Weighted differentiation composition operator from logarithmic Bloch spaces to Bloch-type spaces

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In this paper, we give a new characterization for the boundedness of the weighted differentiation composition operator from logarithmic Bloch spaces to Bloch-type spaces and calculate its essential norm in terms of the n-th power of the induced analytic self-map on the unit disk. From which a sufficient and necessary condition of compactness of the operator follows immediately.

1 Introduction

Denote $H(\mathbb{D})$ the space of all holomorphic functions on the unit disk $\mathbb{D}$ and $S(\mathbb{D})$ the set of all self-maps on $\mathbb{D}$. Throughout this paper, $\log$ denotes the natural logarithm function. Given a bounded, continuous and strictly positive function μ on $\mathbb{D}$, we define the μ-Bloch space $\mathcal{B}_\mu = \mathcal{B}_\mu(\mathbb{D})$, consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_\mu = \sup_{z\in\mathbb{D}} \mu(z)|f'(z)| < \infty.$$

The space $\mathcal{B}_\mu$ is a Banach space under the norm

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + \|f\|_\mu.$$

For $\alpha > 0$ and $\mu : \mathbb{D} \to (0, 1)$ is defined by $\mu(z) = (1 - |z|^2)^\alpha$, then in this case, $\mathcal{B}_\mu$ is denoted by $\mathcal{B}_\alpha$, the so-called α-Bloch space on $\mathbb{D}$. when $\alpha = 1$, $\mathcal{B}_\alpha$ is the classical Bloch space $\mathcal{B}$. Moreover, let $\mu = v_{\log} : \mathbb{D} \to (0, \infty)$ be given by

$$v_{\log}(z) = (1 - |z|) \log\left(\frac{3}{1 - |z|}\right),$$

then we obtain the log-Bloch space and denote $\mathcal{B}_{v_{\log}}(\mathbb{D})$ by $\mathcal{B}_{\log}$. It is well-known that the log-Bloch space $\mathcal{B}_{\log}$ is a Banach space endowed with the norm

$$\|f\|_{\mathcal{B}_{\log}} = |f(0)| + \|f\|_{\log},$$

where

$$\|f\|_{\log} = \sup_{z\in\mathbb{D}}(1 - |z|) \log\left(\frac{3}{1 - |z|}\right) |f'(z)|.$$

We refer the readers to the book [22] by K. H. Zhu, which is excellent source for the development of the theory of function spaces.

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For $\varphi \in S(\mathbb{D})$, the composition operator C_φ is defined by

$$C_\varphi(f) = f \circ \varphi,\ f \in H(\mathbb{D}).$$

It is an interesting topic to provide function-theoretic characterizations of when φ induces a bounded or compact composition operator on various holomorphic function spaces. For this theory, we refer the readers to the books [4] by Cowen and MacCluer, and [14] by Shapiro.

As we all know the differentiation operator is defined as $Df = f'$ for $f \in H(\mathbb{D})$. For $u \in H(\mathbb{D})$, the weighted composition operator uC_φ is given by

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)),\quad f \in H(\mathbb{D}).$$

Now let $m \in \mathbb{N}$, the *weighted differentiation composition operator*, denoted by $D^m_{\varphi,u}$, is defined as follows

$$(D^m_{\varphi,u}f)(z) = u(z).f^{(m)}(\varphi(z)),\quad f \in H(\mathbb{D}).$$

When $m = 0$, then the operator $D^m_{\varphi,u}$ becomes the weighted composition operator uC_φ. That's the reason why we call $D^m_{\varphi,u}$ the *weighted differentiation composition operator*. If $m = 0$ and $u(z) = 1$, then $D^m_{\varphi,u} = C_\varphi$. If $m = 1$ and $u(z) = 1$, then $D^m_{\varphi,u} = C_\varphi D$. If $m = 1$ and $u(z) = \varphi'(z)$, then $D^m_{\varphi,u} = DC_\varphi$. For the operator $D^m_{\varphi,u}$, we refer the readers to the papers [15, 16].

The *essential norm* of a continuous linear operator $T : X \to Y$ is the distance from T to the set of all compact operators, that is,

$$\|T\|_e = \inf\{\|T - K\| :\ K \text{ is compact }\}.$$

Since $\|T\|_e = 0$ if and only if T is compact, so the estimates on $\|T\|_e$ lead to conditions for T to be compact.

Recently there has been a great interest in the new characterizations for the essential norms of composition operator and differentiation operator between Bloch-type spaces on the unit disk. In papers [10] and [11], we respectively studied the new characterizations for the operators $C_\varphi D^m : \mathcal{B}_\alpha \to \mathcal{B}_\beta$ and $DC_\varphi : \mathcal{B}_\alpha \to \mathcal{B}_\beta$. Concerning the composition operator from the log-Bloch space to μ-Bloch space, we refer the readers to the papers [2, 5]. Moreover, the papers [6–9, 13, 19–21] are also about the new subject, which are helpful for our study. Based on the above foundations, the goal of this paper is to give the new characterizations for the *weighted differentiation operator* $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ on the unit disk. In section 2, we list some lemmas. The characterizations for the boundedness and essential norm of $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ are given in section 3 and section 4, respectively.

Throughout the remainder of this paper, the notations $A \asymp B,\ A \preceq B,\ A \succeq B$ mean that there maybe different positive constants C such that $B/C \le A \le CB,\ A \le CB,\ CB \le A$.

2 Some lemmas

Let $L = 1 - \frac{1}{\log 3} \in (0, 1)$ in this paper. In this section, we list some auxiliary facts. We define a sequence $(r_j)_{j\in\mathbb{N}}$ by $r_0 = 0$ and $r_j = 1 - \frac{m-1+L}{j+m-1+L}$ for each $j \in \mathbb{N}$. The sequence $(r_j)_{j\in\mathbb{N}}$ lies in $[0, 1)$ is strictly increasing and satisfies $r_j \to 1^-$ as $j \to \infty$. For $\varphi \in S(\mathbb{D})$ and $j \in \mathbb{N}$, we define the set

$$A^j_\varphi = \{z \in \mathbb{D} :\ r_j \le |\varphi(z)| < r_{j+1}\}.$$

It is obvious that $A^j_\varphi \cap A^k_\varphi = \emptyset$ for $j \ne k$.

Lemma 2.1 *Define* $A : [1, \infty) \to (0, 1]$ *by*

$$A(x) = \left(\frac{x+1}{x+m+L}\right)^x.$$

Then we have that

$$\inf_{x\ge1} A(x) = \lim_{x\to\infty} A(x) = e^{-(m-1+L)}.$$

Proof.

$$\begin{aligned}\lim_{x\to\infty} A(x) &= \lim_{x\to\infty}\left(\frac{x+1}{x+m+L}\right)^x\\ &= \lim_{x\to\infty}\left(1+\frac{-(m-1+L)}{x+m+L}\right)^{\frac{x+m+L}{-(m-1+L)}\frac{-x(m-1+L)}{x+m+L}}\\ &= e^{-(m-1+L)}.\\ (\log A(x))' &= \log\frac{x+1}{x+m+L}+\frac{x(m+L-1)}{(x+1)(x+m+L)}\\ &\le \log\frac{x+1}{x+m+L}+\frac{m+L-1}{x+m+L}\\ &= \log\frac{x+1}{x+m+L}-\frac{x+1}{x+m+L}+1.\end{aligned}$$

The above inequality is negative, due to the function $\log\eta-\eta+1$ takes values $-\infty$ and 0 at $\eta=0$ and $\eta=1$, respectively and is strictly increasing in $\eta\in(0,1)$. Thus $\log A$ is strictly decreasing, in turn the function A is strictly decreasing on $[1,\infty)$. Hence it follows that $\inf_{x\ge1} A(x)=\lim_{x\to\infty} A(x)=e^{-(m-1+L)}$. This ends the proof. □

Lemma 2.2 *Let* $m,\ j\in\mathbb{N}$, *then the function*

$$f_j(x)=x^j(1-x)^m\log\frac{3}{1-x},\quad x\in(0,1),$$

is decreasing on $[r_j,1)$. *Also, we have that*

$$\frac{j^m}{\log\frac{j+1}{m-1+L}}f_j(x)\ge\frac{L^m}{3^m e^{m-1+L}}\quad \textit{for all}\ \ x\in[r_j,r_{j+1}], \tag{1}$$

$$\lim_{j\to\infty}\frac{j^m}{\log\frac{j+1}{m-1+L}}\min_{x\in[r_j,r_{j+1}]} f_j(x)=\frac{(m-1+L)^m}{e^{m-1+L}}. \tag{2}$$

Proof. It suffices to show that f_j is decreasing on $[r_j,r_{j+1}]$. Since

$$f_j'(x)=x^j(1-x)^{m-1}\left((j(1-x)-mx)\log\frac{3}{1-x}+x\right),$$

and $L\in(0,1)$, then we have $j-(j+m)x<0$ holds for all $x\in[r_j,1)$. By the fact $\log\frac{3}{1-x}\ge\log3$ for all $x\in(0,1)$, it follows that

$$f_j'(x)\le x^{j-1}(1-x)^{m-1}\left((j-(j+m)x)\log3+x\right).$$

Since the function

$$h_j(x)=(j-(j+m)x)\log3+x=j\log3-((j+m)\log3-1)x$$

is decreasing for $x\in[r_j,1)$ and $h_j(r_j)=0$, thus $f_j'(x)<0$ for all $x\in(r_j,1)$. That is, f_j is decreasing on $[r_j,1)$. On the other hand, by the first statement in this lemma and lemma 2.1, we get that for each $j\in\mathbb{N}$ and all

$x \in [r_j, r_{j+1}]$,

$$\begin{aligned} & \frac{j^m}{\log \frac{j+1}{m-1+L}} f_j(x) \geq \frac{j^m}{\log \frac{j+1}{m-1+L}} f_j(r_{j+1}) \\ = & \frac{j^m}{\log \frac{j+1}{m-1+L}} \left(1 - \frac{m-1+L}{j+m+L}\right)^j \left(\frac{m-1+L}{j+m+L}\right)^m \log \frac{3(j+m+L)}{m-1+L} \\ = & \frac{\log \frac{3(j+m+L)}{m-1+L}}{\log \frac{j+1}{m-1+L}} \left(\frac{j+1}{j+m+L}\right)^j \left(\frac{(m-1+L)j}{j+m+L}\right)^m \\ \geq & 1 \cdot e^{-(m-1+L)} \left(\frac{m-1+L}{m+1+L}\right)^m \\ \geq & \frac{L^m}{3^m e^{m-1+L}}. \end{aligned}$$

Moreover, we have that

$$\begin{aligned} & \lim_{j\to\infty} \frac{j^m}{\log \frac{j+1}{m-1+L}} \min_{x\in[r_j, r_{j+1}]} f_j(x) = \lim_{j\to\infty} \frac{j^m}{\log \frac{j+1}{m-1+L}} f_j(r_{j+1}) \\ & = \lim_{j\to\infty} \frac{\log \frac{3(j+m+L)}{m-1+L}}{\log \frac{j+1}{m-1+L}} \left(\frac{j+1}{j+m+L}\right)^j \left(\frac{(m-1+L)j}{j+m+L}\right)^m \\ & = \frac{(m-1+L)^m}{e^{m-1+L}}. \end{aligned}$$

This completes the proof. □

A positive continuous function v on $[0, 1)$ is called *normal* (see, e.g. [18]), if there exist three positive constants $0 \leq \delta < 1$, and $0 < a < b < \infty$, such that for $r \in [\delta, 1)$

$$\frac{v(r)}{(1-r)^a} \downarrow 0, \quad \frac{v(r)}{(1-r)^b} \uparrow \infty \quad \text{as} \quad r \to 1.$$

Denoting $v_1(z) = (1-|z|)^m \log\left(\frac{3}{1-|z|}\right)$, it is clear that v_1 is a *normal* weight. From [17, Lemma 3] we obtain that " Assume that v is a *normal* weight, then $\sup_{z\in\mathbb{D}} v(z)|f(z)| \asymp |f(0)| + \sup_{z\in\mathbb{D}} v(z)(1-|z|)|f'(z)|$ for every $f \in H(\mathbb{D})$." Hence the following result holds for the log-Bloch space $\mathcal{B}_{\log}$ on the unit disk:

Lemma 2.3 *For $f \in H(\mathbb{D}), m \in \mathbb{N}$, then*

$$f \in \mathcal{B}_{\log} \Leftrightarrow \|f\|_{\log} \asymp \sup_{z\in\mathbb{D}} (1-|z|)^m \log\left(\frac{3}{1-|z|}\right) |f^{(m)}(z)| < \infty.$$

Hence, $f \in \mathcal{B}_{\log} \Leftrightarrow f^{(m)} \in H^\infty_{v_1} = \{f \in H(\mathbb{D}) : \|f\|_{v_1} = \sup_{z\in\mathbb{D}} v_1(z)|f(z)| < \infty\}$. Moreover, v_1 is also a *essential* weight. In fact, a weight $v : \mathbb{D} \to \mathbb{R}_+$ is called *radial* if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. The so-called associated weights are defined by

$$\tilde{v}(z) = (\sup\{|f(z)| : \ f \in H^\infty_v, \ \|f\|_v \leq 1\})^{-1}.$$

It is evident that $\tilde{v}$ is also a weight. A weight v is called *essential* if there exists a constant $C > 0$ such that

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \ \text{ for each } \ z \in \mathbb{D}.$$

Besides, the following condition $(L1)$ which was introduced by Lusky in [12] plays an important part in deciding whether a weight is *essential* or not,

$$(L1) \quad \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0.$$

Radial weights which satisfy $(L1)$ are always *essential* (see [1]). It is obvious that the weight v_1 is *radial* and satisfies

$$\inf_{n\in\mathbb{N}} \frac{v_1(1-2^{-n-1})}{v_1(1-2^{-n})} = \inf_{n\in\mathbb{N}} \frac{1}{2^m} \frac{\log 6\cdot 2^n}{\log 3\cdot 2^n} > \frac{1}{2^m} > 0.$$

Hence v_1 is *essential*, which will be used to show the following theorem A.

Lemma 2.4 *[2, Lemma 2.3]*

$$\frac{\|z^j\|_{\log}}{\log(j+1)} \asymp \frac{1}{e} \;\; as \;\; j\to\infty. \tag{3}$$

Besides, since

$$\lim_{j\to\infty} \frac{\log(j+1)}{\log\frac{j+1}{m-1+L}} = 1,$$

thus

$$\frac{\|z^j\|_{\log}}{\log\frac{j+1}{m-1+L}} \asymp \frac{1}{e} \;\; as \;\; j\to\infty. \tag{4}$$

Lemma 2.5 *For f_j defined in lemma 2.2, the following statements hold:*
(a) *for all $j\in\mathbb{N}$, there is a unique $x_j\in(0,1)$ such that $f_j(x_j)$ is the absolute maximum of f_j.*
(b) *the sequence $(x_j)_{j\in\mathbb{N}}$ satisfies*

$$\lim_{j\to\infty} x_j = 1^-, \tag{5}$$

where "$-$" denotes that x_j tends to 1 from the left, moreover,

$$\lim_{j\to\infty} j(1-x_j) = m. \tag{6}$$

(c)

$$\lim_{j\to\infty} \frac{j^m}{\log(j+1)} \max_{0<x<1} f_j(x) = \lim_{j\to\infty} \frac{j^m}{\log(j+1)} f_j(x_j) = \frac{m^m}{e^m}. \tag{7}$$

Proof. It is obvious that

$$f_j'(x) = x^j(1-x)^{m-1}\left((j(1-x)-mx)\log\frac{3}{1-x}+x\right).$$

We denote

$$g_j(x) = (j(1-x)-mx)\log\frac{3}{1-x}+x.$$

Then

$$\begin{aligned} g_j'(x) &= j+1+m-\frac{m}{1-x}-(j+m)\log\frac{3}{1-x} \\ &\le j+1+m-m-(j+m)\log 3 \\ &= j+1-(j+m)\log 3 < 0. \end{aligned}$$

Thus g_j is strictly decreasing on $(0,1)$, and since

$$\lim_{x\to 0^+} g_j(x) = j\log j > 0 \;\text{ and }\; \lim_{x\to 1^-} g_j(x) = -\infty,$$

hence there is a unique $x_j\in(0,1)$ such that $g_j(x_j)=0$. It is clear that $g_j(x)>0$ whenever $x\in(0,x_j)$ and that $g_j(x)<0$ whenever $x\in(x_j,1)$. Since

$$f_j'(x) = x^{j-1}(1-x)^{m-1}g_j(x),$$

so the function f_j is increasing on $(0, x_j)$ and decreasing on $(x_j, 1)$. Therefore, f_j has a unique absolute maximum at x_j, which implies (a) holds.

Since $g_j(x_j) = (j(1-x_j) - mx_j)\log\frac{3}{1-x_j} + x_j = 0$, thus

$$\left((1-x_j) - \frac{mx_j}{j}\right)\log\frac{3}{1-x_j} = \frac{-x_j}{j}.$$

Since $x_j \in (0,1)$, then letting $j \to \infty$ in the above equation, it follows that

$$\lim_{j\to\infty}\left((1-x_j) - \frac{mx_j}{j}\right)\log\frac{3}{1-x_j} = 0.$$

However, $\log\frac{3}{1-x_j} \geq \log 3$, then $\lim\limits_{j\to\infty}[(1-x_j) - \frac{mx_j}{j}] = 0$. Hence $\lim\limits_{j\to\infty} x_j = 1^-$. That is, (5) is true.

Using $g_j(x_j) = 0$ again, we have that

$$\lim_{j\to\infty}[j(1-x_j) - mx_j] = \lim_{j\to\infty}\frac{-x_j}{\log\frac{3}{1-x_j}} = 0.$$

Which implies that $\lim\limits_{j\to\infty} j(1-x_j) = \lim\limits_{j\to\infty} mx_j = m$. That is, (6) is true.

Since

$$\lim_{j\to\infty} j\log x_j = \lim_{j\to\infty} j(x_j-1)\frac{\log[1+(x_j-1)]}{x_j-1} = -m,$$

then $\lim\limits_{j\to\infty} x_j^j = e^{-m}$. Further by

$$\lim_{j\to\infty}\frac{\log\frac{3}{1-x_j}}{\log(j+1)} = \lim_{j\to\infty}\frac{\log\frac{3j}{j(1-x_j)}}{\log(j+1)} = \lim_{j\to\infty}\left(\frac{\log 3j}{\log(j+1)} - \frac{\log(j(1-x_j))}{\log(j+1)}\right) = 1,$$

we have that

$$\begin{aligned}\lim_{j\to\infty}\frac{j^m}{\log(j+1)}\max_{0<x<1} f_j(x) &= \lim_{j\to\infty}\frac{j^m}{\log(j+1)}x_j^j(1-x_j)^m\log\frac{3}{1-x_j}\\ &= \lim_{j\to\infty}\frac{\log\frac{3}{1-x_j}}{\log(j+1)}j^m(1-x_j)^m x_j^j = \frac{m^m}{e^m}.\end{aligned}$$

This completes the proof. □

In this paper, we use the following notation. Let $u \in H(\mathbb{D})$ and $f \in H(\mathbb{D})$, define

$$I_u f(z) = \int_0^z f'(\zeta)u(\zeta)d\zeta,\ J_u f(z) = \int_0^z f(\zeta)u'(\zeta)d\zeta.$$

Then it follows that

$$I_u(\varphi^{j+1})(z) = \int_0^z (\varphi^{j+1})'(\zeta)u(\zeta)d\zeta, \quad J_u(\varphi^j)(z) = \int_0^z \varphi^j(\zeta)u'(\zeta)d\zeta. \tag{8}$$

Besides,

$$\|I_u(\varphi^{j+1})\|_\mu = (j+1)\sup_{z\in\mathbb{D}}\mu(z)|\varphi(z)|^j|u(z)\varphi'(z)|, \quad \|J_u(\varphi^j)\|_\mu = \sup_{z\in\mathbb{D}}\mu(z)|\varphi^j(z)u'(z)|.$$

Since $(D^m_{\varphi,u}f)' = u'f^{(m)}\circ\varphi + u\varphi' f^{(m+1)}\circ\varphi$, thus $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is bounded (or compact) if and only if the weighted composition operators $u'C_\varphi : H^\infty_{v_1} \to H^\infty_\mu$ and $[u\varphi']C_\varphi : H^\infty_{v_2} \to H^\infty_\mu$ are bounded (or compact), where $v_2(z) = (1-|z|)^{m+1}\log\left(\frac{3}{1-|z|}\right)$ is also a *essential* weight. Similar to the paper [3, Proposition 3.1] we obtain the following theorem A.

Theorem A Let $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}$ and μ be a weight on $\mathbb{D}$, Then the operator $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is bounded if and only if

$$M_1 := \sup_{z\in\mathbb{D}} \frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|)^m \log\frac{3}{1-|\varphi(z)|}} < \infty,$$

$$M_2 := \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^{m+1} \log\frac{3}{1-|\varphi(z)|}} < \infty.$$

Based on the above result, we will give the new criterion for the boundedness of the operator $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$.

3 The boundedness

Theorem 3.1 *Let $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}$ and μ be a weight on $\mathbb{D}$. Then $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is bounded if and only if $u \in \mathcal{B}_\mu$, $\sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)| < \infty$ and both of the inequalities hold:*

$$\sup_{j\geq 1} \frac{j^m \|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}} < \infty, \tag{9}$$

$$\sup_{j\geq 1} \frac{j^m \|I_u\varphi^{j+1}\|_\mu}{\|z^j\|_{\log}} < \infty. \tag{10}$$

Proof. *Necessity*. Suppose the operator $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is bounded. Since the function $f_1(z) = z^m \in \mathcal{B}_{\log}$ and $f_2(z) = z^{m+1} \in \mathcal{B}_{\log}$, then we have that

$$m! \sup_{z\in\mathbb{D}} \mu(z)|u'(z)| < \infty.$$

$$(m+1)! \sup_{z\in\mathbb{D}} \mu(z)|u'(z)\varphi(z) + u(z)\varphi'(z)| < \infty.$$

From the above inequalities, we obtain that $u \in \mathcal{B}_\mu$ and $\sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)| < \infty$. Next we will show (9) and (10) hold. From (7), we know that there exists a constant $K_1 > 0$ such that

$$\sup_{j\geq 1} \frac{j^m}{\log(j+1)} \max_{0<x<1} x^j(1-x)^m \log\frac{3}{1-x} < K_1. \tag{11}$$

Since the operator $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is bounded, then by theorem A, lemma 2.4 and (11), it follows that

$$\begin{aligned}
\sup_{j\geq 1} \frac{j^m \|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}} &\preceq \sup_{j\geq 1} \frac{j^m}{\log(j+1)} \sup_{z\in\mathbb{D}} \mu(z)|\varphi^j(z)u'(z)| \\
&= \sup_{j\geq 1} \frac{j^m}{\log(j+1)} \sup_{z\in\mathbb{D}} \frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|)^m \log\frac{3}{1-|\varphi(z)|}} |\varphi(z)|^j (1-|\varphi(z)|)^m \log\frac{3}{1-|\varphi(z)|} \\
&\leq M_1 \sup_{j\geq 1} \frac{j^m}{\log(j+1)} \sup_{z\in\mathbb{D}} |\varphi(z)|^j (1-|\varphi(z)|)^m \log\frac{3}{1-|\varphi(z)|} < M_1K_1.
\end{aligned}$$

That is, (9) holds. Similarly, from (7), we obtain that

$$\lim_{j\to\infty} \frac{j^{m+1}}{\log(j+1)} \max_{0<x<1} x^j(1-x)^{m+1} \log\frac{3}{1-x} = \frac{(m+1)^{m+1}}{e^{m+1}}.$$

By the above equation, there exists a constant $K_2 > 0$ such that

$$\sup_{j\geq 1} \frac{j^{m+1}}{\log(j+1)} \max_{0<x<1} x^j(1-x)^{m+1} \log\frac{3}{1-x} < K_2. \tag{12}$$

Similarly, by theorem A, lemma 2.4 and (12),

$$\begin{aligned}
&\sup_{j\ge 1}\frac{j^m\|I_u\varphi^{j+1}\|_\mu}{\|z^j\|_{\log}} \preceq \sup_{j\ge 1}\frac{j^m}{\log(j+1)}\sup_{z\in\mathbb{D}}(j+1)\mu(z)|\varphi^j(z)\varphi'(z)u(z)|\\
&=\sup_{j\ge 1}\frac{j^m(j+1)}{\log(j+1)}\sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}}|\varphi(z)|^j(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}\\
&\le M_2\sup_{j\ge 1}\frac{j^m(j+1)}{\log(j+1)}\sup_{z\in\mathbb{D}}|\varphi(z)|^j(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}\\
&\preceq M_2\sup_{j\ge 1}\frac{j^{m+1}}{\log(j+1)}\sup_{z\in\mathbb{D}}|\varphi(z)|^j(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}\preceq M_2K_2.
\end{aligned}$$

That is, (10) holds.

Sufficiency. Firstly, if $\sup_{z\in\mathbb{D}}|\varphi(z)|<1$, then there is a number $r\in(0,1)$ such that $\sup_{z\in\mathbb{D}}|\varphi(z)|<r$. For every $f\in\mathcal{B}_{\log}$ with $\|f\|_{\mathcal{B}_{\log}}\le 1$, by the condition $u\in\mathcal{B}_\mu$ and $\sup_{z\in\mathbb{D}}\mu(z)|u(z)\varphi'(z)|<\infty$, it follows that

$$\begin{aligned}
&\|D^m_{\varphi,u}f\|_{\mathcal{B}_\mu}=|(D^m_{\varphi,u}f)(0)|+\|D^m_{\varphi,u}f\|_\mu\\
&=|u(0)||f^{(m)}(\varphi(0))|+\sup_{z\in\mathbb{D}}\mu(z)|u'(z)f^{(m)}(\varphi(z))+u(z)f^{(m+1)}(\varphi(z))\varphi'(z)|\\
&\le|u(0)||f^{(m)}(\varphi(0))|+\sup_{z\in\mathbb{D}}\mu(z)|u'(z)f^{(m)}(\varphi(z))|+\sup_{z\in\mathbb{D}}\mu(z)|u(z)f^{(m+1)}(\varphi(z))\varphi'(z)|\\
&\le\frac{|u(0)|\|f\|_{\log}}{(1-|\varphi(0)|)^m\log\frac{3}{1-|\varphi(0)|}}+\sup_{z\in\mathbb{D}}\frac{\mu(z)|u'(z)|\|f\|_{\mathcal{B}_{\log}}}{(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}+\sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)\varphi'(z)|\|f\|_{\mathcal{B}_{\log}}}{(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}}\\
&\le\frac{|u(0)|}{(1-|\varphi(0)|)^m\log\frac{3}{1-|\varphi(0)|}}+\sup_{z\in\mathbb{D}}\frac{\mu(z)|u'(z)|}{(1-r)^m\log\frac{3}{1-r}}+\sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)\varphi'(z)|}{(1-r)^{m+1}\log\frac{3}{1-r}}<\infty,
\end{aligned}$$

which implies the boundedness of $D^m_{\varphi,u}:\mathcal{B}_{\log}\to\mathcal{B}_\mu$.

Secondly, if $\sup_{z\in\mathbb{D}}|\varphi(z)|=1$. For every $f\in\mathcal{B}_{\log}$ with $\|f\|_{\mathcal{B}_{\log}}\le 1$, we have that

$$\begin{aligned}
&\|D^m_{\varphi,u}f\|_{\mathcal{B}_\mu}=|(D^m_{\varphi,u}f)(0)|+\|D^m_{\varphi,u}f\|_\mu\\
&=|u(0)||f^{(m)}(\varphi(0))|+\sup_{z\in\mathbb{D}}\mu(z)|u'(z)f^{(m)}(\varphi(z))+u(z)f^{(m+1)}(\varphi(z))\varphi'(z)|\\
&\le|u(0)||f^{(m)}(\varphi(0))|+\sup_{z\in\mathbb{D}}\mu(z)|u'(z)f^{(m)}(\varphi(z))|+\sup_{z\in\mathbb{D}}\mu(z)|u(z)f^{(m+1)}(\varphi(z))\varphi'(z)|\\
&\le\frac{|u(0)|\|f\|_{\log}}{(1-|\varphi(0)|)^m\log\frac{3}{1-|\varphi(0)|}}+\sup_{z\in\mathbb{D}}\frac{\mu(z)|u'(z)|\|f\|_{\mathcal{B}_{\log}}}{(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}+\sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)\varphi'(z)|\|f\|_{\mathcal{B}_{\log}}}{(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}}\\
&\le\frac{|u(0)|}{(1-|\varphi(0)|)^m\log\frac{3}{1-|\varphi(0)|}}+\sup_{z\in\mathbb{D}}\frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}+\sup_{z\in\mathbb{D}}\frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}}\\
&=I_1+I_2+I_3,
\end{aligned}\tag{13}$$

It is obvious that $I_1<\infty$. We only need to show that I_2 and I_3 is finite. For any integer $j\ge 1$, let

$$A^j_\varphi=\{z\in\mathbb{D}:\ r_j\le|\varphi(z)|<r_{j+1}\},$$

where $r_j=1-\frac{m-1+L}{j+m-1+L}$. Let k be the smallest positive integer such that $A^k_\varphi\ne\emptyset$. Since $\sup_{z\in\mathbb{D}}|\varphi(z)|=1$, hence the set A^j_φ is not empty for every integer $j\ge k$, and $\mathbb{D}=\cup_{j=k}^\infty A^j_\varphi$. By (1), for every $j\in\mathbb{N}$,

$$\frac{j^m}{\log\frac{j+1}{m+1+L}}\min_{x\in[r_j,r_{j+1}]}f_j(x)\ge\frac{L^m}{3^m e^{m-1+L}}=\delta_1.\tag{14}$$

Hence by (14) and (4), it follows that

$$\begin{aligned}
I_2 &= \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)|}{(1-|\varphi(z)|)^m \log\frac{3}{1-|\varphi(z)|}} \\
&= \sup_{k\le j}\sup_{z\in A_\varphi^j} \frac{j^m\mu(z)|u'(z)||\varphi(z)|^j}{\frac{j^m}{\log\frac{j+1}{m+1+L}}|\varphi(z)|^j(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}\log\frac{j+1}{m+1+L}} \\
&\le \frac{1}{\delta_1}\sup_{k\le j}\sup_{z\in A_\varphi^j} \frac{j^m\mu(z)|u'(z)||\varphi(z)|^j}{\log\frac{j+1}{m+1+L}} \\
&\le \frac{1}{\delta_1}\sup_{1\le j}\sup_{z\in\mathbb{D}} \frac{j^m\mu(z)|u'(z)||\varphi(z)|^j}{\log\frac{j+1}{m+1+L}} \\
&\preceq \frac{1}{\delta_1}\sup_{j\ge 1}\sup_{z\in\mathbb{D}} \frac{j^m\mu(z)|u'(z)||\varphi(z)|^j}{\|z^j\|_{\log}} \\
&= \frac{1}{\delta_1}\sup_{j\ge 1}\frac{j^m\|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}}. \qquad (15)
\end{aligned}$$

Similarly, $\frac{j^{m+1}}{\log\frac{j+1}{m+L}} \min_{x\in A_\varphi^j} x^j(1-x)^{m+1}\log\frac{3}{1-x} \ge \frac{L^{m+1}}{3^{m+1}e^{m+L}} = \delta_2$. Hence

$$\begin{aligned}
I_3 &= \sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}} \\
&= \sup_{k\le j}\sup_{z\in A_\varphi^j} \frac{\mu(z)|u(z)\varphi'(z)|j^{m+1}|\varphi(z)|^j}{\frac{j^{m+1}}{\log\frac{j+1}{m+L}}|\varphi(z)|^j(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}\log\frac{j+1}{m+L}} \\
&\le \frac{1}{\delta_2}\sup_{k\le j}\sup_{z\in A_\varphi^j} \frac{\mu(z)|u(z)\varphi'(z)|j^{m+1}|\varphi(z)|^j}{\log\frac{j+1}{m+L}} \\
&\le \frac{1}{\delta_2}\sup_{j\ge 1}\sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)|j^{m+1}|\varphi(z)|^j}{\log\frac{j+1}{m+L}} \\
&\preceq \frac{1}{\delta_2}\sup_{j\ge 1}\sup_{z\in\mathbb{D}} \frac{\mu(z)|u(z)\varphi'(z)|j^{m+1}|\varphi(z)|^j}{\|z^j\|_{\log}} \\
&= \frac{1}{\delta_2}\sup_{j\ge 1}\frac{j^{m+1}\|I_u\varphi^{j+1}\|_\mu}{(j+1)\|z^j\|_{\log}} \\
&\preceq \frac{1}{\delta_2}\sup_{j\ge 1}\frac{j^m\|I_u\varphi^{j+1}\|_\mu}{\|z^j\|_{\log}}. \qquad (16)
\end{aligned}$$

Combining (13), (15) and (16) we obtain the boundedness of $D_{\varphi,u}^m : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ in this case. Now the proof is complete. □

4 The essential norm

In this section, we will give an estimate for the essential norm of $D_{\varphi,u}^m : \mathcal{B}_{\log} \to \mathcal{B}_\mu$. To simplify the notations, we denote

$$A := \limsup_{j\to\infty} \frac{j^m\|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}} \quad\text{and}\quad B := \limsup_{j\to\infty} \frac{j^m\|I_u\varphi^{j+1}\|_\mu}{\|z^j\|_{\log}}.$$

For $r \in [0,1]$, we define the linear dilation operator $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by $K_r f = f_r$, where $f_r(z) = f(rz)$. Then we have:

Lemma 4.1 *[2, Lemma 5.2] Let $r \in [0,1]$. Then the following statements hold:*
(a) $\mathcal{B}_{\log}$ *is a K_r-invariant subspace of $H(\mathbb{D})$; moreover, we have that*

$$\|K_r\|_{\mathcal{B}_{\log}\to\mathcal{B}_{\log}} \le 1.$$

(b) *If $r \neq 1$, then K_r is compact on $\mathcal{B}_{\log}$.*

The following criterion for compactness comes from an easy modification of [4, Proposition 3.11]. Hence we omit the details.

Lemma 4.2 *Let μ_1 and μ_2 be weights on $\mathbb{D}$, $m \in \mathbb{N}$, $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$. Then the operator $D^m_{\varphi,u} : \mathcal{B}_{\mu_1} \to \mathcal{B}_{\mu_2}$ is compact if and only if given any bounded sequence $(f_j)_{j\in\mathbb{N}}$ in $\mathcal{B}_{\mu_1}$ such that f_j converges to 0 uniformly on compact subsets of $\mathbb{D}$, then $\|D^m_{\varphi,u}(f_j)\|_{\mathcal{B}_{\mu_2}} \to 0$ as $j \to \infty$.*

Theorem 4.3 *Let $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}$ and μ be a weight on $\mathbb{D}$. Suppose that the operator $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is bounded. Then*

$$\|D^m_{\varphi,u}\|_e \asymp A + B. \tag{17}$$

P r o o f. Since the operator $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is bounded, then $A < \infty$ and $B < \infty$. Moreover,

$$L_1 := \sup_{z\in\mathbb{D}} \mu(z)||u'(z)| < \infty, \quad L_2 := \sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)| < \infty.$$

Firstly, if $\sup_{z\in\mathbb{D}} |\varphi(z)| < 1$, then there is a number $r \in (0,1)$ such that $\sup_{z\in\mathbb{D}} |\varphi(z)| < r$. It's easy to verify the operator $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is compact by lemma 4.2. In fact, let $\{f_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $\mathcal{B}_{\log}$ converging to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. We denote $L := \sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{B}_{\log}} < \infty$. By Cauchy's integral formula, we obtain that

$$\begin{aligned}
&\|D^m_{\varphi,u} f_k\|_{\mathcal{B}_\mu} = |(D^m_{\varphi,u} f_k)(0)| + \|D^m_{\varphi,u} f_k\|_\mu \\
&= |u(0)||f_k^{(m)}(\varphi(0))| + \sup_{z\in\mathbb{D}} \mu(z)|u'(z) f_k^{(m)}(\varphi(z)) + u(z) f_k^{(m+1)}(\varphi(z))\varphi'(z)| \\
&\le |u(0)||f_k^{(m)}(\varphi(0))| + \sup_{z\in\mathbb{D}} \mu(z)|u'(z) f_k^{(m)}(\varphi(z))| \\
&+ \sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)||f_k^{(m+1)}(\varphi(z))| \\
&\le |u(0)||f_k^{(m)}(\varphi(0))| + L_1 \sup_{z\in\mathbb{D}} |f_k^{(m)}(\varphi(z))| + \sup_{z\in\mathbb{D}} L_2 |f_k^{(m+1)}(\varphi(z))| \\
&\le |u(0)||f_k^{(m)}(\varphi(0))| + L_1 \sup_{|w|\le r} |f_k^{(m)}(w)| + L_2 \sup_{|w|\le r} |f_k^{(m+1)}(w)| \\
&\to 0, \quad \text{as} \quad k \to \infty.
\end{aligned}$$

Employing lemma 4.2, it is clear that the operator $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is compact. Thus

$$\|D^m_{\varphi,u}\|_e = 0. \tag{18}$$

On the other hand, by (3) we obtain that

$$\begin{aligned} A &= \limsup_{j\to\infty} \frac{j^m \|J_u \varphi^j\|_\mu}{\|z^j\|_{\log}} \preceq \limsup_{j\to\infty} \frac{j^m \|J_u \varphi^j\|_\mu}{\log(j+1)} \\ &= \limsup_{j\to\infty} \frac{j^m}{\log(j+1)} \sup_{z\in\mathbb{D}} \mu(z)|u'(z)\varphi^j(z)| \\ &\le L_1 \limsup_{j\to\infty} \frac{j^m}{\log(j+1)} r^j = 0. \qquad (19) \\ B &= \limsup_{j\to\infty} \frac{j^m \|I_u \varphi^{j+1}\|_\mu}{\|z^j\|_{\log}} \preceq \limsup_{j\to\infty} \frac{j^m \|I_u \varphi^{j+1}\|_\mu}{\log(j+1)} \\ &= \limsup_{j\to\infty} \frac{j^m}{\log(j+1)} \sup_{z\in\mathbb{D}} \mu(z)(j+1)|\varphi(z)|^j |\varphi'(z)||u(z)| \\ &\le L_2 \limsup_{j\to\infty} \frac{j^m (j+1)}{\log(j+1)} r^j = 0. \qquad (20) \end{aligned}$$

From (18)-(20), both sides of (17) are zero. Therefore the essential norm formula is true in this case. In the sequel, we assume that $\sup_{z\in\mathbb{D}} |\varphi(z)| = 1$.

The upper bounded. By the boundedness of the operator $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ and lemma 4.2, we know the operator $D^m_{\varphi,u} K_{r_j} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is compact. Then

$$\begin{aligned} \|D^m_{\varphi,u}\|_e &\le \limsup_{j\to\infty} \|D^m_{\varphi,u} - D^m_{\varphi,u} K_{r_j}\| \\ &= \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}_{\log}}\le 1} \|(D^m_{\varphi,u} - D^m_{\varphi,u} K_{r_j}) f\|_{\mathcal{B}_\mu} \\ &\le \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}_{\log}}\le 1} |(D^m_{\varphi,u} - D^m_{\varphi,u} K_{r_j}) f(0)| \\ &+ \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}_{\log}}\le 1} \sup_{z\in\mathbb{D}} \mu(z) |(D^m_{\varphi,u} - D^m_{\varphi,u} K_{r_j})' f(z)| \\ &\le \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}_{\log}}\le 1} |u(0) f^{(m)}(\varphi(0)) - r_j^m u(0) f^{(m)}(r_j \varphi(0))| \\ &+ \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}_{\log}}\le 1} \sup_{z\in\mathbb{D}} \mu(z)|u'(z)||f^{(m)}(\varphi(z)) - r_j^m f^{(m)}(r_j\varphi(z))| \\ &+ \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}_{\log}}\le 1} \sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)||f^{(m+1)}(\varphi(z)) - r_j^{m+1} f^{(m+1)}(r_j\varphi(z))| \\ &\le \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}_{\log}}\le 1} |u(0)||f^{(m)}(\varphi(0)) - r_j^m f^{(m)}(r_j\varphi(0))| \\ &+ J_1 + J_2. \qquad (21) \end{aligned}$$

On the one hand, similar to [2, Lemma 5.3], by $r_j \to 1$ as $j \to \infty$ and Cauchy's integral formula, it follows that

$$\limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}_{\log}}\le 1} |u(0)||f^{(m)}(\varphi(0)) - r_j^m f^{(m)}(r_j\varphi(0))| = 0.$$

On the other hand, for each positive integer $j \in \mathbb{N}$, denote the set $A^j_\varphi = \{z \in \mathbb{D} : \ r_j \le |\varphi(z)| < r_{j+1}\}$. Since $\sup_{z\in\mathbb{D}} |\varphi(z)| = 1$, then there is a $k \in \mathbb{N}$ such that $A^k_\varphi \ne \emptyset$ and A^j_φ is not empty for every integer $j \ge k$. Hence $\mathbb{D} = \bigcup_{j=k}^\infty A^j_\varphi$. By (2), for an arbitrary $\epsilon > 0$, there is $N_1 \in \mathbb{N}$ such that for any integer $j \ge N_1$,

$$\frac{j^m}{\log \frac{j+1}{m-1+L}} \min_{x\in[r_j, r_{j+1}]} x^j (1-x)^m \log \frac{3}{1-x} > \frac{(m-1+L)^m}{e^{m-1+L}} - \epsilon. \qquad (22)$$

Further by

$$\lim_{j\to\infty}\frac{j^{m+1}}{\log\frac{j+1}{m+L}}\min_{x\in[r_j,r_{j+1}]}x^j(1-x)^{m+1}\log\frac{3}{1-x}=\frac{(m+L)^{m+1}}{e^{m+L}},$$

there is $N_2\in\mathbb{N}$ such that for any integer $j\geq N_2$,

$$\frac{j^{m+1}}{\log\frac{j+1}{m+L}}\min_{x\in[r_j,r_{j+1}]}x^j(1-x)^{m+1}\log\frac{3}{1-x}>\frac{(m+L)^{m+1}}{e^{m+L}}-\epsilon. \tag{23}$$

In the following, denote $N=\max\{N_1,\ N_2\}$. In fact (22) is used to prove J_1 and (23) for J_2. We divide J_1 into two parts

$$J_{1,1}=\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{B}_{\log}}\leq 1}\sup_{k\leq j\leq N}\sup_{z\in A_\varphi^j}\mu(z)|u'(z)||f^{(m)}(\varphi(z))-r_j^m f^{(m)}(r_j\varphi(z))|;$$
$$J_{1,2}=\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{B}_{\log}}\leq 1}\sup_{N+1\leq j}\sup_{z\in A_\varphi^j}\mu(z)|u'(z)||f^{(m)}(\varphi(z))-r_j^m f^{(m)}(r_j\varphi(z))|.$$

It is clear that

$$\begin{aligned}J_{1,1}&\leq L_1\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{B}_{\log}}\leq 1}\sup_{r_k\leq|\varphi(z)|\leq r_{N+1}}|f^{(m)}(\varphi(z))-r_j^m f^{(m)}(r_j\varphi(z))|\\&=0.\end{aligned}$$

On the other hand,

$$\begin{aligned}J_{1,2}&\leq\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{B}_{\log}}\leq 1}\sup_{N+1\leq j}\sup_{z\in A_\varphi^j}\mu(z)|u'(z)|(|f^{(m)}(\varphi(z))|+|r_j^m f^{(m)}(r_j\varphi(z))|)\\&\leq\limsup_{j\to\infty}\sup_{\|f\|_{\mathcal{B}_{\log}}\leq 1}\sup_{N+1\leq j}\sup_{z\in A_\varphi^j}\mu(z)|u'(z)|(|f^{(m)}(\varphi(z))|+|f^{(m)}(r_j\varphi(z))|).\end{aligned}$$

Denote the expression $s(\rho)$ for $\rho\in(0,1)$,

$$\begin{aligned}s(\rho)&=\sup_{N+1\leq j}\sup_{z\in A_\varphi^j}\mu(z)|u'(z)||f^{(m)}(\rho\varphi(z))|\\&=\sup_{N+1\leq j}\sup_{z\in A_\varphi^j}\mu(z)|u'(z)|\frac{|f^{(m)}(\rho\varphi(z))|(1-|\rho\varphi(z)|)^m\log\frac{3}{1-|\rho\varphi(z)|}}{(1-|\rho\varphi(z)|)^m\log\frac{3}{1-|\rho\varphi(z)|}}\\&\leq\sup_{N+1\leq j}\sup_{z\in A_\varphi^j}\|f\|_{\mathcal{B}_{\log}}\frac{\mu(z)|u'(z)|}{(1-|\rho\varphi(z)|)^m\log\frac{3}{1-|\rho\varphi(z)|}}\\&\leq\sup_{N+1\leq j}\sup_{z\in A_\varphi^j}\|f\|_{\mathcal{B}_{\log}}\frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}\\&=\sup_{N+1\leq j}\sup_{z\in A_\varphi^j}\|f\|_{\mathcal{B}_{\log}}\frac{j^m\mu(z)|u'(z)||\varphi(z)|^j}{j^m|\varphi(z)|^j(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}\\&\preceq\sup_{N+1\leq j}\|f\|_{\mathcal{B}_{\log}}\frac{j^m\|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}}\sup_{z\in A_\varphi^j}\frac{\log\frac{j+1}{m-1+L}}{j^m|\varphi(z)|^j(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}\\&\leq\frac{1}{\frac{(m-1+L)^m}{e^{m-1+L}}-\epsilon}\sup_{N+1\leq j}\|f\|_{\mathcal{B}_{\log}}\frac{j^m\|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}},\end{aligned}$$

In the above chain of relations, the first inequality follows from the definition of the norm $\|\cdot\|_{\mathcal{B}_{\log}}$ and lemma 2.3. The second inequality is due to the fact that the function $(1-x)^{m+1}\log\frac{3}{1-x}$ is decreasing on $[0,1]$. The third and fourth inequalities follow from (4) and (22), respectively. Since ϵ is arbitrary, then

$$\begin{aligned} J_{1,2} &\leq \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}_{\log}}\leq 1} (s(1)+s(r_j)) \\ &\preceq \limsup_{j\to\infty} \frac{j^m\|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}}. \end{aligned}$$

The above inequalities imply that

$$J_1 \preceq \limsup_{j\to\infty} \frac{j^m\|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}}. \tag{24}$$

Using the similar way by replacing (22) with (23), we can obtain that

$$\begin{aligned} J_2 &= \limsup_{j\to\infty} \sup_{\|f\|_{\mathcal{B}_{\log}}\leq 1} \sup_{z\in\mathbb{D}} \mu(z)|u(z)\varphi'(z)||f^{(m+1)}(\varphi(z)) - r_j^{(m+1)} f^{(m+1)}(r_j\varphi(z))| \\ &\preceq \limsup_{j\to\infty} \frac{j^m\|I_u\varphi^{j+1}\|_\mu}{\|z^j\|_{\log}}. \end{aligned} \tag{25}$$

Combining (21), (24) and (25), it follows the *upper* bounded.

The lower bounded. The assumption $\sup_{z\in\mathbb{D}}|\varphi(z)|=1$ holds. For every compact operator $K:\mathcal{B}_{\log}\to\mathcal{B}_\mu$ and each sequence $\{f_k\}_{k\in\mathbb{N}}\subset\mathcal{B}_{\log}$ with $\sup_{k\in\mathbb{N}}\|f_k\|_{\mathcal{B}_{\log}}<\infty$, and f_k converging to zero on the compact subsets of $\mathbb{D}$, it follows that $\lim_{k\to\infty}\|Kf_k\|_{\mathcal{B}_\mu}=0$. Thus

$$\|D^m_{\varphi,u}-K\| \geq \limsup_{k\to\infty}\|(D^m_{\varphi,u}-K)f_k\|_{\mathcal{B}_\mu} \geq \limsup_{k\to\infty}\|D^m_{\varphi,u}(f_k)\|_{\mathcal{B}_\mu},$$

that is, $\|D^m_{\varphi,u}\|_e \geq \limsup_{k\to\infty}\|D^m_{\varphi,u}(f_k)\|_{\mathcal{B}_\mu}$.

Choosing a sequence $\{a_k\}_{k\in\mathbb{N}}\subset\mathbb{D}$ such that $|\varphi(a_k)|\to 1$ as $k\to\infty$, we define the function sequences $\{f^{(m)}_{1,k}\}_{k\in\mathbb{N}}$ and $\{f^{(m)}_{2,k}\}_{k\in\mathbb{N}}$, respectively.

$$\begin{aligned} f^{(m)}_{1,k}(z) &= \frac{(m+2)(1-|\varphi(a_k)|^2)}{(1-\overline{\varphi(a_k)}z)^{m+1}\log\frac{3}{1-\overline{\varphi(a_k)}z}} - \frac{(m+2)(1-|\varphi(a_k)|^2)^2}{(1-\overline{\varphi(a_k)}z)^{m+2}\log\frac{3}{1-\overline{\varphi(a_k)}z}}; \\ f^{(m)}_{2,k}(z) &= \frac{(m+2)(1-|\varphi(a_k)|^2)}{(1-\overline{\varphi(a_k)}z)^{m+1}\log\frac{3}{1-\overline{\varphi(a_k)}z}} - \frac{(m+1)(1-|\varphi(a_k)|^2)^2}{(1-\overline{\varphi(a_k)}z)^{m+2}\log\frac{3}{1-\overline{\varphi(a_k)}z}}. \end{aligned}$$

Clearly $\{f_{1,k}\}_{k\in\mathbb{N}}\subset\mathcal{B}_{\log}$ and $\{f_{2,k}\}_{k\in\mathbb{N}}\subset\mathcal{B}_{\log}$; moreover, both sequences converge to zero uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Since $f^{(m)}_{1,k}(\varphi(a_k))=0$ and

$$\begin{aligned} f^{(m+1)}_{1,k}(\varphi(a_k)) &= \frac{(m+1)\overline{\varphi(a_k)}}{(1-|\varphi(a_k)|^2)^{m+1}\log\frac{3}{1-|\varphi(a_k)|^2}}, \\ f^{(m)}_{2,k}(\varphi(a_k)) &= \frac{1}{(1-|\varphi(a_k)|^2)^{m}\log\frac{3}{1-|\varphi(a_k)|^2}}, \\ f^{(m+1)}_{2,k}(\varphi(a_k)) &= \frac{\overline{\varphi(a_k)}}{(1-|\varphi(a_k)|^2)^{m+1}\left(\log\frac{3}{1-|\varphi(a_k)|^2}\right)^2}, \end{aligned}$$

we have that

$$\begin{aligned}
\|D^m_{\varphi,u}\|_e &\geq \limsup_{k\to\infty}\|D^m_{\varphi,u}(f_{1,k})\|_{\mathcal{B}_\mu} \geq \limsup_{k\to\infty}\|D^m_{\varphi,u}(f_{1,k})\|_\mu \\
&= \limsup_{k\to\infty}\sup_{z\in\mathbb{D}}\mu(z)|u'(z)f^{(m)}_{1,k}(\varphi(z))+u(z)\varphi'(z)f^{(m+1)}_{1,k}(\varphi(z))| \\
&\geq \limsup_{k\to\infty}\mu(a_k)|u'(a_k)f^{(m)}_{1,k}(\varphi(a_k))+u(a_k)\varphi'(a_k)f^{(m+1)}_{1,k}(\varphi(a_k))| \\
&= \limsup_{k\to\infty}\frac{\mu(a_k)|u(a_k)\varphi'(a_k)|(m+1)|\varphi(a_k)|}{(1-|\varphi(a_k)|^2)^{m+1}\log\frac{3}{1-|\varphi(a_k)|^2}} \\
&\succeq \limsup_{k\to\infty}\frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1-|\varphi(a_k)|^2)^{m+1}\log\frac{3}{1-|\varphi(a_k)|^2}} \qquad (26)\\
&= \limsup_{|\varphi(z)|\to1}\frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{m+1}\log\frac{3}{1-|\varphi(z)|^2}}, \qquad (27)
\end{aligned}$$

and

$$\begin{aligned}
\|D^m_{\varphi,u}\|_e &\geq \limsup_{k\to\infty}\|D^m_{\varphi,u}(f_{2,k})\|_{\mathcal{B}_\mu} \geq \limsup_{k\to\infty}\|D^m_{\varphi,u}(f_{2,k})\|_\mu \\
&= \limsup_{k\to\infty}\sup_{z\in\mathbb{D}}\mu(z)|u'(z)f^{(m)}_{2,k}(\varphi(z))+u(z)\varphi'(z)f^{(m+1)}_{2,k}(\varphi(z))| \\
&\geq \limsup_{k\to\infty}\mu(a_k)|u'(a_k)f^{(m)}_{2,k}(\varphi(a_k))+u(a_k)\varphi'(a_k)f^{(m+1)}_{2,k}(\varphi(a_k))| \\
&\geq \limsup_{k\to\infty}\frac{\mu(a_k)|u'(a_k)|}{(1-|\varphi(a_k)|^2)^{m}\log\frac{3}{1-|\varphi(a_k)|^2}} \\
&- \limsup_{k\to\infty}\frac{\mu(a_k)|u(a_k)\varphi'(a_k)\overline{\varphi(a_k)}|}{(1-|\varphi(a_k)|^2)^{m+1}\left(\log\frac{3}{1-|\varphi(a_k)|^2}\right)^2} \\
&= \limsup_{k\to\infty}\frac{\mu(a_k)|u'(a_k)|}{(1-|\varphi(a_k)|^2)^{m}\log\frac{3}{1-|\varphi(a_k)|^2}} \\
&- \limsup_{k\to\infty}\frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1-|\varphi(a_k)|^2)^{m+1}\left(\log\frac{3}{1-|\varphi(a_k)|^2}\right)^2}.
\end{aligned}$$

Since

$$\log\frac{3}{1-|\varphi(a_k)|^2} \leq \left(\log\frac{3}{1-|\varphi(a_k)|^2}\right)^2,$$

then

$$\begin{aligned}
&\limsup_{k\to\infty}\frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1-|\varphi(a_k)|^2)^{m+1}\log\frac{3}{1-|\varphi(a_k)|^2}} + \|D^m_{\varphi,u}\|_e \\
&\geq \limsup_{k\to\infty}\frac{\mu(a_k)|u(a_k)\varphi'(a_k)|}{(1-|\varphi(a_k)|^2)^{m+1}\left(\log\frac{3}{1-|\varphi(a_k)|^2}\right)^2} + \|D^m_{\varphi,u}\|_e \\
&\geq \limsup_{k\to\infty}\frac{\mu(a_k)|u'(a_k)|}{(1-|\varphi(a_k)|^2)^{m}\log\frac{3}{1-|\varphi(a_k)|^2}}. \qquad (28)
\end{aligned}$$

Combining (26) and (28), it follows that

$$\|D^m_{\varphi,u}\|_e \succeq \limsup_{k\to\infty} \frac{u(a_k)|u'(a_k)|}{(1-|\varphi(a_k)|^2)^m \log\frac{3}{1-|\varphi(a_k)|^2}} = \limsup_{|\varphi(z)|\to 1} \frac{u(z)|u'(z)|}{(1-|\varphi(z)|^2)^m \log\frac{3}{1-|\varphi(z)|^2}}. \tag{29}$$

For $s\in(0,1)$, by (3), we consider

$$\begin{aligned}
&\frac{j^m\|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}} \preceq \frac{j^m}{\log(j+1)}\sup_{z\in\mathbb{D}}\mu(z)|u'(z)||\varphi(z)|^j\\
&\le \frac{j^m}{\log(j+1)}\sup_{|\varphi(z)|\le s}\mu(z)|u'(z)||\varphi(z)|^j + \frac{j^m}{\log(j+1)}\sup_{|\varphi(z)|>s}\mu(z)|u'(z)||\varphi(z)|^j\\
&\le L_1\frac{j^m s^j}{\log(j+1)} + \frac{j^m}{\log(j+1)}\sup_{|\varphi(z)|>s}\mu(z)|u'(z)||\varphi(z)|^j.
\end{aligned}$$

By lemma 2.5, the above inequalities imply that

$$\begin{aligned}
&\frac{j^m}{\log(j+1)}\sup_{|\varphi(z)|>s}\mu(z)|u'(z)||\varphi(z)|^j\\
&= \frac{j^m}{\log(j+1)}\sup_{|\varphi(z)|>s}\mu(z)|u'(z)|\frac{|\varphi(z)|^j(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}{(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}\\
&\le \frac{j^m}{\log(j+1)}f_j(x_j)\sup_{|\varphi(z)|>s}\frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}.
\end{aligned}$$

Thus by (7) we obtain that

$$\begin{aligned}
&\limsup_{j\to\infty}\frac{j^m\|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}} \preceq \limsup_{j\to\infty} L_1\frac{j^m s^j}{\log(j+1)}\\
&+\limsup_{j\to\infty}\frac{j^m}{\log(j+1)}f_j(x_j)\sup_{|\varphi(z)|>s}\frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}\\
&= \frac{m^m}{e^m}\sup_{|\varphi(z)|>s}\frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}.
\end{aligned}$$

Further we have that

$$\begin{aligned}
\limsup_{j\to\infty}\frac{j^m\|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}} &\preceq \lim_{s\to 1}\sup_{|\varphi(z)|>s}\frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}\\
&\le \limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|)^m\log\frac{3}{1-|\varphi(z)|}}\\
&\preceq \limsup_{|\varphi(z)|\to 1}\frac{\mu(z)|u'(z)|}{(1-|\varphi(z)|^2)^m\log\frac{3}{1-|\varphi(z)|^2}}.
\end{aligned} \tag{30}$$

Combining (29) and (30), it's clear that

$$\limsup_{j\to\infty}\frac{j^m\|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}} \preceq \|D^m_{\varphi,u}\|_e.$$

Similarly,

$$\begin{aligned}
&\frac{j^m\|I_u\varphi^{j+1}\|_\mu}{\|z^j\|_{\log}} \preceq \frac{j^m}{\log(j+1)}\sup_{z\in\mathbb{D}}(j+1)\mu(z)|\varphi(z)|^j|u(z)\varphi'(z)| \\
&\preceq \frac{j^{m+1}}{\log(j+1)}\sup_{z\in\mathbb{D}}\mu(z)|\varphi(z)|^j|u(z)\varphi'(z)| \\
&\le L_2\frac{j^{m+1}}{\log(j+1)}\sup_{|\varphi(z)|\le s}|\varphi(z)|^j + \frac{j^{m+1}}{\log(j+1)}\sup_{|\varphi(z)|>s}\mu(z)|\varphi(z)|^j|u(z)\varphi'(z)| \\
&\le L_2\frac{j^{m+1}}{\log(j+1)}s^j + \frac{j^{m+1}}{\log(j+1)}\sup_{|\varphi(z)|>s}\mu(z)|\varphi(z)|^j|u(z)\varphi'(z)|.
\end{aligned}$$

By lemma 2.5, the above inequalities imply that

$$\begin{aligned}
&\frac{j^{m+1}}{\log(j+1)}\sup_{|\varphi(z)|>s}\mu(z)|\varphi(z)|^j|u(z)\varphi'(z)| \\
&= \frac{j^{m+1}}{\log(j+1)}\sup_{|\varphi(z)|>s}\mu(z)|u(z)\varphi'(z)|\frac{|\varphi(z)|^j(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}}{(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}} \\
&\le \frac{j^{m+1}}{\log(j+1)}x_j^j(1-x_j)^{m+1}\log\frac{3}{1-x_j}\sup_{|\varphi(z)|>s}\frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}}.
\end{aligned}$$

Then by (7), we obtain that

$$\begin{aligned}
&\limsup_{j\to\infty}\frac{j^m\|I_u\varphi^{j+1}\|_\mu}{\|z^j\|_{\log}} \preceq \limsup_{j\to\infty}L_2\frac{j^{m+1}}{\log(j+1)}s^j \\
&+\limsup_{j\to\infty}\frac{j^{m+1}}{\log(j+1)}x_j^j(1-x_j)^{m+1}\log\frac{3}{1-x_j}\sup_{|\varphi(z)|>s}\frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}} \\
&= \frac{(m+1)^{m+1}}{e^{m+1}}\sup_{|\varphi(z)|>s}\frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}}.
\end{aligned}$$

Further we get that

$$\begin{aligned}
\limsup_{j\to\infty}\frac{j^m\|I_u\varphi^{j+1}\|_\mu}{\|z^j\|_{\log}} &\preceq \lim_{s\to1}\sup_{|\varphi(z)|>s}\frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}} \\
&\le \limsup_{|\varphi(z)|\to1}\frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|)^{m+1}\log\frac{3}{1-|\varphi(z)|}} \\
&\preceq \limsup_{|\varphi(z)|\to1}\frac{\mu(z)|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{m+1}\log\frac{3}{1-|\varphi(z)|^2}}. \qquad (31)
\end{aligned}$$

Combining (27) and (31), it follows that

$$\limsup_{j\to\infty}\frac{j^m\|I_u\varphi^{j+1}\|_\mu}{\|z^j\|_{\log}} \preceq \|D^m_{\varphi,u}\|_e.$$

This completes the proof. □

The following corollary is a consequence of theorem 4.3 and lemma 2.4.

Corollary 4.4 *Let $u \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, $m \in \mathbb{N}$ and μ be a weight on $\mathbb{D}$. Suppose that the operator $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is bounded. Then $D^m_{\varphi,u} : \mathcal{B}_{\log} \to \mathcal{B}_\mu$ is compact if and only if either of the following statements holds:*

(a)

$$\limsup_{j\to\infty} \frac{j^m \|J_u\varphi^j\|_\mu}{\|z^j\|_{\log}} = 0 \ \textit{ and } \ \limsup_{j\to\infty} \frac{j^m \|I_u\varphi^{j+1}\|_\mu}{\|z^j\|_{\log}} = 0.$$

(b)

$$\limsup_{j\to\infty} \frac{j^m \|J_u\varphi^j\|_\mu}{\log(j+1)} = 0 \ \textit{ and } \ \limsup_{j\to\infty} \frac{j^m \|I_u\varphi^{j+1}\|_\mu}{\log(j+1)} = 0.$$

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