to \infty \), we define the function sequences \( \{f_{1,k}^{(m)}\}_{k \in \mathbb{N}} \) and \( \{f_{2,k}^{(m)}\}_{k \in \mathbb{N}} \), respectively.

\[
f_{1,k}^{(m)}(z) = \frac{(m + 2)(1 - |\varphi(a_k)|^2)^2}{(1 - \varphi(a_k)z)^{m+1} \log \frac{3}{1 - |\varphi(a_k)|z}} - \frac{(m + 2)(1 - |\varphi(a_k)|^2)^2}{(1 - \varphi(a_k)z)^{m+2} \log \frac{3}{1 - |\varphi(a_k)|z}}; \\
f_{2,k}^{(m)}(z) = \frac{(m + 2)(1 - |\varphi(a_k)|^2)^2}{(1 - \varphi(a_k)z)^{m+1} \log \frac{3}{1 - |\varphi(a_k)|z}} - \frac{(m + 1)(1 - |\varphi(a_k)|^2)^2}{(1 - \varphi(a_k)z)^{m+2} \log \frac{3}{1 - |\varphi(a_k)|z}}.
\]

Clearly \( \{f_{1,k}\}_{k \in \mathbb{N}} \subset B_{\text{log}} \) and \( \{f_{2,k}\}_{k \in \mathbb{N}} \subset B_{\text{log}} \); moreover, both sequences converge to zero uniformly on compact subsets of \( \mathbb{D} \) as \( k \to \infty \). Since \( f_{1,k}^{(m)}(\varphi(a_k)) = 0 \) and

\[
f_{1,k}^{(m+1)}(\varphi(a_k)) = \frac{(m + 1)|\varphi(a_k)|}{(1 - |\varphi(a_k)|^2)^{m+1} \log \frac{3}{1 - |\varphi(a_k)|^2}}, \\
f_{2,k}^{(m)}(\varphi(a_k)) = \frac{1}{(1 - |\varphi(a_k)|^2)^m \log \frac{3}{1 - |\varphi(a_k)|^2}}; \\
f_{2,k}^{(m+1)}(\varphi(a_k)) = \frac{\varphi(a_k)}{(1 - |\varphi(a_k)|^2)^m \log \frac{3}{1 - |\varphi(a_k)|^2}}.
\]
we have that

\[ \|D_{\varphi,u}^{m}\|_{e} \geq \limsup_{k \to \infty} \|D_{\varphi,u}^{m}(f_{1,k})\|_{\mathcal{S}_{\mu}} \geq \limsup_{k \to \infty} \|D_{\varphi,u}^{m}(f_{1,k})\|_{\mu} \]

\[ = \limsup_{k \to \infty} \sup_{z \in \mathbb{D}} \mu(z)|u'(z)f_{1,k}^{(m)}(\varphi(z)) + u(z)\varphi'(z)f_{1,k}^{(m+1)}(\varphi(z))| \]

\[ \geq \limsup_{k \to \infty} \mu(a_k)|u'(a_k)f_{1,k}^{(m)}(\varphi(a_k)) + u(a_k)\varphi'(a_k)f_{1,k}^{(m+1)}(\varphi(a_k))| \]

\[ = \limsup_{k \to \infty} \frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1 - |\varphi(a_k)|^2)^m} (m + 1)|\varphi(a_k)| \]

\[ \geq \limsup_{k \to \infty} \frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1 - |\varphi(a_k)|^2)^m} \log \frac{3}{1 - |\varphi(a_k)|^2} \]

\[ = \limsup_{|\varphi(z)| \to 1} \frac{\mu(z)|u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^m} \log \frac{3}{1 - |\varphi(z)|^2} \quad \text{(26)} \]

and

\[ \|D_{\varphi,u}^{m}\|_{e} \geq \limsup_{k \to \infty} \|D_{\varphi,u}^{m}(f_{2,k})\|_{\mathcal{S}_{\mu}} \geq \limsup_{k \to \infty} \|D_{\varphi,u}^{m}(f_{2,k})\|_{\mu} \]

\[ = \limsup_{k \to \infty} \sup_{z \in \mathbb{D}} \mu(z)|u'(z)f_{2,k}^{(m)}(\varphi(z)) + u(z)\varphi'(z)f_{2,k}^{(m+1)}(\varphi(z))| \]

\[ \geq \limsup_{k \to \infty} \mu(a_k)|u'(a_k)f_{2,k}^{(m)}(\varphi(a_k)) + u(a_k)\varphi'(a_k)f_{2,k}^{(m+1)}(\varphi(a_k))| \]

\[ = \limsup_{k \to \infty} \frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1 - |\varphi(a_k)|^2)^m} (m + 1)|\varphi(a_k)| \]

\[ \geq \limsup_{k \to \infty} \frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1 - |\varphi(a_k)|^2)^m} \log \frac{3}{1 - |\varphi(a_k)|^2} \]

\[ = \limsup_{k \to \infty} \frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1 - |\varphi(a_k)|^2)^m} \log \frac{3}{1 - |\varphi(a_k)|^2} \quad \text{(27)} \]

Since

\[ \log \frac{3}{1 - |\varphi(a_k)|^2} \leq \left( \log \frac{3}{1 - |\varphi(a_k)|^2} \right)^2 , \]

then

\[ \limsup_{k \to \infty} \frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1 - |\varphi(a_k)|^2)^m} + \|D_{\varphi,u}^{m}\|_{e} \]

\[ \geq \limsup_{k \to \infty} \frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1 - |\varphi(a_k)|^2)^m} \log \frac{3}{1 - |\varphi(a_k)|^2} + \|D_{\varphi,u}^{m}\|_{e} \]

\[ \geq \limsup_{k \to \infty} \frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1 - |\varphi(a_k)|^2)^m} \log \frac{3}{1 - |\varphi(a_k)|^2} . \quad \text{(28)} \]
Combining (26) and (28), it follows that

$$\|D^m_{\varphi,u}\|_c \geq \limsup_{k \to \infty} \frac{u(a_k)|u'(a_k)|}{(1 - |\varphi(a_k)|^2)^m \log \frac{\frac{3}{1 - |\varphi(a_k)|^2}}{e}}$$

$$= \limsup_{|\varphi(z)| \to 1} \frac{u(z)|u'(z)|}{(1 - |\varphi(z)|^2)^m \log \frac{\frac{3}{1 - |\varphi(z)|^2}}{e}}. \quad (29)$$

For $s \in (0, 1)$, by (3), we consider

$$\frac{j^m \|J_{u}\varphi^j\|_\mu}{\|x\|^j \log(j + 1)} \leq \frac{j^m \|z\|^j \| \varphi(z) \|^j}{\log(j + 1)} \sup_{z \in \mathbb{D}} \mu(z)|u'(z)||\varphi(z)|^j$$

$$\leq \frac{j^m \sup_{|\varphi(z)| > s} \mu(z)|u'(z)||\varphi(z)|^j}{\log(j + 1)} + \frac{j^m \sup_{|\varphi(z)| \leq s} \mu(z)|u'(z)||\varphi(z)|^j}{\log(j + 1)}$$

$$\leq L_1 \frac{j^m s^j}{\log(j + 1)} \sup_{|\varphi(z)| > s} \mu(z)|u'(z)||\varphi(z)|^j.$$ 

By lemma 2.5, the above inequalities imply that

$$\frac{j^m \sup_{|\varphi(z)| > s} \mu(z)|u'(z)||\varphi(z)|^j}{\log(j + 1)}$$

$$= \frac{j^m \sup_{|\varphi(z)| > s} \mu(z)|u'(z)||\varphi(z)|^j}{\log(j + 1)} \frac{|\varphi(z)|^j(1 - |\varphi(z)|)^m \log \frac{\frac{3}{1 - |\varphi(z)|}}{e}}{(1 - |\varphi(z)|)^m \log \frac{\frac{3}{1 - |\varphi(z)|}}{e}}$$

$$\leq L_1 \frac{j^m s^j}{\log(j + 1)} f_j(x_j) \sup_{|\varphi(z)| > s} \mu(z)|u'(z)| \frac{1}{(1 - |\varphi(z)|)^m \log \frac{\frac{3}{1 - |\varphi(z)|}}{e}}.$$ 

Thus by (7) we obtain that

$$\limsup_{j \to \infty} \frac{j^m \|J_{u}\varphi^j\|_\mu}{\|x\|^j \log(j + 1)} \leq \frac{L_1 \frac{j^m s^j}{\log(j + 1)}}{\sup_{j \to \infty} \frac{j^m s^j f_j(x_j)}{(1 - |\varphi(z)|)^m \log \frac{\frac{3}{1 - |\varphi(z)|}}{e}}}$$

$$= \frac{m^m \sup_{|\varphi(z)| > s} \mu(z)|u'(z)|}{\|x\|^j \log \frac{\frac{3}{1 - |\varphi(z)|}}{e}}.$$ 

Further we have that

$$\limsup_{j \to \infty} \frac{j^m \|J_{u}\varphi^j\|_\mu}{\|x\|^j \log(j + 1)} \leq \limsup_{s \to 1} \sup_{|\varphi(z)| > s} \mu(z)|u'(z)| \frac{(1 - |\varphi(z)|)^m \log \frac{\frac{3}{1 - |\varphi(z)|}}{e}}{(1 - |\varphi(z)|)^m \log \frac{\frac{3}{1 - |\varphi(z)|}}{e}}$$

$$\leq \limsup_{|\varphi(z)| \to 1} \frac{\mu(z)|u'(z)|}{(1 - |\varphi(z)|)^m \log \frac{\frac{3}{1 - |\varphi(z)|}}{e}} \frac{3}{1 - |\varphi(z)|}.$$ 

Combining (29) and (30), it’s clear that

$$\limsup_{j \to \infty} \frac{j^m \|J_{u}\varphi^j\|_\mu}{\|x\|^j \log(j + 1)} \leq \|D^m_{\varphi,u}\|_c.$$
Similarly, 
\[
\frac{j^m \| I_u \varphi^{j+1} \|_\mu}{\| z^j \|_{\log}} \leq \frac{j^m}{\log(j+1)} \sup_{z \in \mathcal{B}} (j+1) \mu(z) |\varphi(z)|^2 |u(z)\varphi'(z)|
\]
\[
\leq \frac{\log(j+1)}{\log(j+1)} \sup_{z \in \mathcal{B}} (j+1) \mu(z) |\varphi(z)|^2 |u(z)\varphi'(z)|
\]
\[
\leq L_2 \frac{j^m}{\log(j+1)} \sup_{|\varphi(z)| > s} |\varphi(z)|^2 + \frac{j^m}{\log(j+1)} \sup_{|\varphi(z)| > s} (j+1) \mu(z) |\varphi(z)|^2 |u(z)\varphi'(z)|
\]
\[
\leq L_2 \frac{j^m}{\log(j+1)} s^j + \frac{j^m}{\log(j+1)} \sup_{|\varphi(z)| > s} (j+1) \mu(z) |\varphi(z)|^2 |u(z)\varphi'(z)|.
\]
By lemma 2.5, the above inequalities imply that 
\[
\frac{j^m}{\log(j+1)} \sup_{|\varphi(z)| > s} \mu(z) |\varphi(z)|^2 |u(z)\varphi'(z)|
\]
\[
= \frac{j^m}{\log(j+1)} \sup_{|\varphi(z)| > s} \mu(z) |u(z)\varphi'(z)| \frac{|\varphi(z)|^2 (1 - |\varphi(z)|)^{m+1} \log 3}{1 - |\varphi(z)|}
\]
\[
\leq \frac{j^m}{\log(j+1)} s^j (1 - x_j)^{m+1} \log 3 \sup_{|\varphi(z)| > s} \frac{\mu(z) |u(z)\varphi'(z)|}{(1 - |\varphi(z)|)^{m+1} \log 3}.
\]
Then by (7), we obtain that 
\[
\limsup_{j \to \infty} \frac{j^m \| I_u \varphi^{j+1} \|_\mu}{\| z^j \|_{\log}} \leq \limsup_{j \to \infty} L_2 \frac{j^m}{\log(j+1)} s^j
\]
\[
+ \limsup_{j \to \infty} \frac{j^m}{\log(j+1)} s^j (1 - x_j)^{m+1} \log 3 \sup_{|\varphi(z)| > s} \frac{\mu(z) |u(z)\varphi'(z)|}{(1 - |\varphi(z)|)^{m+1} \log 3}
\]
\[
= \frac{(m+1)^{m+1}}{e^{m+1}} \sup_{|\varphi(z)| > s} \frac{\mu(z) |u(z)\varphi'(z)|}{(1 - |\varphi(z)|)^{m+1} \log 3}.
\]
Further we get that 
\[
\limsup_{j \to \infty} \frac{j^m \| I_u \varphi^{j+1} \|_\mu}{\| z^j \|_{\log}} \leq \limsup_{s \to 1} \frac{\mu(z) |u(z)\varphi'(z)|}{(1 - |\varphi(z)|)^{m+1} \log 3}
\]
\[
\leq \limsup_{|\varphi(z)| \to 1} \frac{\mu(z) |u(z)\varphi'(z)|}{(1 - |\varphi(z)|)^{m+1} \log 3}
\]
\[
\leq \limsup_{|\varphi(z)| \to 1} \frac{\mu(z) |u(z)\varphi'(z)|}{(1 - |\varphi(z)|)^{m+1} \log 3}.
\]
Combining (27) and (31), it follows that 
\[
\limsup_{j \to \infty} \frac{j^m \| I_u \varphi^{j+1} \|_\mu}{\| z^j \|_{\log}} \leq \| D_{\varphi,u}^m \|_\epsilon.
\]
This completes the proof. □
\[\limsup_{j \to \infty} \frac{j^m \|Ju \varphi_j\|_\mu}{\|z^j\|_\log} = 0 \quad \text{and} \quad \limsup_{j \to \infty} \frac{j^m \|Ju \varphi_{j+1}\|_\mu}{\|z^j\|_\log} = 0.\]

\[\limsup_{j \to \infty} \frac{j^m \|Ju \varphi_j\|_\mu}{\log(j+1)} = 0 \quad \text{and} \quad \limsup_{j \to \infty} \frac{j^m \|Ju \varphi_{j+1}\|_\mu}{\log(j+1)} = 0.\]

References