COMPACT INTERTWINING RELATIONS FOR COMPOSITION OPERATORS ON H^∞ AND THE BLOCH SPACES

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Abstract. On the space of bounded analytic functions and the Bloch space on the unit disk, we study the compact intertwining relations for composition operators, whose intertwining operators are Volterra type operators. Further, we consider the compact intertwining relations, which are between the whole collection of composition operators and some Volterra operator, and the whole collection of bounded Volterra operators and some composition operator.

1. Introduction

If X and Y are two Banach spaces, the symbol $\mathscr{B}(X,Y)$ denotes the collection of all bounded linear operators from X to Y. Let $\mathcal{K}(X,Y)$ be the collection of all compact elements of $\mathscr{B}(X,Y)$, and let $\mathscr{Q}(X,Y)$ be the quotient set $\mathscr{B}(X,Y)/\mathcal{K}(X,Y)$.

For linear operators $A \in \mathscr{B}(X,X)$, $B \in \mathscr{B}(Y,Y)$ and $T \in \mathscr{B}(X,Y)$, the phrase "T intertwines A and B in $\mathscr{Q}(X,Y)$" (or "T intertwines A and B compactly") means that

$$TA = BT \mod \mathcal{K}(X,Y) \quad \text{with} \quad T \neq 0. \tag{1.1}$$

The notation $A \propto_K B$ (T) represents the relation in equation (1.1). In fact, if T is an invertible operator on X, then the relation $\propto_K$ is symmetric.

Recall that the essential norm of a bounded linear operator T is the distance from T to the compact operators, that is,

$$\|T\|_e = \inf\{\|T-K\| : K \text{ is compact}\}.$$

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Notice that $\|T\|_e = 0$ if and only if T is compact. So estimates on $\|T\|_e$ lead to conditions for T to be compact.

Let $\mathbb{D}$ be the unit disk in the complex plane. Denote by $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, and $S(\mathbb{D})$ the collection of all the holomorphic self-mappings of $\mathbb{D}$. Every $\varphi \in S(\mathbb{D})$ induces a composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$.

Let $g \in H(\mathbb{D})$. The Volterra operator J_g is defined by

$$J_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta,\ z \in \mathbb{D},\ \ f \in H(\mathbb{D});$$

and another integral operator I_g is defined by

$$I_g f(z) = \int_0^z f'(\zeta) g(\zeta) d\zeta,\ z \in \mathbb{D},\ \ f \in H(\mathbb{D}).$$

The notation $H^\infty(\mathbb{D})$ represents the algebra of bounded holomorphic functions with $\|\cdot\|_\infty$ as its supreme norm. For $\alpha > 0, \beta \in \mathbb{R}$, the Bloch type space $\mathcal{B}_{\alpha,\log^\beta}$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_* := \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha \log^\beta \frac{2}{1-|z|^2} |f'(z)| < \infty.$$

Then $\|\cdot\|_*$ is a complete semi-norm on $\mathcal{B}_{\alpha,\log^\beta}$, which is Möbius invariant. We denote the Banach space associated to $\mathcal{B}_{\alpha,\log^\beta}$ by $\tilde{\mathcal{B}}_{\alpha,\log^\beta}$, where the norm is given by the formula

$$\|f\|_{\tilde{\mathcal{B}}_{\alpha,\log^\beta}} = |f(0)| + \|f\|_*.$$

The little Bloch space, denoted by $\mathcal{B}_{\alpha,\log^\beta;0}$, consists of $f \in \mathcal{B}_{\alpha,\log^\beta}$ for which

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha \log^\beta \frac{2}{1-|z|^2} |f'(z)| = 0.$$

In this paper, we abbreviate $\mathcal{B} = \mathcal{B}_{1,\log^0}$ and $\mathcal{B}_{\log} = \mathcal{B}_{1,\log^1}$.

Composition operators were studied intensively in the past a few decades. A lot of efforts have been made on characterizing bounded and compact composition operators on spaces of analytic functions, for example, [12] for Hardy space and [10] for Bloch spaces. Interested readers may refer to books [4, 13, 25] and some recent papers [22, 23, 24] to learn much more on this subject.

The discussion of J_g first arose in connection with semigroups of composition operators, and readers may refer to [14] for background. Recently, the problem of characterizing the boundedness and compactness of J_g and I_g on various spaces of analytic functions has attracted considerable attention. For example, the boundedness of J_g on Hardy spaces, Bergman spaces, BMOA space, Bloch space and $\mathcal{Q}_p$ space are characterized in [1, 2, 14, 18, 20, 21], respectively. The same problems for the product of composition and Volterra operators on kinds of function spaces on the open unit disk of the plane have also been discussed, see some recent papers [7, 8, 9].

Based on these results, we consider the composition operator $C_\varphi : X \to X$, and the integral-type operator V_g $(= J_g$ or $I_g) : X \to \mathcal{B}$ where X represents H^∞ or $\mathcal{B}$. We are interested in the compact intertwining relations

$$C_\varphi \mid_X \propto_K C_\varphi \mid_{\mathcal{B}} \quad (V_g \mid_{X\to\mathcal{B}}) \tag{1.2}$$

If (1.2) holds for any $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, we may also say that C_φ and V_g *essentially commute*. It is of special interest to us to decide when a given Volterra operator essentially commutes with every composition operator and when a given composition operator essentially commutes with every Volterra operator.

Two main Questions in this paper are:

(Q1): what properties should a non-constant g have, if

$$C_\varphi \mid_X \propto_K C_\varphi \mid_{\mathcal{B}} \quad (V_g \mid_{X\to\mathcal{B}})$$

holds for every $\varphi \in S(\mathbb{D})$; and

(Q2): what properties should a $\varphi \in S(\mathbb{D})$ have, if

$$C_\varphi \mid_X \propto_K C_\varphi \mid_{\mathcal{B}} \quad (V_g \mid_{X\to\mathcal{B}})$$

holds for every bounded V_g?

For simplicity, if g satisfies conditions in **(Q1)**, we write $C \mid_X \propto_K C \mid_{\mathcal{B}} (V_g)$; if φ satisfies conditions in **(Q2)**, we write $C_\varphi \mid_X \propto_K C_\varphi \mid_{\mathcal{B}}$ $(V \mid_{X\to\mathcal{B}})$. By the way, the collections of g satisfying conditions similar as **(Q1)** was called the *universal set* of V_g by the authors in [16, 17]. Our use of the term "universal set" should not be confused with the notion of "universal set" which appears in the dynamical theory of linear operators.

In the following discussion, we write $A \lesssim B$ if there exists an absolute constant C such that $A \le C \cdot B$, and $A \approx B$ represents $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

Before the discussion of our main results, we need some preliminary notation and propositions. We state them without proof.

For $\varphi \in S(\mathbb{D})$, denote the *Schwarz derivative* of φ by

$$\varphi^{\#}(z) := \frac{1-|z|^2}{1-|\varphi(z)|^2}\varphi'(z).$$

From the Schwarz Lemma we know that $|\varphi^{\#}(z)| \le 1$, and the equality holds if and only if φ is an automorphism of the unit disk. The following lemma characterizes bounded and compact composition operators on $\mathcal{B}$ and H^∞(see [10] and [4]).

Lemma 2.1. *If $\varphi \in S(\mathbb{D})$, then*

(1): *Every φ induces an bounded composition operator on $\mathcal{B}$ and H^∞.*

(2): *$C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if C_φ is bounded and*

$$\lim_{|\varphi(z)|\to 1} |\varphi^{\#}(z)| = 0.$$

(3): *$C_\varphi : H^\infty \to H^\infty$ is compact if and only if $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$.*

The following criterion for compactness follows from standard arguments, that the proof is similar to the method of Proposition 3.11 in [4]. Hence we omit the details.

Lemma 2.2. *Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $C_\varphi V_g - V_g C_\varphi$ is compact from X to $\mathcal{B}$ if and only if for any bounded sequence $\{f_k\}$, $k = 1, 2, ...$ in X which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\|(C_\varphi V_g - V_g C_\varphi) f_k\|_* \to 0$ as $k \to \infty$.*

Recall that the notation $\mathbb{C}^N$ represents the N dimensional complex Euclidean space. Denote the unit ball of $\mathbb{C}^N$ by B_N. If $z, w \in B_N$, we define Möbius transform by

$$\Phi_w(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle},$$

where $P_a(z) = \frac{\langle z,a\rangle}{\langle a,a\rangle} a$, $Q_a(z) = z - P_a(z)$ and $s_a = \sqrt{1 - |a|^2}$. The following Lemma will be used in Section 4, which was first presented by Berndtsson in [3].

Lemma 2.3. *Let $\{x_i\}$ be a sequence in the ball B_N satisfying*

$$\prod_{j:j\neq k} |\Phi_{x_j}(x_k)| \ge d > 0 \quad \text{for any} \quad k. \tag{2.1}$$

Then there exists a number $M = M(d) < \infty$ and a sequence of functions $h_k \in H^\infty(B_N)$ such that

$$(a)\ \ h_k(x_j) = \delta_{kj}; \quad (b)\ \ \sum_k |h_k(z)| \le M \quad \text{for} \quad |z| < 1. \tag{2.2}$$

(The symbol δ_{kj} is equal to 1 if $k = j$ and 0 otherwise.)

The next lemma was proved by Carl Toews in [15].

Lemma 2.4. *Let* $\{z_n\} \subset B_N$ *be a sequence with* $|z_n| \to 1$ *as* $n \to \infty$. *Then for any given* $d \in (0,1)$ *there is a subsequence such that* $\{x_i\} := \{z_{n_i}\}$ *satisfies* (2.1).

From this lemma, there is always a subsequence which satisfies (2.1) for every sequence converging to the boundary of B_N, and Lemma 2.2 holds for this subsequence. We just need the result in one dimension.

To get some simple consequences of our main problems, we will consider the situation in the little Bloch setting. The next lemma is well known, see [11].

Lemma 2.5. *A closed set* K *in* $\mathcal{B}_0$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K}(1-|z|^2)|f'(z)| = 0.$$

In our discussion, we will use the boundedness of operators J_g and I_g, from H^∞ or $\mathcal{B}$ to $\mathcal{B}$. Several characterizations are listed below.

Lemma 2.6. *Suppose that* $g \in H(\mathbb{D})$. *Then*

- **(Corollary 4 in** [7]**):** $J_g : H^\infty \to \mathcal{B}$ *is bounded if and only if* $g \in \mathcal{B}$*;*
- **(Theorem 3 in** [9]**):** $J_g : \mathcal{B} \to \mathcal{B}$ *is bounded if and only if* $g \in \mathcal{B}_{\log}$*;*
- **(Corollary 1 in** [7] **and Theorem 14 in** [9]**):** $I_g : H^\infty$ *or* $\mathcal{B} \to \mathcal{B}$ *is bounded if and only if* $g \in H^\infty$.

Some definitions and results in Geometric Function Theory are needed, and interested readers can refer to [5] and [6]. For $\zeta \in \partial\mathbb{D}$ and $M > 1$ the *nontangential approaching region* at ζ is defined by

$$\Gamma(\zeta, M) = \{z \in \mathbb{D} : |z-\zeta| < M(1-|z|^2)\}.$$

A function f is said to have a *nontangential limit* at ζ if $\lim_{z\to\zeta} f(z)$ exists in each nontangential region $\Gamma(\zeta, M)$, and we denote it by $\angle - \lim_{z\to\zeta} f(z)$. If $\varphi \in S(\mathbb{D})$ and $\zeta \in \partial\mathbb{D}$, we will call ζ a *boundary fixed point* of φ if

$$\angle - \lim_{z\to\zeta} \varphi(z) = \zeta.$$

We say φ has a *finite angular derivative* at $\zeta \in \partial\mathbb{D}$ if there is $\eta \in \partial\mathbb{D}$ so that$(\varphi(z)-\eta)/(z-\zeta)$ has finite nontangential limit as $z \to \zeta$. When it exists as a finite complex number, this limit is denoted $\varphi'(\zeta)$. A $\varphi \in S(\mathbb{D})$ is said to be *parabolic type* if φ has a boundary fixed point ζ with $\varphi'(\zeta) = 1$. If φ is parabolic type, $\varphi(z) \to \zeta$ and $\varphi'(z) \to 1$ as $z \to \zeta$ unrestricted in the unit disk, we say ζ is a C^1 parabolic boundary fixed point of φ.

3. The case of intertwining operator I_g

First we consider $C_\varphi I_g - I_g C_\varphi$ as an operator from the Bloch space to itself.

Theorem 3.1. *Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $C_\varphi I_g - I_g C_\varphi$ is bounded on the Bloch space if and only if*

$$\sup_{z\in\mathbb{D}} |\varphi^{\#}(z)| \cdot |g(\varphi(z)) - g(z)| < \infty. \tag{3.1}$$

Proof. Suppose (3.1) holds, we will show that $C_\varphi I_g - I_g C_\varphi$ is bounded on $\mathcal{B}$. For any $f \in \mathcal{B}$,

$$\begin{aligned}((C_\varphi I_g - I_g C_\varphi)f)(z) &= C_\varphi \int_0^z f'(\zeta)g(\zeta)d\zeta - I_g f(\varphi(z))\\ &= \int_0^{\varphi(z)} f'(\zeta)g(\zeta)d\zeta - \int_0^z (f\circ\varphi)'(\zeta)g(\zeta)d\zeta.\end{aligned}$$

$$\begin{aligned}&\|(C_\varphi I_g - I_g C_\varphi)f\|_*\\ =\ & \sup_{z\in\mathbb{D}}(1-|z|^2)\,|\varphi'(z)f'(\varphi(z))g(\varphi(z)) - \varphi'(z)f'(\varphi(z))g(z)|\\ =\ & \sup_{z\in\mathbb{D}}(1-|z|^2)\,|\varphi'(z)f'(\varphi(z))(g(\varphi(z)) - g(z))|\\ =\ & \sup_{z\in\mathbb{D}} |\varphi^{\#}(z)|\,|g(\varphi(z)) - g(z)|\,(1-|\varphi(z)|^2)\,|f'(\varphi(z))|\\ \leq\ & \sup_{z\in\mathbb{D}} |\varphi^{\#}(z)|\,|g(\varphi(z)) - g(z)| \cdot \|f\|_*.\end{aligned}$$

From which we obtain that $C_\varphi I_g - I_g C_\varphi$ is bounded by (3.1).

Now suppose that $C_\varphi I_g - I_g C_\varphi$ is bounded on $\mathcal{B}$, then there is a constant C such that

$$\|(C_\varphi I_g - I_g C_\varphi)f\|_* \leq \|(C_\varphi I_g - I_g C_\varphi)f\|_{\mathcal{B}} < C$$

for $\|f\|_{\mathcal{B}} \leq 1$. We will prove condition (3.1). Suppose not, there exists a sequence $\{w_n\}$ in $\mathbb{D}$ such that

$$\lim_{n\to\infty} |\varphi^{\#}(w_n)||g(\varphi(w_n)) - g(w_n)| = \infty.$$

Let $\alpha_n(z) = \frac{\varphi(w_n)-z}{1-\overline{\varphi(w_n)}z}$, and $\alpha_n'(z) = -\frac{1-|\varphi(w_n)|^2}{(1-\overline{\varphi(w_n)}z)^2}$ for $n = 1, 2, ...$. It is easy to check that $\|\alpha_n\|_* = 1$.

$$\begin{aligned}
& \|(C_\varphi I_g - I_g C_\varphi)\alpha_n\|_* \\
= & \sup_{z\in\mathbb{D}} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)| \cdot (1 - |\varphi(z)|^2)|\alpha_n'(\varphi(z))| \\
\geq & (1 - |\varphi(w_n)|^2)\frac{1 - |\varphi(w_n)|^2}{|(1 - \overline{\varphi(w_n)}\varphi(w_n))^2|} \cdot |\varphi^{\#}(w_n)||g(\varphi(w_n)) - g(w_n)| \\
= & |\varphi^{\#}(w_n)||g(\varphi(w_n)) - g(w_n)| \to \infty.
\end{aligned}$$

That is impossible. So (3.1) holds. □

When we investigate essential commutativity of C_φ and I_g, we need add the condition $g \in H^\infty$ to ensure the boundedness of I_g on the Bloch space, see Lemma 2.5.

Theorem 3.2. *Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H^\infty(\mathbb{D})$. Then C_φ and I_g are essentially commutative on $\mathcal{B}$ if and only if*

$$\lim_{|\varphi(z)|\to 1} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)| = 0. \tag{3.2}$$

Proof. Sufficiency. Note that $g \in H^\infty$ and $\|\varphi^{\#}\|_\infty \leq 1$, we have

$$\sup_{z\in\mathbb{D}} |\varphi^{\#}(z)| \cdot |g(\varphi(z)) - g(z)| < \infty.$$

From which we conclude that $C_\varphi I_g - I_g C_\varphi$ is a bounded operator by Theorem 3.1. For any bounded sequence $\{f_k\}$ in $\mathcal{B}$ converging to zero uniformly on compact subsets of $\mathbb{D}$ with $\|f_k\|_{\mathcal{B}} \leq M$. From (3.2), it follows that for any small $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|\varphi^{\#}(z)||g(\varphi(z)) - g(z)| < \frac{\varepsilon}{M}$$

with $\varphi(z) \in \mathbb{D} \setminus (1 - \delta)\mathbb{D}$.

$$\begin{aligned}
& \|(C_\varphi I_g - I_g C_\varphi)f_k\|_* \\
= & \sup_{z\in\mathbb{D}} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)|(1 - |\varphi(z)|^2)|f_k'(\varphi(z))| \\
\leq & (A) + (B),
\end{aligned} \tag{3.3}$$

where

$$(A) = \sup_{\varphi(z)\in(1-\delta)\mathbb{D}} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)|(1 - |\varphi(z)|^2)|f_k'(\varphi(z))|,$$

and

$$(B) = \sup_{\varphi(z)\in\mathbb{D}\setminus(1-\delta)\mathbb{D}} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)|(1 - |\varphi(z)|^2)|f_k'(\varphi(z))|.$$

It is obvious that $(A) < \varepsilon$ for sufficient large k , and

$$(B) < \frac{\varepsilon}{M} \cdot \sup_{z\in\mathbb{D}}(1-|\varphi(z)|^2)|f_k'(\varphi(z))| = \frac{\varepsilon}{M}\|f_k\|_* < \varepsilon.$$

Thus $C_\varphi I_g - I_g C_\varphi$ is compact on $\mathcal{B}$.

Necessity. If $C_\varphi I_g - I_g C_\varphi$ is compact on $\mathcal{B}$, certainly $C_\varphi I_g - I_g C_\varphi$ is a bounded operator. Therefore

$$\sup_{z\in\mathbb{D}} |\varphi^{\#}(z)| \cdot |g(\varphi(z)) - g(z)| < \infty$$

by Theorem 3.1. Suppose the

$$\sup_{|\varphi(z)|\to 1} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)| \neq 0,$$

then there is a sequence $\{w_n\}$ in $\mathbb{D}$ with $|\varphi(w_n)| \to 1$ and an $\epsilon > 0$ such that

$$|\varphi^{\#}(w_n)||g(\varphi(w_n)) - g(w_n)| > \epsilon. \tag{3.4}$$

Let

$$h_n(z) = \frac{1-|\varphi(w_n)|^2}{1-\overline{\varphi(w_n)}z}. \tag{3.5}$$

A simple computation shows that

$$h_n'(z) = (1-|\varphi(w_n)|^2)\frac{\overline{\varphi(w_n)}}{(1-\overline{\varphi(w_n)}z)^2}.$$

So $\|h_n\|_* \leq 1$ and $\{h_n\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. By Lemma 2.2, it follows that $\|(C_\varphi I_g - I_g C_\varphi)h_n\|_* \to 0$. On the other hand, by (3.4), we have

$$\begin{aligned}
& \|(C_\varphi I_g - I_g C_\varphi)h_n\|_* \\
= & \sup_{z\in\mathbb{D}} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)| \cdot (1-|\varphi(z)|^2)|h_n'(\varphi(z))| \\
= & \sup_{z\in\mathbb{D}} |\varphi^{\#}(z)||g(\varphi(z)) - g(z)| \cdot (1-|\varphi(z)|^2)\frac{|\varphi(w_n)|(1-|\varphi(w_n)|^2)}{|1-\overline{\varphi(w_n)}\varphi(z)|^2} \\
\geq & |\varphi^{\#}(w_n)||g(\varphi(w_n)) - g(w_n)||\varphi(w_n)| > \epsilon
\end{aligned}$$

when $n \to \infty$. And we find a contradiction. So (3.2) holds when C_φ and I_g essentially commute. □

Corollary 3.3. *Let $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, then $C_\varphi I_g - I_g C_\varphi : H^\infty \to \mathcal{B}$ is bounded if and only if (3.1) holds; $C_\varphi I_g - I_g C_\varphi : H^\infty \to \mathcal{B}$ is compact if and only if (3.2) holds.*

Proof. To prove this corollary, we only adjust (3.5), in the proof of Theorem 3.2, as

$$f_n(z) = \frac{1-|\varphi(w_n)|^2}{1-\overline{\varphi(w_n)}z} \cdot \frac{\varphi(w_n)-z}{1-\overline{\varphi(w_n)}z}.$$

The other proofs are similar to that of Theorems 3.1 and 3.2. We omit the details. □

Using Lemma 2.4, we can easily get the following corollary in the little Bloch setting. The method is as before and we omit its proof.

Corollary 3.4. *Let $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$, and let X represent the space H^∞, $\mathcal{B}$ or $\mathcal{B}_0$. The following three statements are equivalent:*

(a): *$C_\varphi I_g - I_g C_\varphi : X \to \mathcal{B}_0$ is bounded;*
(b): *$C_\varphi I_g - I_g C_\varphi : X \to \mathcal{B}_0$ is compact;*
(c): $\lim_{|z|\to 1} |\varphi^\#(z)||g(\varphi(z)) - g(z)| = 0$.

Now we consider **(Q1)** raised in the first section:

- When does $C\mid_X \propto_K C\mid_{\mathcal{B}}\ (V_g)$ hold?

Theorem 3.5. *$C\mid_X \propto_K C\mid_{\mathcal{B}}\ (V_g)$ holds if and only if g is a constant.*

Proof. Sufficiency is obvious. To verify the necessity, just consider φ as any automorphism in condition (3.2). Then maximum modulus theorem implies that g must be a constant. □

We have following result which answers **(Q2)** in part.

Proposition 3.6. *Suppose $\varphi \in S(\mathbb{D})$ and $g \in H^\infty$ and let X denote either $\mathcal{B}$ or H^∞. If φ satisfies*

$$\lim_{|\varphi(z)|\to 1} \frac{|\varphi(z)-z|}{(1-\max\{|z|,|\varphi(z)|\})^2} \left|\varphi^\#(z)\right| = 0, \tag{3.6}$$

then $C_\varphi\mid_X \propto_K C_\varphi\mid_{\mathcal{B}} \quad (I\mid_{X\to\mathcal{B}})$.

Proof. By the Cauchy integral formula, one finds that

$$|g'(z)| \le \frac{\|g\|_\infty}{(1-|z|)^2} \quad (\forall z \in \mathbb{D})$$

for $g \in H^\infty$. The proposition will be proved immediately by noting that

$$\begin{aligned}
& |\varphi^\#(z)||g(\varphi(z)) - g(z)| \\
= \ & |\varphi^\#(z)| \left| \int_z^{\varphi(z)} g'(\zeta) \mathrm{d}\zeta \right| \\
\leq \ & |\varphi^\#(z)| \frac{\|g\|_\infty}{(1 - \max\{|\varphi(z)|, |z|\})^2} \int_z^{\varphi(z)} |\mathrm{d}\zeta| \\
\leq \ & |\varphi^\#(z)| \|g\|_\infty \frac{|\varphi(z) - z|}{(1 - \max\{|\varphi(z)|, |z|\})^2}
\end{aligned}$$

□

According to Lemma 2.6, the operator $I_g : X \to \mathcal{B}$ is bounded if g is analytic in $\mathbb{D}$ and continuous to the boundary. The next theorem will answer **(Q2)** in part. We will get a necessary and sufficient condition of the compact intertwining relation, when g is the function in the disk algebra.

Theorem 3.7. *Let $\mathcal{A}$ be the disk algebra, that is the subspace of H^∞ whose elements are analytic in $\mathbb{D}$ and continuous to the unit circle. Then*

$$C_\varphi \mid_X \propto_K C_\varphi \mid_{\mathcal{B}} \quad (I_g \mid_{X \to \mathcal{B}})$$

holds for all $g \in \mathcal{A}$ if and only if

$$\lim_{|\varphi(z)| \to 1} |\varphi^\#(z)||\varphi(z) - z| = 0. \tag{3.7}$$

Proof. Necessity can be proved immediately by putting $g = \mathrm{id}$ in e-quation (3.2), we just prove sufficiency. Condition (3.7) is sufficient to make

$$C_\varphi \mid_X \propto_K C_\varphi \mid_{\mathcal{B}} \quad (I_g)$$

hold for each monomial $g(z) = z^n$. For each $h \in \mathcal{A}$, there is a polynomial sequence $\{g_n\}$ such that $g_n \to h$ in supreme norm. One has

$$\begin{aligned}
& \|C_\varphi I_h - I_h C_\varphi\|_{e,X\to\mathcal{B}} \\
= \ & \inf\{\|C_\varphi I_h - I_h C_\varphi - K\|_{X\to\mathcal{B}} : \ K \in \mathcal{K}(X,\mathcal{B})\} \\
\le \ & \|C_\varphi I_h - I_h C_\varphi - (C_\varphi I_{g_n} - I_{g_n} C_\varphi)\|_{X\to\mathcal{B}} \\
\le \ & 2\|C_\varphi\|_{X\to\mathcal{B}} \cdot \|I_h - I_{g_n}\|_{X\to\mathcal{B}} \\
= \ & 2\|C_\varphi\|_{X\to\mathcal{B}} \cdot \sup_{\|f\|_X\le 1} \left\| \int_0^z f'(\zeta)(h-g_n)(\zeta)\mathrm{d}\zeta \right\|_{\mathcal{B}} \\
\le \ & 2\|C_\varphi\|_{X\to\mathcal{B}} \cdot \|h-g_n\|_\infty \cdot \sup_{\|f\|_X\le 1} \left\| \int_0^z f'(\zeta)\mathrm{d}\zeta \right\|_{\mathcal{B}} \\
= \ & 2\|C_\varphi\|_{X\to\mathcal{B}} \cdot \|h-g_n\|_\infty
\end{aligned}$$

Hence $\|C_\varphi I_h - I_h C_\varphi\|_{e,X\to\mathcal{B}} = 0$ by taking $n \to \infty$. □

4. The case of intertwining operator J_g

First, we consider the case for $X = H^\infty$ in the main question.

Theorem 4.1. *Suppose that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then*

(1) *$C_\varphi J_g - J_g C_\varphi$ is bounded from H^∞ to $\mathcal{B}$ if and only if*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)|(g\circ\varphi)'(z) - g'(z)| < \infty; \tag{4.1}$$

(2) *$C_\varphi J_g - J_g C_\varphi$ is compact from H^∞ to $\mathcal{B}$ if and only if (4.1) holds and*

$$\lim_{|\varphi(z)|\to 1} (1-|z|^2)|(g\circ\varphi)'(z) - g'(z)| = 0. \tag{4.2}$$

Proof. Being similar to the proofs of Theorems 3.1 and 3.2, we just need some modifications.

$$\begin{aligned}
& \|(C_\varphi J_g - J_g C_\varphi)f\|_* \\
= \ & \left\| \int_0^{\varphi(z)} f(\zeta)g'(\zeta)d\zeta - \int_0^z f(\varphi(\zeta))g'(\zeta)d\zeta \right\|_* \\
= \ & \sup_{z\in\mathbb{D}} (1-|z|^2)|f(\varphi(z))g'(\varphi(z))\varphi'(z) - f(\varphi(z))g'(z)| \\
= \ & \sup_{z\in\mathbb{D}} (1-|z|^2)|(g\circ\varphi)'(z) - g'(z)||f(\varphi(z))|.
\end{aligned} \tag{4.3}$$

Sufficiency of the two items in the theorem is obvious from the last formula in (4.3).

Necessity of boundedness can be proved by computing test functions

$$f_n(z) = 1 - \frac{\varphi(w_n) - z}{1 - \overline{\varphi(w_n)}z},$$

where sequence $\{w_n\}$ violates equation (4.1).

Necessity of compactness. Suppose not, we can find a sequence $\{\varphi(w_n)\}$ converging to the boundary of $\mathbb{D}$ and $\epsilon_0 > 0$ such that

$$\lim_{n\to\infty}(1-|w_n|^2)|(g\circ\varphi)'(w_n)-g'(w_n)| > \epsilon_0. \tag{4.4}$$

Further we may assume that $\{\varphi(w_n)\}$ is interpolating. Then there exist functions $\{h_n\}$ in H^∞ for $\{\varphi(w_n)\}$ such that

$$h_n(\varphi(w_k)) = \begin{cases} 1 & n=k, \\ 0 & n\neq k. \end{cases} \tag{4.5}$$

and

$$\sum_n |h_n(z)| \le M < \infty, \tag{4.6}$$

by Lemma 2.3 and 2.2, or see [5]. Equation (4.6) guarantees that $\{h_n\}$ is bounded in H^∞ and converges to zero on compact subsets of $\mathbb{D}$.

$$\begin{aligned} & \|(C_\varphi J_g - J_g C_\varphi)h_n\|_* \\ = {} & \sup_{z\in\mathbb{D}}(1-|z|^2)|(g\circ\varphi'(z)-g'(z)||h_n(\varphi(z))| \\ \ge {} & (1-|w_n|^2)|(g\circ\varphi'(w_n)-g'(w_n)||h_n(\varphi(w_n))| \\ = {} & (1-|w_n|^2)|(g\circ\varphi)'(w_n)-g'(w_n)|. \end{aligned}$$

The last equation follows by (4.5). Letting $n\to\infty$, we find contradiction by (4.4). □

Corollary 4.2. *Let $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. The three following conditions are equivalent:*

- **(a):** $C_\varphi J_g - J_g C_\varphi : H^\infty \to \mathcal{B}_0$ *is bounded;*
- **(b):** $C_\varphi J_g - J_g C_\varphi : H^\infty \to \mathcal{B}_0$ *is compact;*
- **(c):** $\lim_{|z|\to 1}(1-|z|^2)|(g\circ\varphi)'(z)-g'(z)| = 0.$

For composition operator and J_g, the result of **(Q1)** turns out to be interesting. Note that $g \in \mathcal{B}$ implies that $\sup_{z\in\mathbb{D}}(1-|z|^2)|(g\circ\varphi)'(z) - g'(z)| < \infty$, thus $C_\varphi J_g - J_g C_\varphi$ is a bounded operator from H^∞ to $\mathcal{B}$. And

$$C_\varphi\,|_{H^\infty} \propto_K C_\varphi\,|_{\mathcal{B}} \quad (J_g\,|_{H^\infty\to\mathcal{B}})$$

if and only if (4.2) holds. Now, we can answer **(Q1)**.

Corollary 4.3. *If $g \in \mathcal{B}_0$, then*

$$C\,|_{H^\infty} \propto_K C\,|_{\mathcal{B}} \quad (J_g\,|_{H^\infty\to\mathcal{B}}). \tag{4.7}$$

Proof. Since g is in the little Bloch space, $(1-|z|^2)|g'(z)|$ tends to 0 whenever z tends to the boundary of the disk. So we have that

$$\begin{aligned} & \lim_{|\varphi(z)|\to 1}(1-|z|^2)|g'(\varphi(z))\varphi'(z)-g'(z)| \\ \leq & \lim_{|\varphi(z)|\to 1}|\varphi^{\#}(z)|(1-|\varphi(z)|^2)|g'(\varphi(z))| + \lim_{|\varphi(z)|\to 1}(1-|z|^2)|g'(z)|. \end{aligned}$$

Conditions $g \in \mathcal{B}$ and $|\varphi^{\#}| \leq 1$ imply that $C_\varphi J_g - J_g C_\varphi$ is compact from H^∞ to $\mathcal{B}$ for every self-mapping φ. □

In contrast to composition operators, C_φ, and operators of the form I_g, there are many non-constant functions g, which are actually the little Bloch functions, such that J_g essentially commutes with all the composition operators. Naturally, we are going to ask: Does J_g commuting essentially with all the composition operators imply that g is in the little Bloch space? The answer is positive. The next lemma is the key lemma to study **(Q1)**. The method of proof is the same as in our recent papers [16, 17].

Lemma 4.4. *If g is a Bloch function on the unit disk with the property that, for any rotation $\tau(z) = e^{\mathbf{i}t}z$, $g \circ \tau - g$ is in the little Bloch space, then g itself must be in the little Bloch space.*

Proof. Condition $g \in \mathcal{B}$ is necessary to ensure that $J_g : H^\infty \to \mathcal{B}$ is bounded. Since $g \circ \tau - g$ is in the little Bloch space, we have

$$\lim_{|z|\to 1}(1-|z|^2)\left|g'(e^{\mathbf{i}t}z)e^{\mathbf{i}t} - g'(z)\right| = 0. \tag{4.8}$$

It is necessary to estimate the upper bound of left formula in (4.8).

$$\begin{aligned} & (1-|z|^2)\left|g'(e^{\mathbf{i}t}z)e^{\mathbf{i}t} - g'(z)\right| \\ \leq & (1-|z|^2)\left|g'(e^{\mathbf{i}t}z)e^{\mathbf{i}t}\right| + (1-|z|^2)\left|g'(z)\right| \\ = & (1-|e^{\mathbf{i}t}z|^2)\left|g'(e^{\mathbf{i}t}z)\right| + (1-|z|^2)\left|g'(z)\right| \\ \leq & 2\|g\|_* \end{aligned}$$

Thus the left formula in equation (4.8) is finite independent of t, since $g \in \mathcal{B}$.

Suppose that $g(z)=\sum\limits_{n=0}^{\infty}a_nz^n$, then $g'(z)=\sum\limits_{n=1}^{\infty}na_nz^{n-1}$. Integrating with respect to t from 0 to 2π, and some calculations show that

$$\begin{aligned}
&\int_0^{2\pi}\lim_{|z|\to1}(1-|z|^2)\left|g'(e^{\mathbf{i}t}z)e^{\mathbf{i}t}-g'(z)\right|\mathrm{d}t\\
=\;&\lim_{|z|\to1}(1-|z|^2)\left|\int_0^{2\pi}g'(e^{\mathbf{i}t}z)e^{\mathbf{i}t}-g'(z)\mathrm{d}t\right|\\
=\;&\lim_{|z|\to1}(1-|z|^2)\left|\int_0^{2\pi}\sum_{n=1}^{\infty}\left(na_n(e^{\mathbf{i}t}z)^{n-1}e^{\mathbf{i}t}-na_nz^{n-1}\right)\mathrm{d}t\right|\\
=\;&\lim_{|z|\to1}(1-|z|^2)\left|\sum_{n=1}^{\infty}na_nz^{n-1}\int_0^{2\pi}(e^{\mathbf{i}nt}-1)\mathrm{d}t\right|\\
=\;&2\pi\cdot\lim_{|z|\to1}(1-|z|^2)|g'(z)|,
\end{aligned}$$

where the equation in the second line is true by the dominated convergence theorem. Thus we get $g\in\mathcal{B}_0$ from (4.8). □

Theorem 4.5. *$C\,|_{H^\infty}\propto_K C\,|_{\mathcal{B}}$ $(J_g\,|_{H^\infty\to\mathcal{B}})$ if and only if $g\in\mathcal{B}_0$.*

Proof. Sufficiency is stated in Corollary 4.3. To prove necessity, just consider the rotation of the disk. That is to say, by putting $\varphi(z)=\tau(z)=e^{\mathbf{i}t}z$ in the condition (4.2), and we have $g\in\mathcal{B}_0$ by Lemma 4.4. □

Considering the composition operators C_φ and Volterra operator J_g, we can obtain a necessary condition (Proposition 4.6) and a sufficient condition (Proposition 4.7) to answer **(Q2)** partially.

Proposition 4.6. *Let $\zeta\in\partial\mathbb{D}$, $\varphi\in S(\mathbb{D})$ and $\varphi(z)$ tend to the unit circle when z converges to ζ nontangentially. If*

$$C_\varphi\,|_{H^\infty}\propto_K C_\varphi\,|_{\mathcal{B}}\quad(J\,|_{H^\infty\to\mathcal{B}}),$$

then $\angle-\lim\limits_{z\to\zeta}\varphi(z)=\zeta$.

Proof. By Theorem 4.5, $\lim\limits_{|\varphi(z)|\to1}(1-|z|^2)|(g\circ\varphi-g)'(z)|=0$ holds for every $g\in\mathcal{B}$. Suppose that $\varphi(z)\to\partial\mathbb{D}$ as $z\to\zeta$ in some nontangential approaching region, such that $\eta\neq\zeta$ and $\varphi(z_n)\nrightarrow\eta$. let $g(z)=-\log(1-\bar{\zeta}z)$ in equation (4.7), then

$$\begin{aligned}
&\lim_{z\to\zeta}(1-|z|^2)\left|\frac{\bar{\zeta}\varphi'(z)}{1-\bar{\zeta}\varphi(z)}-\frac{\bar{\zeta}}{1-\bar{\zeta}z}\right|\\
=\;&\lim_{z\to\zeta}\frac{1-|z|^2}{|1-\bar{\zeta}z|}\left|\frac{\zeta-z}{\zeta-\varphi(z)}\varphi'(z)-1\right|=0.
\end{aligned}$$

For any $M>0$, if $z\in\Gamma(\zeta,M)=\{z\in\mathbb{D}:|\zeta-z|<M(1-|z|^2)\}$,

$$\angle-\lim_{z\to\zeta}\left|\frac{\zeta-z}{\zeta-\varphi(z)}\varphi'(z)-1\right|=0.$$

By noting that $|\zeta-z||\varphi'(z)|=\frac{|\zeta-z|}{1-|z|^2}\cdot(1-|\varphi(z)|^2)|\varphi^{\#}(z)|\to 0$, one has

$$\angle-\lim_{z\to\zeta}\varphi(z)=\zeta. \tag{4.9}$$

□

Proposition 4.7. *Let $\varphi\in S(\mathbb{D})$ be of parabolic type with C^1 boundary fixed point $\zeta\in\partial\mathbb{D}$. If*

$$\sup_{z\in\mathbb{D}}\frac{|\varphi(z)-z|}{(1-\max\{|z|,|\varphi(z)|\})^2}\leq\infty, \tag{4.10}$$

then $C_\varphi\mid_{H^\infty}\propto_K C_\varphi\mid_{\mathcal{B}}\ (J\mid_{H^\infty\to\mathcal{B}})$.

Proof. It is well known that $g\in\mathcal{B}$ if and only if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^n|g^{(n)}(z)|<\infty.$$

Note that

$$\begin{aligned}
& (1-|z|^2)|(g\circ\varphi)'(z)-g'(z)| \\
\leq\ & (1-|z|^2)|\varphi'(z)||g'(\varphi(z))-g'(z)|+(1-|z|^2)|g'(z)||\varphi'(z)-1| \\
=\ & (1-|z|^2)|\varphi'(z)|\left|\int_z^{\varphi(z)}g''(\zeta)\mathrm{d}\zeta\right|+(1-|z|^2)|g'(z)||\varphi'(z)-1| \\
\leq\ & (1-|z|^2)|\varphi'(z)|\sup_{z\in\mathbb{D}}(1-|z|^2)^2|g''(z)|\int_z^{\varphi(z)}\frac{|\mathrm{d}\zeta|}{(1-|\zeta|)^2} \\
& +(1-|z|^2)|g'(z)||\varphi'(z)-1| \\
\leq\ & (1-|z|^2)|\varphi'(z)|\sup_{z\in\mathbb{D}}(1-|z|^2)^2|g''(z)|\frac{|\varphi(z)-z|}{(1-\max\{|z|,|\varphi(z)|\})^2} \\
& +(1-|z|^2)|g'(z)||\varphi'(z)-1|
\end{aligned}$$

where the integral path is chosen to be the segment from z to $\varphi(z)$. Then $C_\varphi\mid_{H^\infty}\propto_K C_\varphi\mid_{\mathcal{B}}\ (J\mid_{H^\infty\to\mathcal{B}})$ follows immediately by those conditions in the proposition. □

The next several propositions concern C_φ and J_g as maps from $\mathcal{B}$ to itself.

Proposition 4.8. *Assume that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then $C_\varphi J_g - J_g C_\varphi$ is bounded from $\mathcal{B}$ to itself if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|(g\circ\varphi)'(z)-g'(z)|\log\frac{2}{1-|\varphi(z)|^2}<\infty. \tag{4.11}$$

Proof. Sufficiency can be verified by some straightforward computations and inequalities. Necessity will be proved by choosing the test function

$$f_w(z)=\log\frac{2}{1-\overline{w}z}.$$

□

Proposition 4.9. *Assume that $\varphi \in S(\mathbb{D})$ and $g \in H(\mathbb{D})$. Then the following statements are equivalent:*

(a): *$C_\varphi J_g - J_g C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact and (4.11) holds;*
(b): *$C_\varphi J_g - J_g C_\varphi : \mathcal{B}_0 \to \mathcal{B}_0$ is compact;*
(c): *$C_\varphi J_g - J_g C_\varphi : \mathcal{B}_0 \to \mathcal{B}_0$ is weakly compact;*
(d): *Condition (4.11) holds and*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)|(g\circ\varphi)'(z)-g'(z)|\log\frac{2}{1-|\varphi(z)|^2}=0; \tag{4.12}$$

(e): *$C_\varphi J_g - J_g C_\varphi : \mathcal{B} \to \mathcal{B}_0$ is compact;*
(f): *$C_\varphi J_g - J_g C_\varphi : \mathcal{B} \to \mathcal{B}_0$ is bounded;*

The compactness of operators $C_\varphi J_g$ and $J_g C_\varphi$ are characterized in Theorem 4 and Theorem 11 in [9]. The proofs are similar to those in Theorem 4 in [9], so we omit them.

To continue our discussion, we need an upper bound on the modulus of $\varphi(z)$ (see Corollary 2.40 in [4]).

Lemma 4.10. *If $\varphi \in S(\mathbb{D})$, then $|\varphi(z)| \le \frac{|z|+|\varphi(0)|}{1+|z||\varphi(0)|}$.*

Now we are ready to consider **(Q1)** for C_φ and J_g, where both of them are operators acting on $\mathcal{B}$.

Theorem 4.11. *Assume $g \in H(\mathbb{D})$ is such that J_g is bounded on the Bloch space. Then*

$$C\ |_{\mathcal{B}} \propto_K C\ |_{\mathcal{B}} \quad (J_g\ |_{H^\infty\to\mathcal{B}})$$

if and only if $g \in \mathcal{B}_{\log,0}$, that is

$$\lim_{|z|\to 1}(1-|z|^2)|g'(z)|\log\frac{2}{1-|z|^2}=0. \tag{4.13}$$

Proof. Following the method in the proof of Lemma 4.10, the necessity can be proved similarly.

Now suppose (4.13) holds. We have that

$$\begin{aligned} & (1-|z|^2)|(g\circ\varphi)'(z)-g'(z)|\log\frac{2}{1-|\varphi(z)|^2} \\ \leq\ & |\varphi^{\#}(z)|(1-|\varphi(z)|^2)|g'(\varphi(z)|\log\frac{2}{1-|\varphi(z)|^2} \\ & +(1-|z|^2)|g'(z)|\log\frac{2}{1-|z|^2}\cdot\frac{\log 2-\log(1-|\varphi(z)|^2)}{\log 2-\log(1-|z|^2)}. \end{aligned}$$

Just consider those z's tending to $\partial\mathbb{D}$ with $|\varphi(z)|\to 1$.

$$\begin{aligned} & \lim_{|z|\to 1^-}\frac{\log 2-\log(1-|\varphi(z)|^2)}{\log 2-\log(1-|z|^2)} \\ =\ & \lim_{|z|\to 1^-}\frac{-\log(1-|\varphi(z)|)}{-\log(1-|z|)} \\ \leq\ & \lim_{|z|\to 1^-}\frac{-\log(1-\frac{|z|+|\varphi(0)|}{1+|z||\varphi(0)|})}{-\log(1-|z|)} \\ =\ & \lim_{|z|\to 1^-}\frac{\log(1-|z|)(1-|\varphi(0)|)-\log(1+|z||\varphi(0)|)}{\log(1-|z|)} \\ =\ & \lim_{|z|\to 1^-}\frac{-\frac{1}{1-|z|}-\frac{|\varphi(0)|}{1+|z||\varphi(0)|}}{-\frac{1}{1-|z|}}=1 \end{aligned}$$

where Lemma 4.8 and L'Hospital Law are applied. Thus (4.12) holds for every $\varphi\in S(\mathbb{D})$ by (4.13). The proof is completed by Proposition 4.9. □

By the same method as in Proposition 4.6, ones can obtain a necessary condition for **(Q2)** when $X=\mathcal{B}$. Hence we omit the proof.

Proposition 4.12. *Let $\zeta\in\partial\mathbb{D}$, $\varphi\in S(\mathbb{D})$ and $\varphi(z)$ tend to the unit circle when z converges to ζ nontangentially. If*

$$C_\varphi\,|_{\mathcal{B}}\propto_K C_\varphi\,|_{\mathcal{B}}\quad (J\,|_{\mathcal{B}\to\mathcal{B}}),$$

then $\angle-\lim_{z\to\zeta}\varphi(z)=\zeta$.

Proposition 4.13. *Let $\varphi\in S(\mathbb{D})$ be parabolic type with C^1 boundary fixed point $\zeta\in\partial\mathbb{D}$. If*

$$\sup_{z\in\mathbb{D}}\frac{|\varphi(z)-z|}{1-\max\{|z|,|\varphi(z)|\}}\leq\infty, \tag{4.14}$$

then $C_\varphi\,|_{\mathcal{B}}\propto_K C_\varphi\,|_{\mathcal{B}}\ (J\,|_{\mathcal{B}})$.

Proof. The proposition follows by $\mathcal{B}_{\log} \subset \mathcal{B}$. In fact,

$$\begin{aligned} & \log \frac{2}{1-|\varphi(z)|^2} \approx \log \frac{2}{1-|\varphi(z)|} \leq \log \frac{2(1+|\varphi(z)||z|)}{(1-|z|)(1-|\varphi(z)|)} \\ \lesssim & \log \frac{2}{1-|z|} \approx \log \frac{2}{1-|z|^2} \end{aligned}$$

for z close enough to the unit circle. By estimating that

$$\begin{aligned} & (1-|z|^2)|(g\circ\varphi)'(z) - g'(z)| \log \frac{2}{1-|\varphi(z)|^2} \\ \lesssim & (1-|z|^2)|(g\circ\varphi)'(z) - g'(z)| \log \frac{2}{1-|z|^2} \\ \leq & (1-|z|^2) \log \frac{2}{1-|z|^2} |\varphi'(z)||g'(\varphi(z)) - g'(z)| \\ & + (1-|z|^2) \log \frac{2}{1-|z|^2} |g'(z)||\varphi'(z) - 1|, \end{aligned}$$

the proof will be completed as we did in Proposition 4.7. □

Remark. Question 1 concerns the subclass of the bounded Volterra operators, whose elements' essential commutants contain all the composition operators. Question 2 concerns the subclass of the composition operators, whose elements' essential commutants contain all the bounded Volterra operators. Answers to **(Q1)** are complete, but we can only find some sufficient or necessary conditions for **(Q2)**. It seems very difficult to answer **(Q2)** completely, since the boundary behavior of a function either in H^∞ or $\mathcal{B}$ can be rather wild.

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