Time-consistent Investment-reinsurance Strategies towards Joint Interests of the Insurer and the Reinsurer under CEV models

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Abstract The present paper studies time-consistent solutions to an investment-reinsurance problem under a mean-variance framework. The paper is distinguished from other literature by taking into account the interests of both an insurer and a reinsurer jointly. The claim process of the insurer is governed by a Brownian motion with a drift. A proportional reinsurance treaty is considered and the premium is calculated according to the expected value principle. Both the insurer and the reinsurer are assumed to invest in a risky asset, which is distinct for each other and driven by a constant elasticity of variance model. The optimal decision is formulated on a weighted sum of the insurer's and the reinsurer's surplus processes. Upon a verification theorem, which is established with a formal proof for a more general problem, explicit solutions are obtained for the proposed investment-reinsurance model. Moreover, numerous mathematical analysis and numerical examples are provided to demonstrate those derived results as well as the economic meanings behind.

Keywords Investment-reinsurance problem, Mean-variance analysis, Time-consistent strategy, Constant elasticity of variance model

1 Introduction

The quest for optimal reinsurance design has remained a fascinating problem among insurers, reinsurers and academicians. An appropriate use of reinsurance could reduce the underwriting risk of an insurer and thereby enhance its value. This explains why almost all the insurance companies throughout the world have a reinsurance program. Reinsurance has also been used as an effective mechanism of risk-sharing within an insurance group for various purposes, such as tax alleviation, stabilizing the profitability and satisfying external regulatory capital requirement. The optimal reinsurance design models can primarily be divided into two classes, known as *static models* and *dynamic models*. In the static models, the reinsurance agreement is reached in advance with fixed provisions to apply over a time horizon and therefore they are also called the single period models. Recent literature on the static models include [5], [13], [14], [15], [16], [51] and references therein.

In a dynamic model, the provisions of the reinsurance treaty are adjusted dynamically over time depending on the accumulated information. Typically, the model boils down to solve an optimal control problem. In a dynamic setting, the literature commonly study the optimal reinsurance strategy and investment strategy simultaneously, leading to the optimal investment-reinsurance problem. The optimality criteria adopted in the literature for such problem include minimizing the insurer's ruin probability (e.g., [6], [49] and [50]), maximizing the adjustment coefficient of the insurer's risk process (e.g., [33] and [42]), maximizing the insurer's expected utility (e.g., [7], [34], [44] and [52]), and mean-variance (MV) optimization (e.g., [8], [9] and [20]).

In the present paper, we focus on the MV criterion to study the optimal reinsurance-investment problem. The proportional reinsurance treaty is considered in our model, and moreover, both the insurer and

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reinsurer are assumed to invest on a risky asset, which is distinct for each other, in additional to a risk free asset. The dynamics of the risky assets for the insurer and the reinsurer to invest are driven by the so-called constant elasticity of variance (CEV) model, which has been widely applied due to its ability in capturing the volatility smile of financial asset returns as well as the leverage effect of the equity. The CEV model has also been commonly advocated in actuarial literature, see, for example, [27], [30], [31], [42], [43] and [56] among many others. To solve the proposed reinsurance-investment model, we embed it into a more general dynamic control problem and formally establish a verification theorem for the general problem, which may have potential applications for other relevant problems. Upon the established verification theorem, explicit solutions are obtained for the proposed reinsurance-investment model. Moreover, numerous sensitivity analysis is conducted, both mathematically and numerically with in-depth explanations on the economic meanings behind, to explore how the obtained investment-reinsurance strategies depend on various exogenous parameters, including weight parameter in the weighted sum process, the reinsurance premium rate, as well as those associated with the investment assets.

Along with many others, three main features distinguish the present paper from those existing literature on optimal investment-reinsurance problems. First, the interests of both an insurer and a reinsurer are jointly reflected in our optimal reinsurance model, while a majority of the existing literature study the optimal reinsurance problem (in both static and dynamic settings) merely from the insurer's perspective. Indeed, to the best of our knowledge, [13], [22], [26], [35], [38] and [39] are the only five papers with an optimal reinsurance model jointly reflecting the interests of both parties, and only Li et al. [38], [39] study the problems in dynamic settings with the optimality criterions of maximizing the product of the utilities and the weighted sum of objectives for both parties, respectively. In the present paper, we consider the weighted sum of the insurer's and the reinsurer's surplus processes and study the optimal reinsurance and investment strategies for both parties by a mean-variance analysis on the weighted process. The weight can be viewed as a regularization parameter to stress the importance of each party on the contract in the process of decision, and in the extreme case with the weight being equal to 1, our model reduces to the corresponding problem for a mean-variance insurer only. Thus, our model is more general than other literature in this aspect.

Second, our model can also be viewed as a decision formulation for an insurance group. As we know, in reality only the smallest insurers exist as a single corporation, and most major insurance companies actually exist as insurance groups, say, for example, Allianz SE, Munich Reinsurance Company among many others. Therefore, from the risk management point of view, a risk transfer among an insurance group is a natural risk management way for insurance companies, and such a risk transfer is often realized via certain reinsurance contracts between the insurer and the other members (or another member) within the group. While different member companies may have distinct capacity to absorb risks of different types, all the counter-parties in the reinsurance contracts can be generally viewed as a single identity in terms of absorbing the risk from the insurer. Therefore, the problem can be abstracted to a framework for an insurance group consisting of only two members — one insurer and one reinsurer. Throughout the paper, we shall frequently refer to such a framework to explain the economic meanings of the our results.

Third, the MV analysis for the optimal reinsurance-investment strategies is a time-inconsistent problem in that an optimal solution obtained at a time is no longer optimal as the time goes forward into a future point. In the present paper, we aim to develop time-consistent reinsurance-investment strategy, whereas the solutions obtained in the most previous literature, including those reviewed previously, are time-inconsistent. In the present paper, we view the entire decision process as a non-cooperative game with one player at each time point over the whole investment time horizon, who can be viewed as the future incarnation of the decision-maker, and resort to the (subgame perfect Nash) equilibrium strategy, which is a time-consistent solution. The equilibrium strategy has been popularly advocated for tackling time-inconsistent problems among academia in recent years; see, for instance, [11], [12], [23], [36], [37], [47], [53] and [54] among others. There are also some references on time-consistent solutions to reinsurance-investment problems; see, for example, [40], [41], and [57]. They, however, address the problem only from the insuer's perspective without taking into account any concern from the reinsurer.

The paper proceeds as follows. The model formulation is given in Section 2 along with a verification theorem for a general problem. The equilibrium solutions as well as some mathematical analysis are presented in Section 3, and the numerical analysis is demonstrated in Section 4. Section 5 concludes the paper, and all of the proofs are relegated to Appendices A and B.

2 Model formulation

2.1 Wealth processes

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t, t \geqslant 0\}$ satisfying the usual conditions, i.e., the filtration contains all P-null sets and is right continuous. All stochastic processes in this paper are assumed to be well defined and adapted to the filtration. Consider an insurer with a claim process described by a Brownian motion with a drift as follows

$$\mathrm{d}C(t) = a\mathrm{d}t - b\mathrm{d}\overline{W}(t), \tag{2.1}$$

where a and b are two positive constants and $\overline{W}$ is a standard Brownian motion. The diffusion model for the claim process is a limit of the classical compound Poisson model (see, e.g., [29]), and such an approximation has been widely adopted in actuarial literature (e.g., [6], [30], [40] and [49]). Further suppose that the insurance premium charged on the claims by the insurer is computed according to the expected value principle with a loading factor $\theta > 0$, so that the premium rate for the insurer is $c = (1 + \theta)a$, and this leads to the following surplus process for the insurer

$$\mathrm{d}R_1(t) = \theta a\mathrm{d}t + b\mathrm{d}\overline{W}(t).$$

To proceed, suppose that a proportional reinsurance is applied between the insurer and a reinsurer, and let $p(t)$ denote the proportion covered by the reinsurer at time t. Further suppose that the reinsurance premium is also charged according to the expected value principle with a safety loading $\eta > \theta$. Therefore, in the presence of the proportional reinsurance contract, the wealth processes of the insurer and the reinsurer are respectively given by

$$\begin{aligned} \mathrm{d}R_1(t) &= (1+\theta)a\mathrm{d}t - (1+\eta)ap(t)\mathrm{d}t - (1-p(t))\mathrm{d}C(t) \\ &= (\theta - \eta p(t))\, a\mathrm{d}t + b(1-p(t))\mathrm{d}\overline{W}(t) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \mathrm{d}R_2(t) &= (1+\eta)p(t)a\mathrm{d}t - p(t)\mathrm{d}C(t) \\ &= \eta p(t)a\mathrm{d}t + bp(t)\mathrm{d}\overline{W}(t). \end{aligned} \tag{2.3}$$

Besides the insurance business, both the insurer and the reinsurer are allowed to invest in a risk-free asset and a risky asset. We assume that the insurer and reinsurer specialize in two distinct risky assets in their investment. Asset specialization can be justified by that investors tend to trade only in familiar assets. Since analyzing stock data takes time and effort, each investor is likely to invest in familiar assets which are in a subset of all available assets. The price process of the risk-free asset is given by

$$\mathrm{d}B(t) = rB(t)\mathrm{d}t, \quad B(0) = B_0,$$

where $r > 0$ is the risk-free interest rate. The price process of the risky asset available for the insurer to invest is described by

$$\mathrm{d}S_1(t) = S_1(t)\left[\mu_1\mathrm{d}t + \sigma_1(S_1(t))^{k_1}\mathrm{d}W_1(t)\right], \quad S_1(0) = s_{1,0}, \tag{2.4}$$

and for the reinsurer is given by

$$\mathrm{d}S_2(t) = S_2(t)\left[\mu_2\mathrm{d}t + \sigma_2(S_2(t))^{k_2}\mathrm{d}W_2(t)\right], \quad S_2(0) = s_{2,0}. \tag{2.5}$$

For convenience, assets $S_1(t)$ and $S_2(t)$ will be respectively referred to as "risky asset 1" and "risky asset 2". It is reasonable to assume that the expected return rates $\mu_i > r$ for $i = 1, 2$, and the parameters k_1 and k_2 in the instantaneous volatilities $\sigma_1(S_1(t))^{k_1}$ and $\sigma_2(S_2(t))^{k_2}$ satisfy the general condition $k_1 \geqslant 0$ and $k_2 \geqslant 0$ as in [18], [21], [25] and [28]. W_1 and W_2 are the randomness sources associated with the financial assets S_1 and S_2. In addition, we assume $\overline{W}$, W_1 and W_2 are independent of each other.

Processes (2.4) and (2.5) are called the CEV models, which are proposed by [17] to model the dynamics of financial assets and is the first one to relax the constant volatility assumption of the celebrated [10] model. In models (2.4) and (2.5), the existence of the exponent $k_i, i = 1, 2$, called the elasticity parameters, allows the instantaneous conditional variance of asset returns to depend on the asset price level, thus exhibiting an implied volatility smile similar to volatility curves empirically observed. Moreover, the CEV model also reflects the so-called leverage effect of the equity market, i.e., the existence of a negative correlation between stock returns and realized stock volatility. Due to the various advantages of the CEV model, it has been widely adopted for modelling financial asset's price processes in the literature; see, for instance, [19], [46], [55] among many others. In recent years, the CEV models have also been widely used in actuarial literature, including [27], [30], [31], [42], [43] and [56] among many others.

The primary objective of the present paper is to investigate the optimal reinsurance between the insurer and the reinsurer as well as their optimal investment strategies in an integrated way. To proceed, let $\pi_1(t)$ and $\pi_2(t)$ respectively denote the money amounts invested in risky asset 1 by the insurer and in risky asset 2 by the reinsurer at time t, $t \geqslant 0$. Then, the triplet

$$u(t) = (\pi_1(t), \pi_2(t), p(t))$$

encompasses the reinsurance strategy as well as the investment strategies for both the insurer and the reinsurer. Hereafter the triplet $u(t)$ is referred to as a trading strategy and the primary objective of the present paper is to explore the optimal trading strategies from certain perspective.

Given a trading strategy $u(t)$, it is easy to obtain the following dynamics for the wealth process X^u of the insurer and Y^u of the reinsurer:

$$\begin{aligned} \mathrm{d}X^u(t) &= [rX^u(t) + a(\theta - \eta p(t)) + \pi_1(t)(\mu_1 - r)]\,\mathrm{d}t \\ &\quad + b(1-p(t))\mathrm{d}\overline{W}(t) + \pi_1(t)\sigma_1(S_1(t))^{k_1}\mathrm{d}W_1(t), \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \mathrm{d}Y^u(t) &= [rY^u(t) + a\eta p(t) + \pi_2(t)(\mu_2 - r)]\,\mathrm{d}t \\ &\quad + bp(t)d\overline{W}(t) + \pi_2(t)\sigma_2(S_2(t))^{k_2}\mathrm{d}W_2(t), \end{aligned} \tag{2.7}$$

respectively.

In the present paper, we consider the following weighted sum process for decision making:

$$Z^u(t) = \alpha X^u(t) + \beta Y^u(t) \text{ for some } \alpha, \beta \in (0,1], \tag{2.8}$$

where α and β are two constants from interval $[0,1]$. $Z^u(t)$ in equation (2.8) can be interpreted in at least two distinct ways, which indeed constitute our motivation for the present paper. First, it can be observed that an insurer and a reinsurer may belong to the same corporations. In such case, Z^u in (2.8) can be interpreted as the total surplus of the corporations, if the corporations own $100\alpha\%$ shares of the insurer and $100\beta\%$ shares of the reinsurer. When the shares on both the insurance and reinsurance companies the corporations hold are enough to dominate the management boards of both companies, the decision-making on both companies is up to the corporations, and the analysis of optimal investment-reinsurance strategies should be based on the weighted sum process $Z^u(t)$ from the interests of the corporations.

Second, if we set $\beta = 1 - \alpha$ in the weighted sum process $Z^u(t)$, then the parameter α takes a role of balancing the interests between the insurer and the reinsurer in deciding the optimal investment-reinsurance strategies. In particular, Z^u is simply the half of total surplus of the insurer's and reinsurer's for $\alpha = 1/2, \beta = 1/2$, whereas it reduces to the wealth process of the insurer with $\alpha = 1$. For the above reasons, α and β may be referred to as *decision weights*.

Combining (2.6)-(2.8) yields the following dynamics for the weighted sum process:

$$\begin{aligned} \mathrm{d}Z^u(t) &= [rZ^u(t) + \alpha\pi_1(t)(\mu_1 - r) + \beta\pi_2(t)(\mu_2 - r) + a\alpha\theta - a\eta p(t)(\alpha - \beta)]\,\mathrm{d}t \\ &\quad + \alpha\pi_1(t)\sigma_1(S_1(t))^{k_1}\mathrm{d}W_1(t) + \beta\pi_2(t)\sigma_2(S_2(t))^{k_2}\mathrm{d}W_2(t) \\ &\quad + [\alpha b - bp(t)(\alpha - \beta)]\,\mathrm{d}\overline{W}(t). \end{aligned} \tag{2.9}$$

Definition 2.1. A trading strategy $\{u(v) := (\pi_1(v), \pi_2(v), p(v)), v \in [t,T]\}$ is said to be admissible with respect to an initial condition $(t, z, s_1, s_2) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, if for any $t \in [0,T]$, it satisfies the following conditions:

(a) $\forall (z, s_1, s_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, equation (2.9) has a pathwise unique solution $\{Z^u(v)\}_{v\in[t,T]}$ with $Z^u(t) = z, S_1(t) = s_1, S_2(t) = s_2$;

(b) $\forall v \in [t,T], p(v) \in [0,1]$;

(c) $\forall \rho \in [1,\infty], \forall (t,z,s_1,s_2) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathrm{E}_{t,z,s_1,s_2}\left(\sup\limits_{v\in[t,T]} |Z^u(v)|^\rho\right) < \infty$;

(d) $\forall v \in [t,T], \mathrm{E}\left[\int_t^T \left(|\pi_1(v)|^4 + |\pi_2(v)|^4\right)\mathrm{d}v\right] < \infty$;

where $\mathrm{E}_{t,z,s_1,s_2}(\cdot) = E(\cdot | Z(t) = z, S_1(t) = s_1, S_2(t) = s_2)$.

Hereafter, $U(t, z, s_1, s_2)$ denotes the set of all admissible strategies with respect to the initial condition $(t, z, s_1, s_2) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$.

2.2 Problem formulation and a verification theorem

Most existing literature (e.g., [8], [32]) on mean-variance problem only consider the optimality of solutions at the initial time, with a formulation corresponding to the weighted sum process Z^u as follows:

$$\sup_{u\in U}\left\{\mathrm{E}_{0,Z^u(0),S_1(0),S_2(0)}\left[Z^u(T)\right] - \frac{\gamma}{2}\mathrm{Var}_{0,Z^u(0),S_1(0),S_2(0)}\left[Z^u(T)\right]\right\}, \tag{2.10}$$

where U denotes the corresponding admissible set and γ is the risk aversion coefficient of the decision-maker. A formulation like (2.10) is a static optimization problem in the sense that the objective are purely based on information at the initial time 0, and thus the resulting solutions are precommitment in that they are not updated with the information accumulated over time. As explained by [37] and [40], the target in a pragmatic decision is often varying over time, making the solutions obtained from formulation (2.10) time-inconsistent in that they are optimal only at time 0 and no longer as time moves forwards into the future. Time-consistency is a basic requirement for rational investors on a decision, and thus, from a practical point of view, a solution based on formulation (2.10) is not usable.

The present paper considers the following formulation with a time-varying objective instead: For any $(t, z, s_1, s_2) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, the insurer aims to derive

$$\sup_{u \in U(t,z,s_1,s_2)} \left\{ \mathrm{E}_{t,z,s_1,s_2}\left[Z^u(T)\right] - \frac{\gamma}{2}\mathrm{Var}_{t,z,s_1,s_2}\left[Z^u(T)\right] \right\}. \tag{2.11}$$

Because the variance term in the objective lacks the iterated expectation property, the value function for problem (2.11) do not satisfy Bellman principle of optimality and this makes it a time-inconsistent problem in that the solutions obtained at time $t \in [0, T]$ is not optimal for a future time $v > t$. To develop a time-consistent solution to problem (2.11), we resort to the so-called equilibrium strategy as defined in Definition 2.2 below. Roughly speaking, the decision in the equilibrium strategy established at time t agree with the one derived at time $t + \Delta t$ for an infinitesimal $\Delta t > 0$, which is time-consistent.

For the equilibrium strategy to the mean-variance problem (2.11), a more general problem, which includes (2.11) as a special case, will be studied using a game theoretic perspective. To proceed, let $O = [0, T] \times \mathbb{R}^n$ for integer $n \geqslant 1$ and denote

$$\begin{aligned} C^{1,2}(O) = \quad & \{\phi(t,x) | \phi(t,\cdot) \text{ is once continuously differentiable on } [0,T] \text{ and } \phi(\cdot,x) \text{ is} \\ & \text{twice continuously differentiable on } \mathbb{R}^n.\}, \\ D^{1,2}(O) = \quad & \{\phi(t,x) | \phi(t,x) \in C^{1,2}(O) \text{ and all once partial derivatives of } \phi(\cdot,x) \text{ satisfy} \\ & \text{the polynomial growth condition on } \mathbb{R}^n.\}. \end{aligned}$$

The general problem is defined for any function $f \in D^{1,2}([0,T] \times \mathbb{R}^5)$ as follows:

$$\sup_{u \in U(t,z,s_1,s_2)} f\left(t, z, s_1, s_2, g^u(t,z,s_1,s_2), h^u(t,z,s_1,s_2)\right), \quad t \in [0,T], \tag{2.12}$$

where $(z, s_1, s_2) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$,

$$g^u(t,z,s_1,s_2) = \mathrm{E}_{t,z,s_1,s_2}[Z^u(T)], \tag{2.13}$$

$$h^u(t,z,s_1,s_2) = \mathrm{E}_{t,z,s_1,s_2}[(Z^u(T))^2], \tag{2.14}$$

and $U(t, z, s_1, s_2)$ is the set of admissible strategies at state (t, z, s_1, s_2) with the precise definition given in the preceding subsection. In particular, with

$$f(t,z,s_1,s_2,g,h) = g - \frac{\gamma}{2}(h - g^2), \tag{2.15}$$

problem (2.12) reduces to problem (2.11).

For a time-consistent solution to a dynamic problem such as (2.12), [23] proposed the following definition of equilibrium strategy.

Definition 2.2. Given an admissible strategy $u^*(t) := (\pi_1^*(t), \pi_2^*(t), p^*(t)) \in U(t, z, s_1, s_2)$, construct strategy

$$u_\tau(v) = \begin{cases} (\widetilde{\pi}_1, \widetilde{\pi}_2, \widetilde{p}), & t \leqslant v < t + \tau, \\ u^*(v), & t + \tau \leqslant v < T, \end{cases}$$

where $\widetilde{\pi}_1, \widetilde{\pi}_2 \in \mathbb{R}$, $\widetilde{p} \in [0,1]$ and $v > 0$. If

$$\liminf_{\tau \to 0} \frac{f(t,z,s_1,s_2,g^{u^*},h^{u^*}) - f(t,z,s_1,s_2,g^{u_\tau},h^{u_\tau})}{\tau} \geqslant 0 \tag{2.16}$$

for all $(\widetilde{\pi}_1, \widetilde{\pi}_2, \widetilde{p}) \in \mathbb{R} \times \mathbb{R} \times [0,1]$ and $(t, z, s_1, s_2) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$, then u^* is said to be an equilibrium strategy. Correspondingly, the equilibrium value function is defined by

$$V(t,z,s_1,s_2) = f(t,z,s_1,s_2,g^{u^*}(t,z,s_1,s_2),h^{u^*}(t,z,s_1,s_2)).$$

To develop an equilibrium strategy for the mean-variance problem (2.11), a verification theorem, which gives the extended Hamilton-Jacobi-Bellman (HJB) equations for the general problem (2.12), seems necessary. To proceed, denote $Q := \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$ and a variational operator

$$\begin{aligned}\mathcal{A}^u\phi(t,z,s_1,s_2) \quad := \quad & \phi_t + [rz + \alpha\pi_1(\mu_1 - r) + \beta\pi_2(\mu_2 - r) + a\alpha\theta - a\eta p(\alpha-\beta)]\,\phi_z \\ & + \mu_1 s_1 \phi_{s_1} + \mu_2 s_2 \phi_{s_2} \\ & + \frac{1}{2}\left\{\alpha^2\pi_1^2\sigma_1^2 s_1^{2k_1} + \beta^2\pi_2^2\sigma_2^2 s_2^{2k_2} + [\alpha b - bp(\alpha-\beta)]^2\right\}\phi_{zz} \\ & + \frac{1}{2}\sigma_1^2 s_1^{2k_1+2}\phi_{s_1s_1} + \frac{1}{2}\sigma_2^2 s_2^{2k_2+2}\phi_{s_2s_2} + \alpha\pi_1\sigma_1^2 s_1^{2k_1+1}\phi_{zs_1} \\ & + \beta\pi_2\sigma_2^2 s_2^{2k_2+1}\phi_{zs_2} \end{aligned} \tag{2.17}$$

for any $\phi(t,z,s_1,s_2) \in C^{1,2}([0,T]\times Q)$.

Theorem 2.3. *(Verification Theorem). Consider problem (2.12). If there exist three real value functions $F(t,z,s_1,s_2), G(t,z,s_1,s_2), H(t,z,s_1,s_2) \in D^{1,2}([0,T]\times Q)$ satisfy*

$$\sup_{u\in U(t,z,s_1,s_2)} \{\mathcal{A}^u F(t,z,s_1,s_2) - \xi^u(t,z,s_1,s_2)\} = 0, \quad F(T,z,s_1,s_2) = f(T,z,s_1,s_2,z,z^2), \tag{2.18}$$

$$\mathcal{A}^{u^*}G(t,z,s_1,s_2) = 0, \quad G(T,z,s_1,s_2) = z, \tag{2.19}$$

$$\mathcal{A}^{u^*}H(t,z,s_1,s_2) = 0, \quad H(T,z,s_1,s_2) = z^2, \tag{2.20}$$

with

$$u^* := \arg \sup_{u\in U(t,z,s_1,s_2)} \{\mathcal{A}^u F(t,z,s_1,s_2) - \xi^u(t,z,s_1,s_2)\},$$

then

$$\begin{aligned} V(t,z,s_1,s_2) &= F(t,z,s_1,s_2), \\ g^{u^*}(t,z,s_1,s_2) &= G(t,z,s_1,s_2), \\ h^{u^*}(t,z,s_1,s_2) &= H(t,z,s_1,s_2), \end{aligned} \tag{2.21}$$

and $u^* = (\pi_1^*, \pi_2^*, p^*)$ is an equilibrium strategy to problem (2.12), where

$$\begin{aligned}\xi^u(t,z,s_1,s_2) \quad = \quad & f_t + [rz + \alpha\pi_1(\mu_1-r) + \beta\pi_2(\mu_2-r) + a\alpha\theta - a\eta p(\alpha-\beta)]\,f_z \\ & + \mu_1 s_1 f_{s_1} + \mu_2 s_2 f_{s_2} + \frac{1}{2}\Big[\alpha^2\pi_1^2\sigma_1^2 s_1^{2k_1} + \beta^2\pi_2^2\sigma_2^2 s_2^{2k_2} \\ & + (\alpha b - bp(\alpha-\beta))^2\Big]\Pi_1^u + \frac{1}{2}\sigma_1^2 s_1^{2k_1+2}\Pi_2^u + \frac{1}{2}\sigma_2^2 s_2^{2k_2+2}\Pi_3^u \\ & + \alpha\pi_1\sigma_1^2 s_1^{2k_1+1}\Pi_4^u + \beta\pi_2\sigma_2^2 s_2^{2k_2+1}\Pi_5^u, \end{aligned} \tag{2.22}$$

$$\begin{cases} \Pi_1^u = f_{zz} + 2f_{zg}g_z^u + 2f_{zh}h_z^u + f_{hh}(h_z^u)^2 + f_{gg}(g_z^u)^2 + 2f_{gh}g_z^u h_z^u, \\ \Pi_2^u = f_{s_1s_1} + 2f_{gs_1}g_{s_1}^u + 2f_{hs_1}h_{s_1}^u + f_{gg}(g_{s_1}^u)^2 + f_{hh}(h_{s_1}^u)^2 + 2f_{gh}g_{s_1}^u h_{s_1}^u, \\ \Pi_3^u = f_{s_2s_2} + 2f_{gs_2}g_{s_2}^u + 2f_{hs_2}h_{s_2}^u + f_{gg}(g_{s_2}^u)^2 + f_{hh}(h_{s_2}^u)^2 + 2f_{gh}g_{s_2}^u h_{s_2}^u, \\ \Pi_4^u = f_{zs_1} + f_{zg}g_{s_1}^u + f_{zh}h_{s_1}^u + f_{gs_1}g_z^u + f_{hs_1}h_z^u + f_{gg}g_z^u g_{s_1}^u + f_{hh}h_z^u h_{s_1}^u \\ \qquad + f_{gh}g_{s_1}^u h_z^u + f_{gh}g_z^u h_{s_1}^u, \\ \Pi_5^u = f_{zs_2} + f_{zg}g_{s_2}^u + f_{zh}h_{s_2}^u + f_{gs_2}g_z^u + f_{hs_2}h_z^u + f_{gg}g_z^u g_{s_2}^u + f_{hh}h_z^u h_{s_2}^u \\ \qquad + f_{gh}g_{s_2}^u h_z^u + f_{gh}g_z^u h_{s_2}^u \end{cases} \tag{2.23}$$

and

$$f = f(t,z,s_1,s_2,g,h),\ g^u = g^u(t,z,s_1,s_2),\ h^u = h^u(t,z,s_1,s_2).$$

Proof. See Appendix A. □

3 Optimal time-consistent strategy for the mean-variance problem

By using Theorem 2.3, the equilibrium strategy $\{u^*(t)=(\pi_1^*(t),\pi_2^*(t),p^*(t)),t\in[0,T]\}$ for problem (2.11) can be obtained analytically as summarized in Theorems 3.1 and 3.2 for $\alpha\neq\beta$ and $\alpha=\beta$, respectively.

Theorem 3.1. *Denote*

$$t_1=T+\frac{1}{r}\min\left(\ln\frac{\alpha\gamma b^2}{a\eta},\ln\frac{\beta\gamma b^2}{a\eta}\right), \tag{3.1}$$

and

$$t_2=T+\frac{1}{r}\max\left(\ln\frac{\alpha\gamma b^2}{a\eta},\ln\frac{\beta\gamma b^2}{a\eta}\right). \tag{3.2}$$

For problem (2.11) with $\alpha\neq\beta$, an equilibrium investment strategy for the insurer and the reinsurer is

$$\pi_1^*(t)=\frac{(\mu_1-r)\mathrm{e}^{-r(T-t)}}{\gamma\alpha\sigma_1^2(S_1(t))^{2k_1}}\left[1+(\mu_1-r)\left(1-\mathrm{e}^{-2k_1r(T-t)}\right)\right],\ \ 0\leqslant t\leqslant T, \tag{3.3}$$

and

$$\pi_2^*(t)=\frac{(\mu_2-r)\mathrm{e}^{-r(T-t)}}{\gamma\beta\sigma_2^2(S_2(t))^{2k_2}}\left[1+(\mu_2-r)\left(1-\mathrm{e}^{-2k_2r(T-t)}\right)\right],\ \ 0\leqslant t\leqslant T, \tag{3.4}$$

and an equilibrium reinsurance strategy and value function are given depending on the model parameters as in the follows cases:

(1) When the parameters satisfy the conditions of Case III in Table 1,

$$p^*(t)=\frac{\alpha}{\alpha-\beta}-\frac{a\eta}{\gamma b^2(\alpha-\beta)}\mathrm{e}^{-r(T-t)},\ \ 0\leqslant t\leqslant T, \tag{3.5}$$

and the equilibrium value function is given by (B.11) with $A_1(t)$, $B_1(t)$, $C_1(t)$ and $D_1(t)$ in (B.23);

(2) When the parameters satisfy the conditions of Case V in Table 1,

$$p^*(t)=\begin{cases}0, & 0\leqslant t<t_1,\\ \dfrac{\alpha}{\alpha-\beta}-\dfrac{a\eta}{\gamma b^2(\alpha-\beta)}\mathrm{e}^{-r(T-t)}, & t_1\leqslant t\leqslant T,\end{cases}$$

and the equilibrium value function is given by (B.24);

(3) When the parameters satisfy the conditions of Case VI in Table 1, $p^*(t)=0$, $0\leqslant t\leqslant T$, and the equilibrium value function is given by the function F in (B.27);

(4) When the parameters satisfy the conditions in any of Cases IV and IX in Table 1,

$$p^*(t)=\begin{cases}0, & 0\leqslant t<t_1,\\ \dfrac{\alpha}{\alpha-\beta}-\dfrac{a\eta}{\gamma b^2(\alpha-\beta)}\mathrm{e}^{-r(T-t)}, & t_1\leqslant t<t_2,\\ 1, & t_2\leqslant t\leqslant T,\end{cases}$$

and the equilibrium value function is given by (B.29);

(5) When the parameters satisfy the conditions in any of Cases II and VIII in Table 1,

$$p^*(t)=\begin{cases}\dfrac{\alpha}{\alpha-\beta}-\dfrac{a\eta}{\gamma b^2(\alpha-\beta)}\mathrm{e}^{-r(T-t)}, & 0\leqslant t<t_2,\\ 1, & t_2\leqslant t\leqslant T,\end{cases}$$

and the equilibrium value function is expressed in (B.33);

Table 1 Classification for optimal reinsurance strategy.

parameters	$\alpha,\quad \beta$	Case
$\frac{a\eta}{\gamma b^2}\mathrm{e}^{-rT} < \frac{a\eta}{\gamma b^2} \leqslant 1$	$\max(\alpha,\beta) \leqslant \frac{a\eta}{\gamma b^2}\mathrm{e}^{-rT}$	I
	$\min(\alpha,\beta) \leqslant \frac{a\eta}{\gamma b^2}\mathrm{e}^{-rT} < \max(\alpha,\beta) < \frac{a\eta}{\gamma b^2}$	II
	$\min(\alpha,\beta) \leqslant \frac{a\eta}{\gamma b^2}\mathrm{e}^{-rT} < \frac{a\eta}{\gamma b^2} \leqslant \max(\alpha,\beta)$	III
	$\frac{a\eta}{\gamma b^2}\mathrm{e}^{-rT} < \min(\alpha,\beta) < \max(\alpha,\beta) < \frac{a\eta}{\gamma b^2}$	IV
	$\frac{a\eta}{\gamma b^2}\mathrm{e}^{-rT} < \min(\alpha,\beta) < \frac{a\eta}{\gamma b^2} \leqslant \max(\alpha,\beta)$	V
	$\frac{a\eta}{\gamma b^2} \leqslant \min(\alpha,\beta)$ and $\alpha \neq \beta$	VI
$\frac{a\eta}{\gamma b^2}\mathrm{e}^{-rT} \leqslant 1 < \frac{a\eta}{\gamma b^2}$	$\max \alpha,\beta \leqslant \frac{a\eta}{\gamma b^2}\mathrm{e}^{-rT}$ and $\alpha \neq \beta$	VII
	$\min(\alpha,\beta) \leqslant \frac{a\eta}{\gamma b^2}\mathrm{e}^{-rT} < \max(\alpha,\beta)$	VIII
	$\frac{a\eta}{\gamma b^2}\mathrm{e}^{-rT} < \min(\alpha,\beta)$ and $\alpha \neq \beta$	IX
$1 < \frac{a\eta}{\gamma b^2}\mathrm{e}^{-rT} < \frac{a\eta}{\gamma b^2}$	$\alpha \neq \beta$	X

(6) When the parameters satisfy the conditions in any of Cases I, VII and X in Table 1, $p^*(t) = 1, 0 \leqslant t \leqslant T$, *and the equilibrium value function is given by the function* F *in (B.35).*

Proof. The proof is relegated to Appendix B. □

Theorem 3.2. *For problem (2.11) with* $\alpha = \beta$, (π_1^*, π_2^*) *given in* (3.3) *and* (3.4) *is also an equilibrium investment strategies for the insurer and the reinsurer, and moreover, any measurable function* $p^*(t) : [0,T] \to [0,1]$ *is one equilibrium reinsurance treaty. The equilibrium value function is*

$$F(t,z,s_1,s_2) = A_1(t)z + \frac{B_1(t)}{\gamma}s_1^{-2k_1} + \frac{C_1(t)}{\gamma}s_2^{-2k_2} + \frac{L_1(t)}{\gamma},$$

where

$$L_1(t) = \frac{\gamma a\alpha\theta}{r}[\mathrm{e}^{r(T-t)} - 1] + \frac{\gamma^2\alpha^2 b^2}{4r}[1 - \mathrm{e}^{2r(T-t)}] + \sigma_1^2 k_1(2k_1+1)\int_t^T B_1(s)\mathrm{d}s$$

and $A_1(t), B_1(t), C_1(t)$ *are given in (B.23).*

Proof. The proof is similar to that of Theorem 3.1 and thus we omit it. □

Indeed, when $\alpha = \beta$, the joint mean-variance utility between the insurer and the reinsurer is irrelevant to the reinsurance proportion $p(t)$ and this implies that the joint utility can not be enhanced via any proportional reinsurance treaty between the two parties. This explains why any measurable function $p^*(t) : [0,T] \to [0,1]$ is one solution as equilibrium reinsurance treaty.

Remark 3.3. An immediate observation from Theorems 3.1 and 3.2 is that the equilibrium investment strategies for both the insurer and reinsurer are related to the proportional reinsurance treaty only via the decision weight α and β in the model.

Moreover, expressions (3.3) and (3.4) show that the equilibrium investment strategies for both the insurer and the reinsurer can be decomposed into two parts (below are just for the insurer's strategy):

$$\frac{\mu_1 - r}{\gamma\alpha\sigma_1^2(S_1(t))^{2k_1}}\mathrm{e}^{-r(T-t)},$$

which is similar to the optimal strategy under the GBM model except for the stochastic volatility, and

$$\frac{(\mu_1 - r)^2\mathrm{e}^{-r(T-t)}}{\gamma\alpha r\sigma_1^2(S_1(t))^{2k_1}}\left(1 - \mathrm{e}^{-2k_1 r(T-t)}\right),$$

which is a supplementary term resulted from the changes of the volatility from GBM model to the CEV model and reflects the insurer's decision to hedge the volatility risk.

Remark 3.4. Theorem 3.1 shows an obvious dependence of the equilibrium strategy on the decision weights α and β as a consequence of its presence in weighted sum process (2.8). For $p^*(t) \neq 0, 1$, the solution developed in Theorem 3.1 is given by

$$p^*(t) = \frac{\alpha}{\alpha - \beta} - \frac{a\eta}{\gamma b^2(\alpha - \beta)}\mathrm{e}^{-r(T-t)}$$

in all the cases. Therefore, the relationship between the reinsurance strategy and α is attributed to two factors. On one hand, the term $-\frac{a\eta e^{-r(T-t)}}{\gamma b^2(\alpha-\beta)}$ increases the value of $p^*(t)$ as α increases, and on the other hand, the term $\frac{\alpha}{\alpha-\beta}$ decreases its value as α increases. Moreover, equations (3.3) and (3.4) respectively show that α exerts a negative effect on the insurer's investment strategy and a positive effect on the reinsurer's investment strategy. Due to the insurer's attitude of risk aversion, with a larger α assigned, more weight is given to the preference of the insurer in the decision formulation, and this leads to a reduction in the risky investment by the insurer and an increase in the reinsurer's risky investment to potentially create more wealth for the whole insurance group.

Remark 3.5. Based on the results established in Theorem 3.1, the sensitivity of the equilibrium reinsurance proportion $p^*(\cdot)$ to each model parameter can be analyzed as follows.

(a) From (3.5), we get

$$\frac{\partial p^*}{\partial \alpha} = \frac{1}{(\alpha-\beta)^2}\left[\frac{a\eta}{\gamma b^2}e^{-r(T-t)} - \beta\right], \frac{\partial p^*}{\partial \beta} = \frac{1}{(\alpha-\beta)^2}\left[\alpha - \frac{a\eta}{\gamma b^2}e^{-r(T-t)}\right]. \tag{3.6}$$

Then the optimal reinsurance strategy increases with α when $\beta < \frac{a\eta}{\gamma b^2}e^{-r(T-t)}$ and increases with β when $\alpha > \frac{a\eta}{\gamma b^2}e^{-r(T-t)}$.

(b) It is easy to derive

$$\frac{\partial p^*}{\partial t} = -\frac{ar\eta e^{-r(T-t)}}{\gamma b^2(\alpha-\beta)} \text{ for } p^*(t) \neq 0, 1.$$

This means that the equilibrium reinsurance proportion decreases with time t for $\alpha > \beta$ and increases for $\alpha < \beta$. This may be explained as follows. As time goes by, the surpluses of both the insurer and the reinsurer are expected to increase on average due to the potential gains from investments in financial assets and this enhances the insurance risk absorbing capacity of both over time. With $\alpha > \beta$, more weight is given to the insurer's preference in decision, and thus, according to the insurer's willingness, more insurance business is retained by the insurer itself at a later period than an earlier time, leading the reinsurance proportion decreasing as a function of time t. For $\alpha < \beta$, the reinsurer's preference is given with more priority, and thus the decision allows the reinsurer to gradually undertake more insurance business from the insurer over time, making the reinsurance proportion $p^*(t)$ increasing over time.

(c) The effects of market parameters on the reinsurance strategy depend on the decision weights α and β too. Equation (3.5) shows that

$$\frac{\partial p^*}{\partial a} = -\frac{\eta e^{-r(T-t)}}{\gamma b^2(\alpha-\beta)} \text{ for } p^*(t) \neq 0, 1,$$

which implies that the equilibrium reinsurance strategy increases with a for $\alpha < \beta$ and decreases with a for $\alpha > \beta$. The parameter a reflects the expectation of the claim size and thus, with the other model parameters (particularly the volatility b in the claim process (2.1)) fixed, an increase in the expected claim a reduces the risk per dollar of the insurance liability and makes the insurance business more attractive to the writer. Therefore, when more weight is given to the insurer in decision, the reinsurance proportion is reduced and more insurance policies are retained by the insurer in response of an increase of a. In contrast, if $\alpha < \beta$, the reinsurer is given more weight in decision, and thus, more insurance policies are transferred to the reinsurer with an increase in a.

(d) Since

$$\frac{\partial p^*}{\partial \eta} = -\frac{a e^{-r(T-t)}}{\gamma b^2(\alpha-\beta)} \text{ for } p^*(t) \neq 0, 1,$$

the reinsurer's safety loading η exerts a positive effect on $p^*(\cdot)$ for $\alpha < \beta$ and a negative effect on $p^*(\cdot)$ for $\alpha > \beta$. This is consistent with our intuition. A larger η means more expensive reinsurance. Therefore, with $\alpha < \beta$ the decision-maker places more consideration on the reinsurer, whereby the reinsurance proportion is increased so that the reinsurer can make more profits in response of an increase in η. With $\alpha > \beta$, the insurer's preference is laid with more weights so that more risk is retained by the insurer itself when the reinsurance becomes more expensive, i.e., η increases.

(e) Equation (3.5) shows that

$$\frac{\partial p^*}{\partial \gamma} = \frac{a\eta e^{-r(T-t)}}{\gamma^2 b^2(\alpha-\beta)} \text{ for } p^*(t) \neq 0, 1,$$

which indicates that the equilibrium reinsurance proportion $p^*(\cdot)$ decreases with the risk aversion coefficient γ of the decision-maker for $\alpha < \beta$ and increases with γ for $\alpha > \beta$. With a decision weight $\alpha < \beta$, the choice of the decision-maker relies on the preference of the reinsurer more than the insurer. Thus, the more risk averse the reinsurer is, the less reinsurance is taken by the reinsurer, leading $p^*(\cdot)$ decreasing in γ. Similarly, when $\alpha > \beta$, the preference of the insurer is reflected more in decision, and thus the more risk averse the insurer is, the more risk is ceded to the reinsurer.

(f) From equation (3.5), we obtain

$$\frac{\partial p^*}{\partial b} = \frac{2a\eta \mathrm{e}^{-r(T-t)}}{\gamma b^3(\alpha-\beta)} \quad \text{for} \quad p^*(t) \neq 0, 1.$$

If $\alpha < \beta$, there is a negative relationship between $p^*(\cdot)$ and the volatility b of the claim process C, whereas the relationship becomes positive for $\alpha > \beta$. This can be interpreted by the change of the risk per dollar of insurance as similarly commented in part (b). With a larger b, the risk per dollar of insurance is larger and thus, the insurance business becomes less attractive to an insurance writer, with more is expected to be ceded to the reinsurer in case a weight $\alpha > \beta$ is assigned in decision. Similar explanation applies for the case $\alpha < \beta$.

(g) Equation (3.5) also shows that

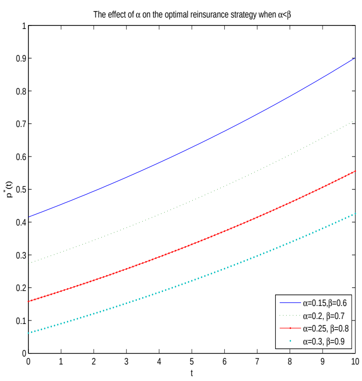
$$\frac{\partial p^*}{\partial r} = \frac{a\eta(T-t)\mathrm{e}^{-r(T-t)}}{\gamma b^2(\alpha-\beta)} \quad \text{for} \quad p^*(t) \neq 0, 1.$$

This indicates that $p^*(\cdot)$ decreases with respect to the risk-free interest rate r for $\alpha < \beta$ and increases in r for $\alpha > \beta$. As r increases, both the insurer and the reinsurer have the expectation to gain more in the financial market, driving the capitals out of the insurance market. When $\alpha < \beta$, the interests of the reinsurer dominate the decision on the trading strategy of the insurance group, and the reinsurer prefers to reduce its investment in insurance market and move more investments to the financial market. Similarly, when $\alpha > \beta$ the insurer's interests dominate the decision and it is preferred by the insurer that more insurance risk is taken by the reinsurer so that it have more wealth to invest in the financial market.

4 Numerical analysis

In this section, a numerical example will be presented to illustrate the effects of parameters on the equilibrium investment-reinsurance strategy derived in Theorem 3.1. Unless otherwise stated, the parameter values are given by $a = 0.5$, $b = 0.6$, $\eta = 0.2$, $r = 0.05$, $\mu_1 = 0.12$, $\sigma_1 = 0.2$, $k_1 = 0.9$, $S_1 = 0.5$, $\mu_2 = 0.15$, $\sigma_2 = 0.3$, $k_2 = 1.1$, $S_2 = 0.6$, $\alpha = 0.3$, $\beta = 0.7$, $\gamma = 0.5$, and $T = 10$.

a



b

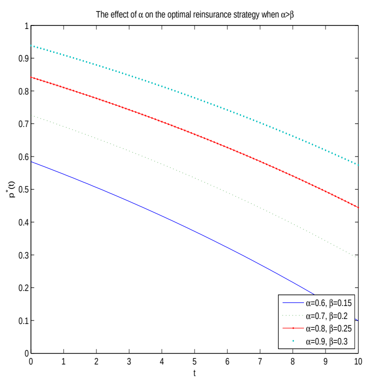


Figure 1 Effect of decision weights α and β on equilibrium reinsurance strategy $p^*(t)$.

Figure 1 shows that the insurance proportion $p^*(t)$ is increasing along with time t for $\alpha < \beta$ and decreasing for $\alpha > \beta$. Such an observation can be explained as follows. The desire of both the insurer and the reinsurer to bear insurance risk becomes gradually strengthen as a result of their wealth accumulation over time. Therefore, both prefer to absorb less insurance risk at the first stage (before time 8) and become more interested in taking insurance business at the second stage with more capital available. Consequently, with $\alpha < \beta$, more voices are heard from the reinsurer in the decision, and according to the desire of the reinsurer, more insurance risk is transferred to the reinsurer at a later period than an earlier period. In contrast, when $\alpha > \beta$, the insurer has more voices over the decision, and therefore, according to the desire of the insurer, more insurance risk is transferred to the reinsurer at an earlier period than a later period. Furthermore, Figure 1 also illustrates that the response of reinsurance strategy $p^*(t)$ to the weights α and β also depends on the relative magnitudes of α and β. For $\alpha < \beta$, $p^*(t)$ is decreasing in both α and β. In contrast, $p^*(t)$ is increasing in both α and β for $\alpha > \beta$.

Figures 2-5 show the impacts of market parameters on the equilibrium reinsurance strategy, and the results consistently confirm our comments presented in Remark 3.5. The effects depend on the value of the decision weight parameters α and β as well. For $\alpha > \beta$, the equilibrium reinsurance proportion increases with γ, b, r and decreases with t, a, η. In contrast, for $\alpha < \beta$, the reinsurance proportion shows an opposite response to these parameter. It increases with t, a, η and decreases with γ, b, r in this case.

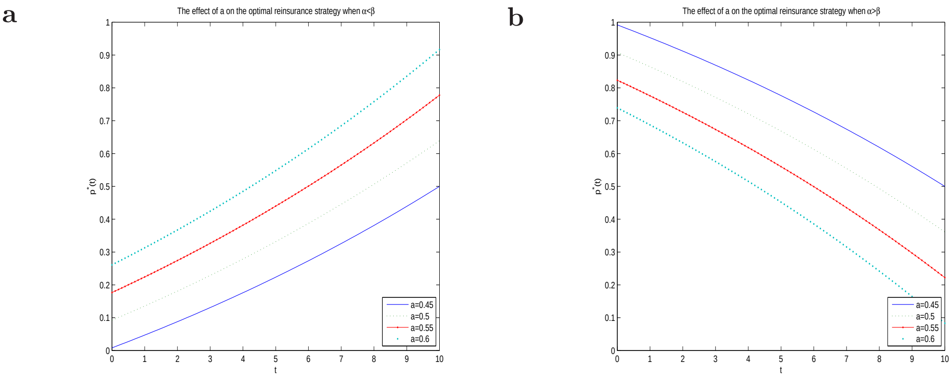


Figure 2 Effect of a on equilibrium reinsurance strategy $p^*(t)$.

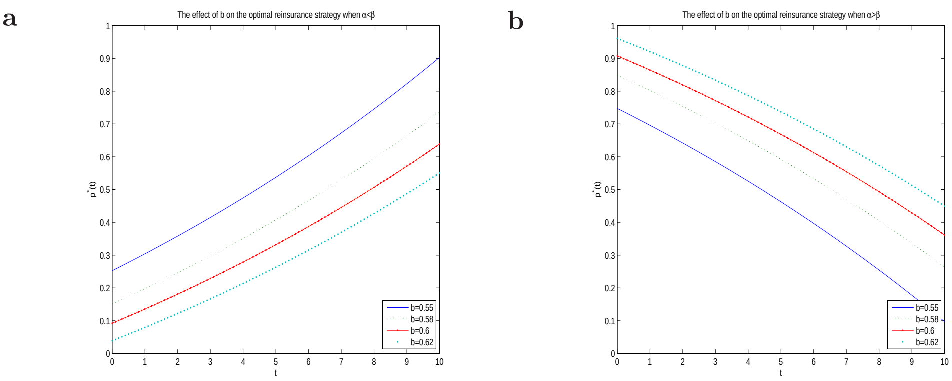


Figure 3 Effect of b on equilibrium reinsurance strategy $p^*(t)$.

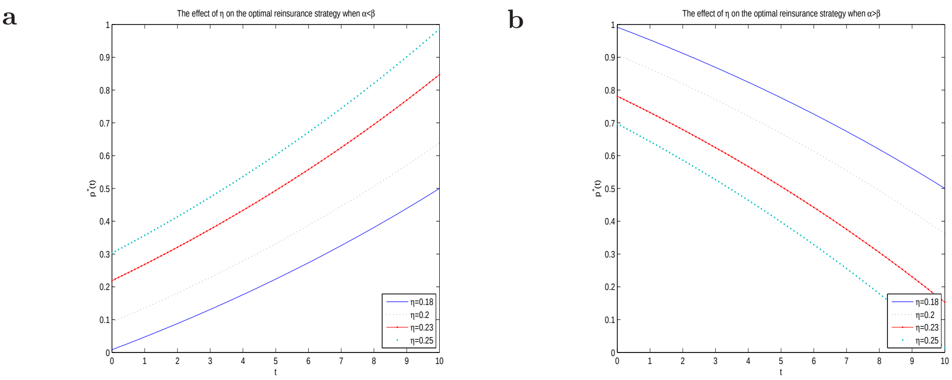


Figure 4 Effect of η on equilibrium reinsurance strategy $p^*(t)$.

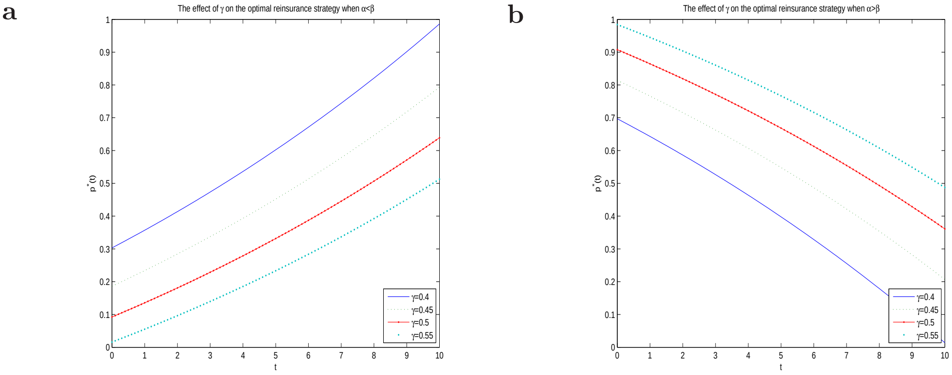


Figure 5 Effect of γ on equilibrium reinsurance strategy $p^*(t)$.

Figures 6-9 depict the sensitivity of the equilibrium investment strategy $(\pi_1^*(t), \pi_2^*(t))$ to various model parameters, where, for simplicity but without loss of generality, the investment strategy is computed only for time 0. Figure 6(a) shows that the risk aversion coefficient γ exerts a negative effect on both $\pi_1^*(t)$ and $\pi_2^*(t)$. The larger γ is, the more risk averse the insurer and the reinsurer are. Thus, as γ increases, the insurer and the reinsurer choose to reduce their investments in the risky assets to control their risk.

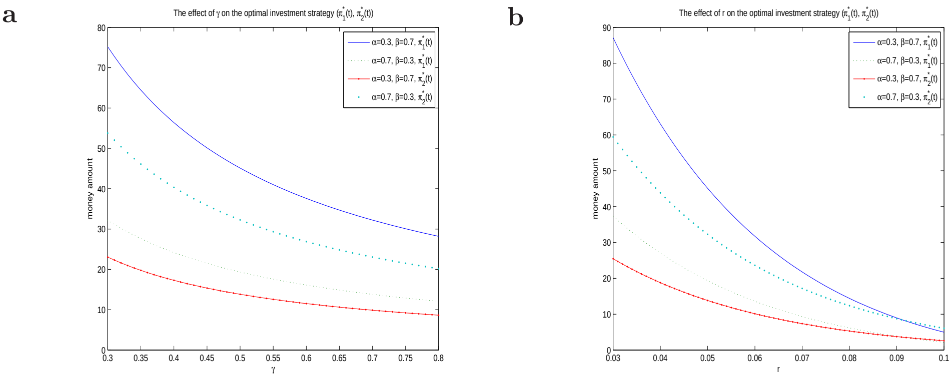


Figure 6. (a) Effect of γ on equilibrium investment strategy $(\pi_1^*(t), \pi_2^*(t))$. (b) Effect of r on equilibrium investment strategy $(\pi_1^*(t), \pi_2^*(t))$.

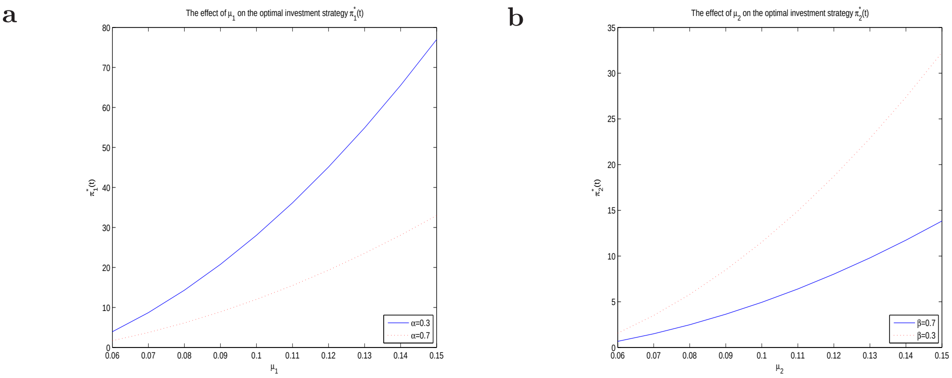


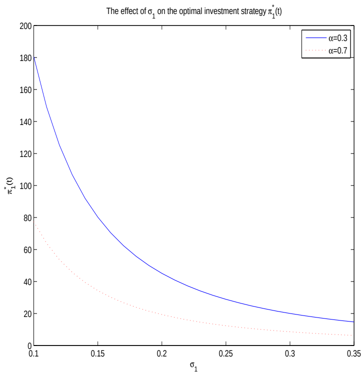
Figure 7. (a) Effect of μ_1 on insurer's equilibrium investment strategy $\pi_1^*(t)$. (b) Effect of μ_2 on reinsurer's equilibrium investment strategy $\pi_1^*(t)$.

Figure 6(b) demonstrates the negative effect from the return rate of the risk-free asset on the equilibrium investment strategy $(\pi_1^*(t), \pi_2^*(t))$, in accordance to the comments in part (f) of Remark 3.5. An increase of the risk-free return rate r changes the interests of both the insurer and the reinsurer in financial market and drives capitals out of the insurance market.

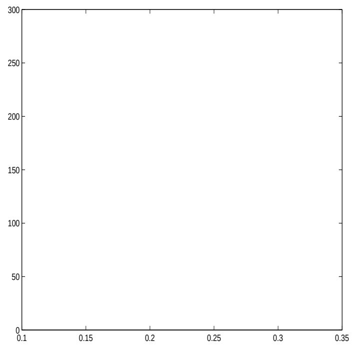
Figure 7 demonstrates an increasing trend of $\pi_1^*(t)$ and $\pi_2^*(t)$ in response of an increase of μ_1 and μ_2 respectively. Such a phenomenon is natural, since μ_1 and μ_2 are respectively the expected return rates of risky assets. With the volatility fixed, an asset with a higher expected return is naturally more attractive to the investors. The decreasing trend of $\pi_1^*(t)$ and $\pi_2^*(t)$ with respect to the corresponding volatility as demonstrated in Figure 8 can be explained by the same reason.

Figure 9 shows that $\pi_1^*(t)$ and $\pi_2^*(t)$ are increasing functions of k_1 and k_2 respectively. Such an observation is consistent to the economic meanings of the parameters k_1 and k_2 in models (2.4) and (2.5). Larger values for k_1 and k_2 lead to more probability of larger positive movement in prices of the risky assets, and hence encourage the insurer and the reinsurer to increase their positions in the risky assets.

a



b



Appendix A Proof of Theorem 2.3

The proof will be obtained in the following three steps. for the case that $\alpha = \frac{1}{2}$, the proof is similar.
Step 1: Consider an arbitrary strategy $u \in U(t, z, s_1, s_2)$. We first show that if $Y(t, z, s_1, s_2) \in D^{1,2}([0, T] \times Q)$, then

$$\int_t^T Y_z(v, Z^u(v), S_1(v), S_2(v))\alpha\sigma_1\pi_1(v)\left[S_1(v)\right]^{k_1} \mathrm{d}W_1(v), \tag{A.1}$$

$$\int_t^T Y_z(v, Z^u(v), S_1(v), S_2(v))\beta\sigma_2\pi_2(v)\left[S_2(v)\right]^{k_2} \mathrm{d}W_2(v), \tag{A.2}$$

$$\int_t^T Y_{s_1}(v, Z^u(v), S_1(v), S_2(v))\sigma_1\left[S_1(v)\right]^{k_1+1} \mathrm{d}W_1(v), \tag{A.3}$$

and

$$\int_t^T Y_{s_2}(v, Z^u(v), S_1(v), S_2(v))\sigma_2\left[S_2(v)\right]^{k_2+1} \mathrm{d}W_2(v) \tag{A.4}$$

are all martingales for $\forall (z, s_1, s_2) \in Q$ and $\forall u \in U(t, z, s_1, s_2)$. In fact, since $Y(t, z, s_1, s_2) \in D^{1,2}([0, T] \times Q)$, there must exist two constants $\kappa > 0$ and $\rho \geqslant 1$ such that

$$|Y_z(t, z, s_1, s_2)| \leqslant \kappa(1 + |z|^\rho + |s_1|^\rho + |s_2|^\rho).$$

Moreover, for $(t, z, s_1, s_2) \in [0, T] \times Q$ and $u \in U(t, z, s_1, s_2)$, (2.8) implies

$$\mathrm{E}_{t,z,s_1,s_2}\left(\sup_{v\in[t,T]} |Z^u(v)|^\rho\right) < \infty, \quad \forall \rho \geqslant 1. \tag{A.5}$$

Applying Itô's formula to $m(t) := (S_1(t))^{-2k_1}$ yields

$$\mathrm{d}m(t) = \left(k_1(2k_1 + 1)\sigma_1^2 - 2k_1\mu_1 m(t)\right)\mathrm{d}t - 2k_1\sigma_1\sqrt{m(t)}\mathrm{d}W_1(t). \tag{A.6}$$

This implies that $m(t)$ is a process similar to ν_t defined in equation (10.48) in [48]. Thus according to Theorem 10.34 in [48], there exists a unique non-negative strong solution to equation (A.6). Moreover, let $\tau = \inf\{t \geqslant 0 | m(t) = 0\}$ and $\tau = \infty$ if $m(t) > 0$ for any t. Proposition 10.35 in [48] shows that for any $m(0) = (S_1(0))^{-2k_1} > 0$, if $k_1 \geqslant 0$, $\tau = \infty, a.s.$, which means that $m(t) = (S_1(t))^{-2k_1}$ is almost surely inaccessible to zero. Therefore, $S_1(t) < \infty$. Similarly, we can show that $S_2(t) < \infty$ for $k_2 \geqslant 0$. We combine all the above analysis to obtain

$$\begin{aligned}
&\mathrm{E}_{t,z,s_1,s_2}\left\{\int_t^T |Y_z(v, Z^u(v), S_1(v), S_2(v))\alpha\sigma_1\pi_1(v)\left(S_1(v)\right)^{k_1}|^2 \mathrm{d}v\right\}\\
\leqslant\ & \tfrac{1}{2}\mathrm{E}_{t,z,s_1,s_2}\left\{\int_t^T \left[\left(Y_z(v, Z^u(v), S_1(v), S_2(v))\alpha\sigma_1\left(S_1(v)\right)^{k_1}\right)^4 + (\pi_1(v))^4\right]\mathrm{d}v\right\}\\
\leqslant\ & \tfrac{1}{2}\mathrm{E}_{t,z,s_1,s_2}\left\{\int_t^T \left[\alpha^4\sigma_1^4\left((Y_z(v, Z^u(v), S_1(v), S_2(v)))^8 + (S_1(v))^{8k_1}\right) + (\pi_1(v))^4\right]\mathrm{d}v\right\}\\
\leqslant\ & \tfrac{1}{2}\mathrm{E}_{t,z,s_1,s_2}\left\{\int_t^T \left[\alpha^4\sigma_1^4\left(\kappa^8\left(1 + |Z^u(v)|^\rho + |S_1(v)|^\rho + |S_2(v)|^\rho\right)^8 + (S_1(v))^{8k_1}\right) + (\pi_1(v))^4\right]\mathrm{d}v\right\}\\
<\ & \infty.
\end{aligned}$$

Similarly, we can prove that

$$\mathrm{E}_{t,z,s_1,s_2}\left\{\int_t^T |Y_z(v, Z^u(v), S_1(v), S_2(v))\beta\sigma_2\pi_2(v)\left(S_2(v)\right)^{k_2}|^2 \mathrm{d}v\right\} < \infty,$$

$$\mathrm{E}_{t,z,s_1,s_2}\left\{\int_t^T |Y_{s_1}(v, Z^u(v), S_1(v), S_2(v))\sigma_1\left(S_1(v)\right)^{k_1+1}|^2 \mathrm{d}v\right\} < \infty,$$

$$\mathrm{E}_{t,z,s_1,s_2}\left\{\int_t^T |Y_{s_2}(v, Z^u(v), S_1(v), S_2(v))\sigma_2\left(S_2(v)\right)^{k_2+1}|^2 \mathrm{d}v\right\} < \infty.$$

Thus, (A.1), (A.2), (A.3) and (A.4) are all martingales.

Next, we show that if a real value function $Y(t,z,s_1,s_2)\in D^{1,2}([0,T]\times Q)$ satisfies

$$\mathcal{A}^u Y(t,z,s_1,s_2)=0,\ \ Y(T,z,s_1,s_2)=z,\ \ \forall(t,z,s_1,s_2)\in([0,T]\times\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+), \tag{A.7}$$

then

$$Y(t,z,s_1,s_2)=g^u(t,z,s_1,s_2). \tag{A.8}$$

Using (2.9) and Itô's formula, we derive

$$\begin{aligned}
&Y(t,Z^u(t),S_1(t),S_2(t))\\
=&\,Y(T,Z^u(T),S_1(T),S_2(T))-\int_t^T \mathrm{d}Y(v,Z^u(v),S_1(v),S_2(v))\\
=&\,Y(T,Z^u(T),S_1(T),S_2(T))-\int_t^T \mathcal{A}^u Y(v,Z^u(v),S_1(v),S_2(v))\mathrm{d}v\\
&-\int_t^T\Big\{Y_z(v,Z^u(v),S_1(v),S_2(v))\Big[\alpha\sigma_1\pi_1(v)\left(S_1(v)\right)^{k_1}\mathrm{d}W_1(v)\\
&+\beta\sigma_2\pi_2(v)\left(S_2(v)\right)^{k_2}\mathrm{d}W_2(v)+(\alpha b-b(\alpha-\beta)p(v))\,\mathrm{d}\overline{W}(v)\Big]\\
&+Y_{s_1}(v,Z^u(v),S_1(v),S_2(v))\sigma_1\left(S_1(v)\right)^{k_1+1}\mathrm{d}W_1(v)\\
&+Y_{s_2}(v,Z^u(v),S_1(v),S_2(v))\sigma_2\left(S_2(v)\right)^{k_2+1}\mathrm{d}W_2(v)\Big\}.
\end{aligned}\tag{A.9}$$

Since (A.1), (A.2), (A.3) and (A.4) are martingales, taking the conditional expectation on both sides of equation (A.9) and applying equations (A.7) and (2.13), we have

$$\begin{aligned}
Y(t,z,s_1,s_2)&=\mathrm{E}_{t,z,s_1,s_2}\left[Y(t,Z^u(t),S_1(t),S_2(t))\right]\\
&=\mathrm{E}_{t,z,s_1,s_2}\left[Y(T,Z^u(T),S_1(T),S_2(T))\right]\\
&=\mathrm{E}_{t,z,s_1,s_2}\left[Z^u(T)\right]=g^u(t,z,s_1,s_2).
\end{aligned}\tag{A.10}$$

Similar to the above derivation, we can show that if there exists a real function $J(t,z,s_1,s_2)\in D^{1,2}([0,T]\times Q)$ such that

$$\mathcal{A}^u J(t,z,s_1,s_2)=0\ \text{ and }\ J(T,z,s_1,s_2)=z^2,\ \ \forall(t,z,s_1,s_2)\in([0,T]\times Q), \tag{A.11}$$

then

$$J(t,z,s_1,s_2)=h^u(t,z,s_1,s_2). \tag{A.12}$$

Thirdly, we use (A.7) and (A.11) to develop an expression for

$$f^u(t):=f(t,Z^u(t),S_1(t),S_2(t),Y^u(t),J^u(t)).$$

where $Y^u(t)=Y(t,Z^u(t),S_1(t),S_2(t))$ and $J^u(t)=J(t,Z^u(t),S_1(t),S_2(t))$. In view of (A.8) and (A.12), we get

$$f(t,Z^u(t),S_1(t),S_2(t),g^u(t,Z^u(t),S_1(t),S_2(t)),h^u(t,Z^u(t),S_1(t),S_2(t)))=f^u(t).$$

Let $\xi^u(t)=\xi^u\left(t,Z^u(t),S_1(t),S_2(t)\right)$. We apply Itô's formula and equation (2.9) to obtain

$$\begin{aligned}
f^u(T)&=f^u(t)+\int_t^T \mathrm{d}f^u(v)\\
&=f^u(t)+\int_t^T\left[f_g^u(v)\mathcal{A}^uY^u(v)+f_g^u(v)\mathcal{A}^uJ^u(v)+\xi^u(v)\right]\mathrm{d}v\\
&+\int_t^T\Big\{\left(f_z^u(v)+f_g^u(v)Y_z^u(v)+f_h^u(v)J_z^u(v)\right)\Big[(\alpha b-\alpha bp(v)+b\beta p(v))\,\mathrm{d}\overline{W}(v)\\
&+\alpha\sigma_1\pi_1(v)\left(S_1(v)\right)^{k_1}\mathrm{d}W_1(v)+\beta\sigma_2\pi_2(v)\left(S_2(v)\right)^{k_2}\mathrm{d}W_2(v)\Big]\\
&+\left(f_{s_1}^u(v)+f_g^u(v)Y_{s_1}^u(v)+f_h^u(v)J_{s_1}(v)\right)\sigma_1\left(S_1(v)\right)^{k_1+1}\mathrm{d}W_1(v)\\
&+\left(f_{s_2}^u(v)+f_g^u(v)Y_{s_2}(v)+f_h^u(v)J_{s_2}(v)\right)\sigma_2\left(S_2(v)\right)^{k_2+1}\mathrm{d}W_2(v)\Big\}.
\end{aligned}$$

Inputting (A.7) and (A.11) into the above equation implies

$$\begin{aligned} f^u(T) &= f^u(t) + \int_t^T \xi^u(v)\mathrm{d}v + \int_t^T \Big\{ \left(f_z^u(v) + f_g^u(v)Y_z^u(v) + f_h^u(v)J_z^u(v)\right) \\ &\times\Big[\left(\alpha b - \alpha b p(v) + b\beta p(v)\right)\mathrm{d}\overline{W}(v) + \alpha\sigma_1\pi_1(v)\left(S_1(v)\right)^{k_1}\mathrm{d}W_1(v) \\ &+\beta\sigma_2\pi_2(v)\left(S_2(v)\right)^{k_2}\mathrm{d}W_2(v)\Big] + \left(f_{s_1}^u(v) + f_g^u(v)Y_{s_1}^u(v)\right. \\ &\left.+f_h^u(v)J_{s_1}^u(v)\right)\sigma_1\left(S_1(v)\right)^{k_1+1}\mathrm{d}W_1(v) \\ &+\left(f_{s_2}^u(v) + f_g^u(v)Y_{s_2}^u(v) + f_h^u(v)J_{s_2}^u(v)\right)\sigma_2\left(S_2(v)\right)^{k_2+1}\mathrm{d}W_2(v)\Big\}. \end{aligned} \tag{A.13}$$

Finally, we show that $\forall(t,z,s_1,s_2)\in([0,T]\times Q)$,

$$F(t,z,s_1,s_2) \geqslant \sup_{u\in U(t,z,s_1,s_2)} f(t,z,s_1,s_2,g^u(t,z,s_1,s_2),h^u(t,z,s_1,s_2)). \tag{A.14}$$

Let $F^u(t) := F(t, Z^u(t), S_1(t), S_2(t))$ for short. We apply (2.9) to obtain

$$\begin{aligned} F^u(t) &= F^u(T) - \int_t^T \mathrm{d}F^u(v) \\ &= F^u(T) - \int_t^T \mathcal{A}^u F^u(v)\mathrm{d}v - \int_t^T \Big\{ F_z^u(v)\Big[\left(\alpha b - \alpha b p(v) + b\beta p(v)\right)\mathrm{d}\overline{W}(v) \\ &+\alpha\sigma_1\pi_1(v)\left(S_1(v)\right)^{k_1}\mathrm{d}W_1(v) + \beta\sigma_2\pi_2(v)\left(S_2(v)\right)^{k_2}\mathrm{d}W_2(v)\Big] \\ &+F_{s_1}^u(v)\sigma_1\left(S_1(v)\right)^{k_1+1}\mathrm{d}W_1(v) + F_{s_2}^u(v)\sigma_2\left(S_2(v)\right)^{k_2+1}\mathrm{d}W_2(v)\Big\}. \end{aligned}$$

Besides, (2.18) implies that $\mathcal{A}^u F(t,z,s_1,s_2) \leqslant \xi^u(t,z,s_1,s_2)$ for $\forall(t,z,s_1,s_2)\in([0,T]\times Q)$, and hence,

$$\begin{aligned} F^u(t) &\geqslant F^u(T) - \int_t^T \xi^u(v)\mathrm{d}v - \int_t^T \Big\{ F_z^u(v)\Big[\left(\alpha b - \alpha b p(v) + b\beta p(v)\right)\mathrm{d}\overline{W}(v) \\ &+\alpha\sigma_1\pi_1(v)\left(S_1(v)\right)^{k_1}\mathrm{d}W_1(v) + \beta\sigma_2\pi_2(v)\left(S_2(v)\right)^{k_2}\mathrm{d}W_2(v)\Big] \\ &+F_{s_1}^u(v)\sigma_1\left(S_1(v)\right)^{k_1+1}\mathrm{d}W_1(v) + F_{s_2}^u(v)\sigma_2\left(S_2(v)\right)^{k_2+1}\mathrm{d}W_2(v)\Big\}. \end{aligned} \tag{A.15}$$

Moreover, equation (2.18) also implies that $F^u(T) = f^u(T)$. Hence, inputting (A.13) into (A.15) gives

$$\begin{aligned} F^u(t) &\geqslant f^u(t) + \int_t^T \Big\{ \left(f_z^u(v) + f_g^u(v)Y_z^u(v) + f_h^u(v)J_z^u(v) - F_z^u(v)\right) \\ &\times\Big[\left(\alpha b - \alpha b p(v) + b\beta p(v)\right)\mathrm{d}\overline{W}(v) + \alpha\sigma_1\pi_1(v)\left(S_1(v)\right)^{k_1}\mathrm{d}W_1(v) \\ &+\beta\sigma_2\pi_2(v)\left(S_2(v)\right)^{k_2}\mathrm{d}W_2(v)\Big] + \left(f_{s_1}^u(v) + f_g^u(v)Y_{s_1}^u(v) + f_h^u(v)J_{s_1}^u(v)\right. \\ &\left.-F_{s_1}^u(v)\right)\sigma_1\left(S_1(v)\right)^{k_1+1}\mathrm{d}W_1(v) + \left(f_{s_2}^u(v) + f_g^u(v)Y_{s_2}^u(v) + f_h^u(v)J_{s_2}^u(v)\right. \\ &\left.-F_{s_2}^u(v)\right)\sigma_2\left(S_2(v)\right)^{k_2+1}\mathrm{d}W_2(v)\Big\}. \end{aligned} \tag{A.16}$$

Taking conditional expectation on both sides of the above equation and supremum over $U(t,z,s_1,s_2)$, we get (A.14).

Step 2: Consider the specific admissible strategy π^*. The assumptions of Theorem 2.3 show that $G(t,z,s_1,s_2)$ and $H(t,z,s_1,s_2)$ satisfy (A.7) and (A.11) with strategy u^*. Thus, $G(t,z,s_1,s_2) = g^{u^*}(t,z,s_1,s_2)$

and $H(t,z,s_1,s_2)=h^{u^*}(t,z,s_1,s_2)$. Moreover, the inequality (A.16) at $u=u^*$ becomes an equation, i.e.,

$$
\begin{aligned}
F^{u^*}(t) \quad &= f^{u^*}(t)+\int_t^T\Big\{\left(f_z^{u^*}(v)+f_g^{u^*}(v)Y_z^{u^*}(v)+f_h^{u^*}(v)J_z^{u^*}(v)-F_z^{u^*}(v)\right)\\
&\times\Big[(\alpha b-\alpha bp(v)+b\beta p(v))\,\mathrm{d}\overline{W}(v)+\alpha\sigma_1\pi_1^*(v)\left(S_1(v)\right)^{k_1}\mathrm{d}W_1(v)\\
&+\beta\sigma_2\pi_2^*(v)\left(S_2(v)\right)^{k_2}\mathrm{d}W_2(v)\Big]+\left(f_{s_1}^{u^*}(v)+f_g^{u^*}(v)Y_{s_1}^{u^*}(v)\right.\\
&\left.+f_h^{u^*}(v)J_{s_1}^{u^*}(v)-F_{s_1}^{u^*}(v)\right)\sigma_1\left(S_1(v)\right)^{k_1+1}\mathrm{d}W_1(v)+\left(f_{s_2}^{u^*}(v)+f_g^{u^*}(v)Y_{s_2}^{u^*}(v)\right.\\
&\left.+f_h^{u^*}(v)J_{s_2}^{u^*}(v)-F_{s_2}^{u^*}(v)\right)\sigma_2\left(S_2(v)\right)^{k_2+1}\mathrm{d}W_2(v)\Big\}.
\end{aligned}
\tag{A.17}
$$

By taking conditional expectation on both sides of (A.17), equation (A.17) reduces to

$$
F(t,z,s_1,s_2)=f(t,z,s_1,s_2,g^{u^*}(t),h^{u^*}(t))\leqslant\sup_{u\in U(t,z,s_1,s_2)}f(t,z,s_1,s_2,g^u(t),h^u(t)),
$$

which together with (A.14) gives

$$
F(t,z,s_1,s_2)=\sup_{u\in U(t,z,s_1,s_2)}f(t,z,s_1,s_2,g^u(t),h^u(t)).
$$

Thus, u^* is optimal and the supremum of $f^u(t)$ is $F(t,z,s_1,s_2)$.

Step 3: Prove u^* is indeed an equilibrium strategy. For any u_τ defined in Definition 2.2 and $t+\tau$, we rewrite (A.13) as

$$
\begin{aligned}
&f^{u_\tau}(t+\tau)=f^{u_\tau}(t)+\int_t^{t+\tau}\xi^{u_\tau}(v)\mathrm{d}v+\int_t^{t+\tau}\Big\{\left(f_z^{u_\tau}(v)+f_g^{u_\tau}(v)Y_z^{u_\tau}(v)\right.\\
&\left.+f_h^{u_\tau}(v)J_z^{u_\tau}(v)\right)\Big[(\alpha b-\alpha bp(v)+b\beta p(v))\,\mathrm{d}\overline{W}(v)+\alpha\sigma_1\pi_1(v)\left(S_1(v)\right)^{k_1}\mathrm{d}W_1(v)\\
&+\beta\sigma_2\pi_2(v)\left(S_2(v)\right)^{k_2}\mathrm{d}W_2(v)\Big]+\left(f_{s_1}^{u_\tau}(v)+f_g^{u_\tau}(v)Y_{s_1}^{u_\tau}(v)\right.\\
&\left.+f_h^{u_\tau}(v)J_{s_1}^{u_\tau}(v)\right)\sigma_1\left(S_1(v)\right)^{k_1+1}\mathrm{d}W_1(v)+\left(f_{s_2}^{u_\tau}(v)+f_g^{u_\tau}(v)Y_{s_2}^{u_\tau}(v)\right.\\
&\left.+f_h^{u_\tau}(v)J_{s_2}^{u_\tau}(v)\right)\sigma_2\left(S_2(v)\right)^{k_2+1}\mathrm{d}W_2(v)\Big\}.
\end{aligned}
\tag{A.18}
$$

Replacing T by $t+\tau$ and u by u^* in (A.15), we obtain

$$
\begin{aligned}
F^{u^*}(t)\quad &\geqslant F^{u^*}(t+\tau)-\int_t^{t+\tau}\xi^{u^*}(v)\mathrm{d}v-\int_t^{t+\tau}\Big\{F_z^{u^*}(v)\left[(\alpha b-\alpha bp(v)+b\beta p(v))\,\mathrm{d}\overline{W}(v)\right.\\
&\left.+\alpha\sigma_1\pi_1(v)\left(S_1(v)\right)^{k_1}\mathrm{d}W_1(v)+\beta\sigma_2\pi_2(v)\left(S_2(v)\right)^{k_2}\mathrm{d}W_2(v)\right]\\
&+F_{s_1}^{u^*}(v)\sigma_1\left(S_1(v)\right)^{k_1+1}\mathrm{d}W_1(v)+F_{s_2}^{u^*}(v)\sigma_2\left(S_2(v)\right)^{k_2+1}\mathrm{d}W_2(v)\Big\}.
\end{aligned}
\tag{A.19}
$$

Based on the definition of u_τ, we have $f^{u_\tau}(t+\tau)\leqslant F^{u^*}((t+\tau))$. Hence, substituting (A.18) into (A.19) gives

$$\begin{aligned}F^{u^*}(t) \quad &\geqslant f^{u_\tau}(t)+\int_t^{t+\tau}\xi^{u_\tau}(v)\mathrm{d}v-\int_t^{t+\tau}\xi^{u^*}(v)\mathrm{d}v+\int_t^{t+\tau}\Big\{\left(f_z^{u_\tau}(v)+f_g^{u_\tau}(v)Y_z^{u_\tau}(v)\right.\\
&\left.+f_h^{u_\tau}(v)J_z^{u_\tau}(v)\right)\Big[(\alpha b-\alpha bp(v)+b\beta p(v))\,\mathrm{d}\overline{W}(v)+\alpha\sigma_1\pi_1(v)\left(S_1(v)\right)^{k_1}\mathrm{d}W_1(v)\\
&+\beta\sigma_2\pi_2(v)\left(S_2(v)\right)^{k_2}\mathrm{d}W_2(v)\Big]+\left(f_{s_1}^{u_\tau}(v)+f_g^{u_\tau}(v)Y_{s_1}^{u_\tau}(v)\right.\\
&\left.+f_h^{u_\tau}(v)J_{s_1}^{u_\tau}(v)\right)\sigma_1\left(S_1(v)\right)^{k_1+1}\mathrm{d}W(v)+\left(f_{s_2}^{u_\tau}(v)+f_g^{u_\tau}(v)Y_{s_2}^{u_\tau}(v)+f_h^{u_\tau}(v)J_{s_2}^{u_\tau}(v)\right)\\
&\times\sigma_2\left(S_2(v)\right)^{k_2+1}\mathrm{d}W(v)\Big\}-\textstyle\int_t^{(t+\tau)}\Big\{F_z^{u^*}(v)\left[(\alpha b-\alpha bp(v)+b\beta p(v))\,\mathrm{d}\overline{W}(v)\right.\\
&\left.+\alpha\sigma_1\pi_1(v)\left(S_1(v)\right)^{k_1}\mathrm{d}W_1(v)+\beta\sigma_2\pi_2(v)\left(S_2(v)\right)^{k_2}\mathrm{d}W_2(v)\right]\\
&+F_{s_1}^{u^*}(v)\sigma_1\left(S_1(v)\right)^{k_1+1}\mathrm{d}W_1(v)+F_{s_2}^{u^*}(v)\sigma_2\left(S_2(v)\right)^{k_2+1}\mathrm{d}W_2(v)\Big\}.\end{aligned}\tag{A.20}$$

Noting that $F^{u^*}(t)=f^{u^*}(t)$ and taking conditional expectation on both sides of (A.20), we obtain

$$f^{u^*}(t)\geqslant f^{u_\tau}(t)+\mathrm{E}_{t,z,s_1,s_2}\left[\int_t^{t+\tau}\xi^{u_\tau}(v)\mathrm{d}v-\int_t^{t+\tau}\xi^{u^*}(v)\mathrm{d}v\right],$$

which implies

$$\liminf_{\tau\to0}\frac{f^{u^*}(t)-f^{u_\tau}(t)}{\tau}\geqslant0,$$

and thus the proof is complete. □

Appendix B Proof of Theorem 3.1

Suppose that three real functions $F(t,z,s_1,s_2),G(t,z,s_1,s_2)$ and $H(t,z,s_1,s_2)$ satisfy the conditions given in Theorem 2.3 and $\Pi_1^u>F_{zz}(t,z,s_1,s_2)$ for all $(t,z,s_1,s_2)\in[0,T]\times\mathbb{R}\times Q$ and $u\in U(t,z,s_1,s_2)$. Equation (2.15) indicates that

$$\begin{aligned}&f_t=f_z=f_{s_1}=f_{s_2}=f_{zz}=f_{zg}=f_{zh}=f_{gh}=f_{zs_1}=f_{gs_1}=f_{hs_1}=f_{zs_2}=f_{gs_2}=f_{hs_2}\\
&=f_{hh}=f_{s_1s_1}=f_{s_2s_2}=0,\quad f_g=1+\gamma g,\quad f_h=-\frac{\gamma}{2},\quad f_{gg}=\gamma.\end{aligned}\tag{B.1}$$

From (2.22) and (2.23), we obtain

$$\begin{aligned}&\Pi_1^u=\gamma(g_z^u)^2,\ \ \Pi_2^u=\gamma(g_{s_1}^u)^2\ \ \Pi_3^u=\gamma(g_{s_2}^u)^2,\ \ \Pi_4^u=\gamma g_z^ug_{s_1}^u,\ \ \Pi_5^u=\gamma g_z^ug_{s_2}^u,\\
&\xi^u(t,z,s_1,s_2)=\frac{\gamma}{2}\left[\alpha^2\pi_1^2\sigma_1^2s_1^{2k_1}+\beta^2\pi_2^2\sigma_2^2s_2^{2k_2}+(\alpha b-bp(\alpha-\beta))^2\right](g_z^u)^2\\
&+\frac{\gamma}{2}\sigma_1^2s_1^{2k_1+2}(g_{s_1}^u)^2+\frac{\gamma}{2}\sigma_2^2s_2^{2k_2+2}(g_{s_2}^u)^2+\gamma\alpha\pi_1\sigma_1^2s_1^{2k_1+1}g_z^ug_{s_1}^u+\gamma\beta\pi_2\sigma_2^2s_2^{2k_2+1}g_z^ug_{s_2}^u.\end{aligned}\tag{B.2}$$

Inputting (2.17) and (B.2) into (2.18), we have

$$\begin{aligned}&\sup\nolimits_{u\in U(t,z,s_1,s_2)}\Big\{F_t+\left[rz+\alpha\pi_1(\mu_1-r)+\beta\pi_2(\mu_2-r)+a\alpha\theta-a\eta p(\alpha-\beta)\right]F_z+\mu_1s_1F_{s_1}\\
&+\mu_2s_2F_{s_2}+\tfrac12\left[\alpha^2\pi_1^2\sigma_1^2s_1^{2k_1}+\beta^2\pi_2^2\sigma_2^2s_2^{2k_2}+\alpha^2b^2-2\alpha b^2p(\alpha-\beta)+b^2p^2(\alpha-\beta)^2\right]F_{zz}\\
&+\tfrac12\sigma_1^2s_1^{2k_1+2}F_{s_1s_1}+\tfrac12\sigma_2^2s_2^{2k_2+2}F_{s_2s_2}+\alpha\pi_1\sigma_1^2s_1^{2k_1+1}F_{zs_1}+\beta\pi_2\sigma_2^2s_2^{2k_2+1}F_{zs_2}\\
&-\tfrac{\gamma}{2}\left(\alpha^2\pi_1^2\sigma_1^2s_1^{2k_1}+\beta^2\pi_2^2\sigma_2^2s_2^{2k_2}+\left(\alpha b-bp(\alpha-\beta)^2\right)^2\right)G_z^2-\tfrac{\gamma}{2}\sigma_1^2s_1^{2k_1+2}G_{s_1}^2-\tfrac{\gamma}{2}\sigma_2^2s_2^{2k_2+2}G_{s_2}^2\\
&-\gamma\alpha\pi_1\sigma_1^2s_1^{2k_1+1}G_zG_{s_1}-\gamma\beta\pi_2\sigma_2^2s_2^{2k_2+1}G_zG_{s_2}\Big\}=0.\end{aligned}$$

Differentiating (2.18) with respect to π_1, π_2, p respectively and using (2.21), we obtain the following first-order optimality conditions:

$$\begin{cases} \pi_1^* = \dfrac{-(\mu_1 - r)F_z - \sigma_1^2 s_1^{2k_1+1} F_{zs_1} + \gamma\sigma_1^2 s_1^{2k_1+1} G_z G_{s_1}}{\alpha\sigma_1^2 s_1^{2k_1}[F_{zz} - \gamma G_z^2]}, \\ \pi_2^* = \dfrac{-(\mu_2 - r)F_z - \sigma_2^2 s_2^{2k_2+1} F_{zs_2} + \gamma\sigma_2^2 s_2^{2k_2+1} G_z G_{s_2}}{\beta\sigma_2^2 s_2^{2k_2}[F_{zz} - \gamma G_z^2]}, \end{cases} \tag{B.3}$$

and

$$p^0 = \frac{\alpha}{\alpha - \beta} + \frac{a\eta F_z}{(\alpha - \beta)b^2[F_{zz} - \gamma G_z^2]}. \tag{B.4}$$

The expressions for π_1* and π_2^* in (B.3) depend on function F and G. As it can be seen shortly, however, they eventually have a uniform expressions as given in (3.3) and (3.4).

To proceed, define sets

$$\begin{aligned} \mathcal{A}_1 &= \left\{(t, z, s_1, s_2) \in [0, T] \times Q;\ 0 \leqslant p^0 \leqslant 1\right\}, \\ \mathcal{A}_2 &= \left\{(t, z, s_1, s_2) \in [0, T] \times Q;\ p^0 < 0\right\}, \\ \mathcal{A}_3 &= \left\{(t, z, s_1, s_2) \in [0, T] \times Q;\ p^0 > 1\right\}. \end{aligned}$$

For $(t, z, s_1, s_2) \in \mathcal{A}_1$, the supremum of (2.18) over p is attained at p^0 given by (B.3). Introducing (B.3) and (B.4) into (2.18) and (2.19) gives

$$\begin{aligned} &F_t + (rz + a\alpha(\theta - \eta))F_z + \mu_1 s_1 F_{s_1} + \mu_2 s_2 F_{s_2} + \frac{1}{2}\sigma_1^2 s_1^{2k_1+2} F_{s_1 s_1} \\ &+\frac{1}{2}\sigma_2^2 s_2^{2k_2+2} F_{s_2 s_2} - \frac{\gamma}{2}\sigma_1^2 s_1^{2k_1+2} G_{s_1}^2 - \frac{\gamma}{2}\sigma_2^2 s_2^{2k_2+2} G_{s_2}^2 - \frac{a^2\eta^2 F_z^2}{2b^2(F_{zz} - \gamma G_z^2)} \\ &-\frac{\left[(\mu_1 - r)F_z + \sigma_1^2 s_1^{2k_1+1} F_{zs_1} - \gamma\sigma_1^2 s_1^{2k_1+1} G_z G_{s_1}\right]^2}{2\sigma_1^2 s_1^{2k_1}(F_{zz} - \gamma G_z^2)} \\ &-\frac{\left[(\mu_2 - r)F_z + \sigma_2^2 s_2^{2k_2+1} F_{zs_2} - \gamma\sigma_2^2 s_2^{2k_2+1} G_z G_{s_2}\right]^2}{2\sigma_2^2 s_2^{2k_2}(F_{zz} - \gamma G_z^2)} = 0, \end{aligned} \tag{B.5}$$

and

$$\begin{aligned} &G_t + (rz + a\alpha(\theta - \eta))G_z + \mu_1 s_1 G_{s_1} + \mu_2 s_2 G_{s_2} + \frac{1}{2}\sigma_1^2 s_1^{2k_1+2} G_{s_1 s_1} + \frac{1}{2}\sigma_2^2 s_2^{2k_2+2} G_{s_2 s_2} \\ &-\frac{a^2\eta^2 F_z}{b^2(F_{zz} - \gamma G_z^2)}\left(G_z - \frac{F_z G_{zz}}{2(F_{zz} - \gamma G_z^2)}\right) - \frac{(\mu_1 - r)F_z + \sigma_1^2 s_1^{2k_1+1} F_{zs_1} - \gamma\sigma_1^2 s_1^{2k_1+1} G_z G_{s_1}}{(F_{zz} - \gamma G_z^2)} \\ &\times\left[\frac{\mu_1 - r}{\sigma_1^2 s_1^{2k_1}} G_z + s_1 G_{zs_1} - \frac{\left[(\mu_1 - r)F_z + \sigma_1^2 s_1^{2k_1+1} F_{zs_1} - \gamma\sigma_1^2 s_1^{2k_1+1} G_z G_{s_1}\right]}{2\sigma_1^2 s_1^{2k_1}(F_{zz} - \gamma G_z^2)} G_{zz}\right] \\ &-\frac{(\mu_2 - r)F_z + \sigma_2^2 s_2^{2k_2+1} F_{zs_2} - \gamma\sigma_2^2 s_2^{2k_2+1} G_z G_{s_2}}{(F_{zz} - \gamma G_z^2)}\left[\frac{\mu_2 - r}{\sigma_2^2 s_2^{2k_2}} G_z + s_2 G_{zs_2}\right. \\ &\left.-\frac{(\mu_2 - r)F_z + \sigma_2^2 s_2^{2k_2+1} F_{zs_2} - \gamma\sigma_2^2 s_2^{2k_2+1} G_z G_{s_2}}{2\sigma_2^2 s_2^{2k_2}}(F_{zz} - \gamma G_z^2) G_{zz}\right] = 0. \end{aligned} \tag{B.6}$$

For $(t, z, s_1, s_2) \in \mathcal{A}_2$, (2.18) reaches its maximum at $p^* = 0$. Consequently, (2.18) and (2.19) respectively become

$$
\begin{aligned}
&F_t+(rz+a\alpha\theta)F_z+\mu_1 s_1 F_{s_1}+\mu_2 s_2 F_{s_2}+\frac{1}{2}\sigma_1^2 s_1^{2k_1+2}F_{s_1 s_1}+\frac{1}{2}\sigma_2^2 s_2^{2k_2+2}F_{s_2 s_2}\\
&-\frac{\gamma}{2}\sigma_1^2 s_1^{2k_1+2}G_{s_1}^2-\frac{\gamma}{2}\sigma_2^2 s_2^{2k_2+2}G_{s_2}^2+\frac{1}{2}\alpha^2 b^2 F_{zz}-\frac{\gamma}{2}\alpha^2 b^2 G_z^2\\
&-\frac{\left[(\mu_1-r)F_z+\sigma_1^2 s_1^{2k_1+1}F_{zs_1}-\gamma\sigma_1^2 s_1^{2k_1+1}G_z G_{s_1}\right]^2}{2\sigma_1^2 s_1^{2k_1}(F_{zz}-\gamma G_z^2)}\\
&-\frac{\left[(\mu_2-r)F_z+\sigma_2^2 s_2^{2k_2+1}F_{zs_2}-\gamma\sigma_2^2 s_2^{2k_2+1}G_z G_{s_2}\right]^2}{2\sigma_2^2 s_2^{2k_2}(F_{zz}-\gamma G_z^2)}=0
\end{aligned}
\tag{B.7}
$$

and

$$
\begin{aligned}
&G_t+(rz+a\alpha\theta)G_z+\mu_1 s_1 G_{s_1}+\mu_2 s_2 G_{s_2}+\frac{1}{2}\sigma_1^2 s_1^{2k_1+2}G_{s_1 s_1}+\frac{1}{2}\sigma_2^2 s_2^{2k_2+2}G_{s_2 s_2}\\
&+\frac{1}{2}\alpha^2 b^2 G_{zz}-\frac{(\mu_1-r)F_z+\sigma_1^2 s_1^{2k_1+1}F_{zs_1}-\gamma\sigma_1^2 s_1^{2k_1+1}G_z G_{s_1}}{(F_{zz}-\gamma G_z^2)}\\
&\times\left[\frac{\mu_1-r}{\sigma_1^2 s_1^{2k_1}}G_z+s_1 G_{zs_1}-\frac{\left[(\mu_1-r)F_z+\sigma_1^2 s_1^{2k_1+1}F_{zs_1}-\gamma\sigma_1^2 s_1^{2k_1+1}G_z G_{s_1}\right]}{2\sigma_1^2 s_1^{2k_1}(F_{zz}-\gamma G_z^2)}G_{zz}\right]\\
&-\frac{(\mu_2-r)F_z+\sigma_2^2 s_2^{2k_2+1}F_{zs_2}-\gamma\sigma_2^2 s_2^{2k_2+1}G_z G_{s_2}}{(F_{zz}-\gamma G_z^2)}\\
&\times\left[\frac{\mu_2-r}{\sigma_2^2 s_2^{2k_2}}G_z+s_2 G_{zs_2}-\frac{(\mu_2-r)F_z+\sigma_2^2 s_2^{2k_2+1}F_{zs_2}-\gamma\sigma_2^2 s_2^{2k_2+1}G_z G_{s_2}}{2\sigma_2^2 s_2^{2k_2}(F_{zz}-\gamma G_z^2)}G_{zz}\right]=0.
\end{aligned}
\tag{B.8}
$$

Similarly, the maximum of (2.18) over p on $\mathcal{A}_3$ is attained at $p^*=1$. Putting (B.3) and $p^*=1$ into (2.18) and (2.19) gives

$$
\begin{aligned}
&F_t+(rz+a\alpha(\theta-2\eta)+a\eta)F_z+\mu_1 s_1 F_{s_1}+\mu_2 s_2 F_{s_2}+\frac{1}{2}\sigma_1^2 s_1^{2k_1+2}F_{s_1 s_1}+\frac{1}{2}\sigma_2^2 s_2^{2k_2+2}F_{s_2 s_2}\\
&-\frac{\gamma}{2}\sigma_1^2 s_1^{2k_1+2}G_{s_1}^2-\frac{\gamma}{2}\sigma_2^2 s_2^{2k_2+2}G_{s_2}^2+\frac{1}{2}\left[\alpha^2 b^2-2\alpha b^2(\alpha-\beta)+b^2(\alpha-\beta)^2\right](F_{zz}-\gamma G_z^2)\\
&-\frac{\left[(\mu_1-r)F_z+\sigma_1^2 s_1^{2k_1+1}F_{zs_1}-\gamma\sigma_1^2 s_1^{2k_1+1}G_z G_{s_1}\right]^2}{2\sigma_1^2 s_1^{2k_1}(F_{zz}-\gamma G_z^2)}\\
&-\frac{\left[(\mu_2-r)F_z+\sigma_2^2 s_2^{2k_2+1}F_{zs_2}-\gamma\sigma_2^2 s_2^{2k_2+1}G_z G_{s_2}\right]^2}{2\sigma_2^2 s_2^{2k_2}(F_{zz}-\gamma G_z^2)}=0
\end{aligned}
\tag{B.9}
$$

and

$$
\begin{aligned}
&G_t + (rz + a\alpha(\theta - 2\eta) + a\eta)G_z + \mu_1 s_1 G_{s_1} + \mu_2 s_2 G_{s_2} + \frac{1}{2}\sigma_1^2 s_1^{2k_1+2} G_{s_1 s_1} \\
&+\frac{1}{2}\sigma_2^2 s_2^{2k_2+2} G_{s_2 s_2} + \frac{1}{2}\left[\alpha^2 b^2 - 2\alpha b^2(\alpha - \beta) + b^2(\alpha - \beta)^2\right] G_{zz} \\
&-\frac{(\mu_1 - r)F_z + \sigma_1^2 s_1^{2k_1+1} F_{zs_1} - \gamma\sigma_1^2 s_1^{2k_1+1} G_z G_{s_1}}{(F_{zz} - \gamma G_z^2)} \\
&\times\left[\frac{\mu_1 - r}{\sigma_1^2 s_1^{2k_1}} G_z + s_1 G_{zs_1} - \frac{\left[(\mu_1 - r)F_z + \sigma_1^2 s_1^{2k_1+1} F_{zs_1} - \gamma\sigma_1^2 s_1^{2k_1+1} G_z G_{s_1}\right]}{2\sigma_1^2 s_1^{2k_1}(F_{zz} - \gamma G_z^2)} G_{zz}\right] \\
&-\frac{(\mu_2 - r)F_z + \sigma_2^2 s_2^{2k_2+1} F_{zs_2} - \gamma\sigma_2^2 s_2^{2k_2+1} G_z G_{s_2}}{(F_{zz} - \gamma G_z^2)} \\
&\times\left[\frac{\mu_2 - r}{\sigma_2^2 s_2^{2k_2}} G_z + s_2 G_{zs_2} - \frac{(\mu_2 - r)F_z + \sigma_2^2 s_2^{2k_2+1} F_{zs_2} - \gamma\sigma_2^2 s_2^{2k_2+1} G_z G_{s_2}}{2\sigma_2^2 s_2^{2k_2}(F_{zz} - \gamma G_z^2)} G_{zz}\right] = 0.
\end{aligned}
\tag{B.10}
$$

The above three pairs of equations can be solved by the same procedure and we demonstrate the procedure with the pair of (B.5) and (B.6) only. To proceed, we conjecture the solutions in the following form:

$$
F(t, z, s_1, s_2) = A_1(t)z + \frac{B_1(t)}{\gamma} s_1^{-2k_1} + \frac{C_1(t)}{\gamma} s_2^{-2k_2} + \frac{D_1(t)}{\gamma}, \tag{B.11}
$$

$$
G(t, z, s_1, s_2) = A_2(t)z + \frac{B_2(t)}{\gamma} s_1^{-2k_1} + \frac{C_2(t)}{\gamma} s_2^{-2k_2} + \frac{D_2(t)}{\gamma} \tag{B.12}
$$

with boundary conditions $A_1(T) = A_2(T) = 1, B_1(T) = C_1(T) = D_1(T) = B_2(T) = C_2(T) = D_2(T) = 0$. Then

$$
\begin{aligned}
&F_t = A_1'(t)z + \tfrac{B_1'(t)}{\gamma} s_1^{-2k_1} + \tfrac{C_1'(t)}{\gamma} s_2^{-2k_2} + \tfrac{D_1'(t)}{\gamma}, \quad F_z = A_1(t), \quad F_{s_1} = \tfrac{-2k_1 B_1(t)}{\gamma} s_1^{-2k_1-1}, \\
&F_{s_2} = \tfrac{-2k_2 C_1(t)}{\gamma} s_2^{-2k_2-1}, \quad F_{s_1 s_1} = \tfrac{2k_1(2k_1+1)B_1(t)}{\gamma} s_1^{-2k_1-2}, \quad F_{s_2 s_2} = \tfrac{2k_2(2k_2+1)C_1(t)}{\gamma} s_2^{-2k_2-2}, \\
&F_{zz} = F_{zs_1} = F_{zs_2} = F_{s_1 s_2} = 0, \quad G_t = A_2'(t)z + \tfrac{B_2'(t)}{\gamma} s_1^{-2k_1} + \tfrac{C_2'(t)}{\gamma} s_2^{-2k_2} + \tfrac{D_2'(t)}{\gamma}, \\
&G_z = A_2(t), \quad G_{s_1 s_1} = \tfrac{2k_1(2k_1+1)B_2(t)}{\gamma} s_1^{-2k_1-2}, \quad G_{s_2 s_2} = \tfrac{2k_2(2k_2+1)C_2(t)}{\gamma} s_2^{-2k_2-2}, \\
&G_{s_1} = \tfrac{-2k_1 B_2(t)}{\gamma} s_1^{-2k_1-1}, \quad G_{s_2} = \tfrac{-2k_2 C_2(t)}{\gamma} s_2^{-2k_2-1}, \quad G_{zz} = G_{zs_1} = G_{zs_2} = G_{s_1 s_2} = 0.
\end{aligned}
$$

Substituting the above derivatives into (B.5) and (B.6) yields

$$
\begin{aligned}
&\left(A_1'(t) + rA_1(t)\right) z + \Bigg(B_1'(t) - 2k_1\mu_1 B_1(t) - 2\sigma_1^2 k_1^2 B_2^2(t) \\
&+\frac{[(\mu_1 - r)A_1(t) + 2\sigma_1^2 k_1 A_2(t)B_2(t)]^2}{2\sigma_1^2 A_2^2(t)}\Bigg) \frac{s_1^{-2k_1}}{\gamma} + \Bigg(C_1'(t) - 2k_2\mu_2 C_1(t) - 2\sigma_2^2 k_2^2 C_2^2(t) \\
&+\frac{[(\mu_2 - r)A_1(t) + 2\sigma_2^2 k_2 A_2(t)C_2(t)]^2}{2\sigma_2^2 A_2^2(t)}\Bigg) \frac{s_2^{-2k_2}}{\gamma} + \frac{D_1'(t)}{\gamma} + a\alpha(\theta - \eta)A_1(t) \\
&+\frac{1}{\gamma}\sigma_1^2 k_1(2k_1 + 1)B_1(t) + \frac{1}{\gamma}\sigma_2^2 k_2(2k_2 + 1)C_1(t) + \frac{a^2\eta^2 A_1^2(t)}{2\gamma b^2 A_2^2(t)} = 0
\end{aligned}
\tag{B.13}
$$

and

$$
\begin{aligned}
&(A_2'(t)+rA_2(t))\,z+\Bigg(B_2'(t)-2k_1\mu_1B_2(t)\\
&+\frac{(\mu_1-r)^2A_1(t)+2k_1\sigma_1^2(\mu_1-r)A_2(t)B_2(t)}{\sigma_1^2A_2(t)}\Bigg)\frac{s_1^{-2k_1}}{\gamma}+\Bigg(C_2'(t)-2k_2\mu_2C_2(t)\\
&+\frac{(\mu_2-r)^2A_1(t)+2k_2\sigma_2^2(\mu_2-r)A_2(t)C_2(t)}{\sigma_2^2A_2(t)}\Bigg)\frac{s_2^{-2k_2}}{\gamma}+\frac{D_2'(t)}{\gamma}+a\alpha(\theta-\eta)A_2(t)\\
&+\frac{1}{\gamma}\sigma_1^2k_1(2k_1+1)B_2(t)+\frac{1}{\gamma}\sigma_2^2k_2(2k_2+1)C_2(t)+\frac{a^2\eta^2A_1^2(t)}{\gamma b^2A_2(t)}=0.
\end{aligned}
\tag{B.14}
$$

In order to eliminate the dependence on z, s_1 and s_2, we can decompose (B.13) and (B.14) into

$$A_1'(t)+rA_1(t)=0, \tag{B.15}$$

$$B_1'(t)-2k_1\mu_1B_1(t)-2\sigma_1^2k_1^2B_2^2(t)+\frac{[(\mu_1-r)A_1(t)+2\sigma_1^2k_1A_2(t)B_2(t)]^2}{2\sigma_1^2A_2^2(t)}=0, \tag{B.16}$$

$$C_1'(t)-2k_2\mu_2C_1(t)-2\sigma_2^2k_2^2C_2^2(t)+\frac{[(\mu_2-r)A_1(t)+2\sigma_2^2k_2A_2(t)C_2(t)]^2}{2\sigma_2^2A_2^2(t)}=0, \tag{B.17}$$

$$\frac{D_1'(t)}{\gamma}+a\alpha(\theta-\eta)A_1(t)+\frac{1}{\gamma}\sigma_1^2k_1(2k_1+1)B_1(t)+\frac{1}{\gamma}\sigma_2^2k_2(2k_2+1)C_1(t)+\frac{a^2\eta^2A_1^2(t)}{2\gamma b^2A_2^2(t)}=0, \tag{B.18}$$

$$A_2'(t)+rA_2(t)=0, \tag{B.19}$$

$$B_2'(t)-2k_1\mu_1B_2(t)+\frac{(\mu_1-r)^2A_1(t)+2k_1\sigma_1^2(\mu_1-r)A_2(t)B_2(t)}{\sigma_1^2A_2(t)}=0, \tag{B.20}$$

$$C_2'(t)-2k_2\mu_2C_2(t)+\frac{(\mu_2-r)^2A_1(t)+2k_2\sigma_2^2(\mu_2-r)A_2(t)C_2(t)}{\sigma_2^2A_2(t)}=0, \tag{B.21}$$

$$\frac{D_2'(t)}{\gamma}+a\alpha(\theta-\eta)A_2(t)+\frac{1}{\gamma}\sigma_1^2k_1(2k_1+1)B_2(t)+\frac{1}{\gamma}\sigma_2^2k_2(2k_2+1)C_2(t)+\frac{a^2\eta^2A_1^2(t)}{\gamma b^2A_2(t)}=0. \tag{B.22}$$

Considering the boundary conditions, we obtain

$$
\begin{aligned}
&A_1(t)=A_2(t)=\mathrm{e}^{r(T-t)},\\
&B_1(t)=\frac{(\mu_1-r)^2}{2k_1\sigma_1^2r}\left[\mathrm{e}^{-2k_1\mu_1(T-t)}-\mathrm{e}^{-2k_1r(T-t)}\right]+\frac{(\mu_1-r)^2(2\mu_1-r)}{4k_1\mu_1r\sigma_1^2}\left[1-\mathrm{e}^{-2k_1\mu_1(T-t)}\right],\\
&C_1(t)=\frac{(\mu_2-r)^2}{2k_2\sigma_2^2r}\left[\mathrm{e}^{-2k_2\mu_2(T-t)}-\mathrm{e}^{-2k_2r(T-t)}\right]+\frac{(\mu_2-r)^2(2\mu_2-r)}{4k_2\mu_2r\sigma_2^2}\left[1-\mathrm{e}^{-2k_2\mu_2(T-t)}\right],\\
&D_1(t)=\frac{\gamma a\alpha(\theta-\eta)}{r}\left(\mathrm{e}^{r(T-t)}-1\right)+\frac{a^2\eta^2}{2b^2}(T-t)+\sigma_1^2k_1(2k_1+1)\int_t^TB_1(s)\mathrm{d}s\\
&\qquad+\sigma_2^2k_2(2k_2+1)\int_t^TC_1(s)\mathrm{d}s.\\
&B_2(t)=\frac{(\mu_1-r)^2}{2k_1\sigma_1^2r}\left(1-\mathrm{e}^{-2k_1r(T-t)}\right),\quad C_2(t)=\frac{(\mu_2-r)^2}{2k_2\sigma_2^2r}\left(1-\mathrm{e}^{-2k_2r(T-t)}\right),\\
&D_2(t)=\frac{\gamma a\alpha(\theta-\eta)}{r}\left(\mathrm{e}^{r(T-t)}-1\right)+\frac{a^2\eta^2}{2b^2}(T-t)+\sigma_1^2k_1(2k_1+1)\int_t^TB_2(s)\mathrm{d}s\\
&\qquad+\sigma_2^2k_2(2k_2+1)\int_t^TC_2(s)\mathrm{d}s.
\end{aligned}
\tag{B.23}
$$

In the rest of the proof, we shall discuss the equilibrium strategies in each of the cases as listed in Table 1. We shall rely on the first-order conditions given in B.3 and B.4, the procedure as demonstrated in the above for solving the three pairs of equations, as well as the continuity of the functions F and G. To proceed, we note that the supreme in (2.18) is attained at $p^*(t)=0$ for $p^0(t)<0$ and $p^*(t)=1$ for $p^0(t)>1$; moreover, according to (B.4), $0\leqslant p^0\leqslant 1$ if and only if $t_1\leqslant t\leqslant t_2$, where t_1 and t_2 are given in ((3.1)) and ((3.2)) respectively. The solutions are obtained by checking when the condition $0 \leqslant p^0 \leqslant 1$ is satisfied or not over the investment time horizon as given below for each case outlined in Table 1.

Case III: In each of these four cases, t_1 and t_2 satisfy $t_1 < 0 < T \leqslant t_2$ and therefore the equilibrium reinsurance strategy is

$$p^*(t) = p^0(t), \quad 0 \leqslant t \leqslant T.$$

The above derivation gives F and G expressed in (B.11) and (B.12) with $A_1(t), B_1(t), C_1(t), D_1(t), A_2(t), B_2(t), C_2(t), D_2(t)$ given in (B.23). Inserting (B.11) and (B.12) into (B.3), we derive (3.3) and (3.4).

Case V: In each of the two cases, we have $0 \leqslant t_1 < T \leqslant t_2$ and hence,

$$p^*(t) = \begin{cases} 0, & 0 \leqslant t < t_1, \\ p^0(t), & t_1 \leqslant t \leqslant T. \end{cases}$$

Similar to the preceding case, we obtain the expression of F and G as following:

$$F(t,z,s_1,s_2) = \begin{cases} A_1(t)z + \frac{B_1(t)}{\gamma}s_1^{-2k_1} + \frac{C_1(t)}{\gamma}s_2^{-2k_2} + \frac{\overline{\tilde{D}}_1(t)}{\gamma}, & 0 \leqslant t < t_1, \\ A_1(t)z + \frac{B_1(t)}{\gamma}s_1^{-2k_1} + \frac{C_1(t)}{\gamma}s_2^{-2k_2} + \frac{D_1(t)}{\gamma}, & t_1 \leqslant t \leqslant T, \end{cases} \tag{B.24}$$

and

$$G(t,z,s_1,s_2) = \begin{cases} A_2(t)z + \frac{B_2(t)}{\gamma}s_1^{-2k_1} + \frac{C_2(t)}{\gamma}s_2^{-2k_2} + \frac{\overline{\tilde{D}}_2(t)}{\gamma}, & 0 \leqslant t < t_1, \\ A_2(t)z + \frac{B_2(t)}{\gamma}s_1^{-2k_1} + \frac{C_2(t)}{\gamma}s_2^{-2k_2} + \frac{D_2(t)}{\gamma}, & t_1 \leqslant t \leqslant T, \end{cases} \tag{B.25}$$

where $A_1(t)$, $B_1(t)$, $C_1(t)$, $D_1(t)$, $A_2(t)$, $B_2(t)$, $C_2(t)$ and $D_2(t)$ are given in (B.23) and

$$\begin{aligned} \overline{\tilde{D}}_1(t) &= \frac{\gamma a\alpha\theta}{r}\left[e^{r(T-t)} - 1\right] + \frac{\gamma a\alpha\eta}{r}\left[1 - e^{r(T-t_1)}\right] + \frac{\gamma^2\alpha^2 b^2}{4r}\left[e^{2r(T-t_1)} - e^{2r(T-t)}\right] \\ &\quad + \frac{a^2\eta^2}{2b^2}(T-t_1) + \sigma_1^2 k_1(2k_1+1)\int_t^T B_1(s)ds + \sigma_2^2 k_2(2k_2+1)\int_t^T C_1(s)ds, \\ \overline{\tilde{D}}_2(t) &= \frac{\gamma a\alpha\theta}{r}\left[e^{r(T-t)} - 1\right] + \frac{\gamma a\alpha\eta}{r}\left[1 - e^{r(T-t_1)}\right] + \frac{a^2\eta^2}{b^2}(T-t_1) \\ &\quad + \sigma_1^2 k_1(2k_1+1)\int_t^T B_2(s)ds + \sigma_2^2 k_2(2k_2+1)\int_t^T C_2(s)ds. \end{aligned} \tag{B.26}$$

Equations (B.24) and (B.25) give the equilibrium investment strategies expressed as in (3.3) and (3.4).

Case VI: In this case, $0 < T \leqslant t_1 < t_2$ and thus,

$$p^*(t) = 0, \quad 0 \leqslant t \leqslant T.$$

For $0 \leqslant t \leqslant T$, F and G satisfy equations (B.7) and (B.8). Considering the boundary conditions in (2.18) and (2.19), we obtain

$$\begin{aligned} F(t,z,s_1,s_2) &= A_1(t)z + \frac{B_1(t)}{\gamma}s_1^{-2k_1} + \frac{C_1(t)}{\gamma}s_2^{-2k_2} + \frac{\tilde{D}_1(t)}{\gamma}, \\ G(t,z,s_1,s_2) &= A_2(t)z + \frac{B_2(t)}{\gamma}s_1^{-2k_1} + \frac{C_2(t)}{\gamma}s_2^{-2k_2} + \frac{\tilde{D}_2(t)}{\gamma}, \end{aligned} \tag{B.27}$$

where

$$\begin{aligned} \tilde{D}_1(t) &= \frac{\gamma a\alpha\theta}{r}\left[e^{r(T-t)} - 1\right] + \frac{\gamma^2\alpha^2 b^2}{4r}\left[1 - e^{2r(T-t)}\right] + \sigma_1^2 k_1(2k_1+1)\int_t^T B_1(s)ds \\ &\quad + \sigma_2^2 k_2(2k_2+1)\int_t^T C_1(s)ds, \\ \tilde{D}_2(t) &= \frac{\gamma a\alpha\theta}{r}\left[e^{r(T-t)} - 1\right] + \sigma_1^2 k_1(2k_1+1)\int_t^T B_2(s)ds + \sigma_2^2 k_2(2k_2+1)\int_t^T C_2(s)ds \end{aligned} \tag{B.28}$$

and $A_1(t)$, $B_1(t)$, $C_1(t)$, $A_2(t)$, $B_2(t)$ and $C_2(t)$ are defined in (B.23). Inserting (B.27) into (B.3) yields (3.3) and (3.4).

Cases IV, IX: In each of the two cases, $0 \leqslant t_1 < t_2 \leqslant T$ and thus,

$$p^*(t) = \begin{cases} 0, & 0 \leqslant t < t_1, \\ p^0(t), & t_1 \leqslant t < t_2, \\ 1, & t_2 \leqslant t \leqslant T. \end{cases}$$

Solving (B.7) and (B.8) for $0 \leqslant t < t_1$ and solving (B.9) and (B.10) for $t_2 \leqslant t \leqslant T$, noting the continuity of F and G and taking the boundary conditions into account, we can similarly obtain

$$F(t,z,s_1,s_2) = \begin{cases} A_1(t)z + \frac{B_1(t)}{\gamma} s_1^{-2k_1} + \frac{C_1(t)}{\gamma} s_2^{-2k_2} + \frac{\overline{D}_1(t)}{\gamma}, & 0 \leqslant t < t_1, \\ A_1(t)z + \frac{B_1(t)}{\gamma} s_1^{-2k_1} + \frac{C_1(t)}{\gamma} s_2^{-2k_2} + \frac{\widetilde{D}_1(t)}{\gamma}, & t_1 \leqslant t < t_2, \\ A_1(t)z + \frac{B_1(t)}{\gamma} s_1^{-2k_1} + \frac{C_1(t)}{\gamma} s_2^{-2k_2} + \frac{\hat{D}_1(t)}{\gamma}, & t_2 \leqslant t \leqslant T, \end{cases} \tag{B.29}$$

and

$$G(t,z,s_1,s_2) = \begin{cases} A_2(t)z + \frac{B_2(t)}{\gamma} s_1^{-2k_1} + \frac{C_2(t)}{\gamma} s_2^{-2k_2} + \frac{\overline{D}_2(t)}{\gamma}, & 0 \leqslant t < t_1, \\ A_2(t)z + \frac{B_2(t)}{\gamma} s_1^{-2k_1} + \frac{C_2(t)}{\gamma} s_2^{-2k_2} + \frac{\widetilde{D}_2(t)}{\gamma}, & t_1 \leqslant t < t_2, \\ A_2(t)z + \frac{B_2(t)}{\gamma} s_1^{-2k_1} + \frac{C_2(t)}{\gamma} s_2^{-2k_2} + \frac{\hat{D}_2(t)}{\gamma}, & t_2 \leqslant t \leqslant T, \end{cases} \tag{B.30}$$

where

$$\begin{aligned} \overline{D}_1(t) &= \frac{\gamma a\alpha\theta}{r}\left[\mathrm{e}^{r(T-t)} - 1\right] + \frac{\gamma^2\alpha^2 b^2}{4r}\left[\mathrm{e}^{2r(T-t_1)} - \mathrm{e}^{2r(T-t)}\right] + \sigma_1^2 k_1(2k_1+1)\int_t^T B_1(s)\mathrm{d}s \\ &\quad + \sigma_2^2 k_2(2k_2+1)\int_t^T C_1(s)\mathrm{d}s - \frac{\gamma a\alpha\eta}{r}\left[2 - \mathrm{e}^{r(T-t_1)} - \mathrm{e}^{r(T-t_2)}\right] + \frac{a^2\eta^2}{2b^2}(t_2 - t_1) \\ &\quad + \frac{\gamma a\eta}{r}\left[\mathrm{e}^{r(T-t_2)} - 1\right] + \frac{\gamma^2[\alpha^2 b^2 - 2\alpha b^2(\alpha-\beta) + b^2(\alpha-\beta)^2]}{4r}\left[1 - \mathrm{e}^{2r(T-t_2)}\right], \\ \widetilde{D}_1(t) &= \frac{\gamma a\alpha(\theta-\eta)}{r}\left[\mathrm{e}^{r(T-t)} - 1\right] + \frac{a^2\eta^2}{2b^2}(t_2 - t) + \frac{\gamma a\beta\eta}{r}\left[\mathrm{e}^{r(T-t_2)} - 1\right] \\ &\quad + \frac{\gamma^2[\alpha^2 b^2 - 2\alpha b^2(\alpha-\beta) + b^2(\alpha-\beta)^2]}{4r}\left[1 - \mathrm{e}^{2r(T-t_2)}\right] \\ &\quad + \sigma_1^2 k_1(2k_1+1)\textstyle\int_t^T B_1(s)\mathrm{d}s + \sigma_2^2 k_2(2k_2+1)\int_t^T C_1(s)\mathrm{d}s, \\ \hat{D}_1(t) &= \frac{\gamma[a\alpha(\theta-2\eta) + a\eta]}{r}[\mathrm{e}^{r(T-t)} - 1] + \frac{\gamma^2[\alpha^2 b^2 - 2\alpha b^2(\alpha-\beta) + b^2(\alpha-\beta)^2]}{4r} \\ &\quad \times[1 - \mathrm{e}^{2r(T-t)}] + \sigma_1^2 k_1(2k_1+1)\int_t^T B_1(s)\mathrm{d}s + \sigma_2^2 k_2(2k_2+1)\int_t^T C_1(s)\mathrm{d}s, \end{aligned} \tag{B.31}$$

$$
\begin{aligned}
\overline{D}_2(t) &= \frac{\gamma a\alpha\theta}{r}\left[\mathrm{e}^{r(T-t)}-1\right]+\frac{a^2\eta^2}{b^2}(t_2-t_1)+\frac{\gamma a\alpha\eta}{r}\left[2-\mathrm{e}^{r(T-t_1)}-\mathrm{e}^{r(T-t_2)}\right]\\
&\quad+\frac{\gamma a\eta}{r}\left[\mathrm{e}^{r(T-t_2)}-1\right]+\sigma_1^2k_1(2k_1+1)\int_t^T B_2(s)\mathrm{d}s+\sigma_2^2k_2(2k_2+1)\int_t^T C_2(s)\mathrm{d}s,\\
\widetilde{D}_2(t) &= \frac{\gamma a\alpha(\theta-\eta)}{r}\left[\mathrm{e}^{r(T-t)}-1\right]+\frac{a^2\eta^2}{b^2}(t_2-t)+\frac{\gamma a(1-\alpha)\eta}{r}\left[\mathrm{e}^{r(T-t_2)}-1\right]\\
&\quad+\sigma_1^2k_1(2k_1+1)\int_t^T B_2(s)\mathrm{d}s+\sigma_2^2k_2(2k_2+1)\int_t^T C_2(s)\mathrm{d}s,\\
\hat{D}_2(t) &= \frac{\gamma[a\alpha(\theta-2\eta)+a\eta]}{r}\left(\mathrm{e}^{r(T-t)}-1\right)+\sigma_1^2k_1(2k_1+1)\int_t^T B_2(s)\mathrm{d}s\\
&\quad+\sigma_2^2k_2(2k_2+1)\int_t^T C_2(s)\mathrm{d}s
\end{aligned}
\tag{B.32}
$$

and $A_1(t)$, $B_1(t)$, $C_1(t)$, $A_2(t)$, $B_2(t)$ and $C_2(t)$ are given in (B.23). Putting (B.29) and (B.30) into (B.3) yields (3.3) and (3.4).

Cases II, VIII: In each of the three cases, $t_1 < 0 \leqslant t_2 < T$ and therefore,

$$
p^*(t)=\begin{cases} p^0(t), & 0\leqslant t<t_2,\\ 1, & t_2\leqslant t\leqslant T.\end{cases}
$$

Similar to the preceding cases, F and G can be solved explicitly as follows:

$$
F(t,z,s_1,s_2)=\begin{cases} A_1(t)z+\frac{B_1(t)}{\gamma}s_1^{-2k_1}+\frac{C_1(t)}{\gamma}s_2^{-2k_2}+\frac{\widetilde{D}_1(t)}{\gamma}, & 0\leqslant t<t_2,\\ A_1(t)z+\frac{B_1(t)}{\gamma}s_1^{-2k_1}+\frac{C_1(t)}{\gamma}s_2^{-2k_2}+\frac{\hat{D}_1(t)}{\gamma}, & t_2\leqslant t\leqslant T,\end{cases}
\tag{B.33}
$$

and

$$
G(t,z,s_1,s_2)=\begin{cases} A_2(t)z+\frac{B_2(t)}{\gamma}s_1^{-2k_1}+\frac{C_2(t)}{\gamma}s_2^{-2k_2}+\frac{\widetilde{D}_2(t)}{\gamma}, & 0\leqslant t<t_2,\\ A_2(t)z+\frac{B_2(t)}{\gamma}s_1^{-2k_1}+\frac{C_2(t)}{\gamma}s_2^{-2k_2}+\frac{\hat{D}_2(t)}{\gamma}, & t_2\leqslant t\leqslant T\end{cases}
\tag{B.34}
$$

with $A_1(t), B_1(t), C_1(t), A_2(t), B_2(t), C_2(t)$ and $\widetilde{D}_1(t), \hat{D}_1(t), \widetilde{D}_2(t), \hat{D}_2(t)$ are given in (B.23), (B.31) and (B.32). Inserting (B.33) and (B.34) into (B.3), we obtain (3.3) and (3.4).

Cases I, VII, X: In each of the three cases, t_1 and t_2 satisfy $t_1 < t_2 < 0 < T$, and therefore,

$$
p^*(t)=1,\quad 0\leqslant t\leqslant T.
$$

Similarly, we can obtain

$$
\begin{aligned}
F(t,z,s_1,s_2) &= A_1(t)z+\frac{B_1(t)}{\gamma}s_1^{-2k_1}+\frac{C_1(t)}{\gamma}s_2^{-2k_2}+\frac{\hat{D}_1(t)}{\gamma},\\
G(t,z,s_1,s_2) &= A_2(t)z+\frac{B_2(t)}{\gamma}s_1^{-2k_1}+\frac{C_2(t)}{\gamma}s_2^{-2k_2}+\frac{\hat{D}_2(t)}{\gamma}
\end{aligned}
\tag{B.35}
$$

with $A_1(t)$, $B_1(t)$, $C_1(t)$, $A_2(t)$, $B_2(t)$, $C_2(t)$, $\hat{D}_1(t)$ and $\hat{D}_2(t)$ defined in (B.23), (B.31) and (B.32). Using (B.35) and (B.3), we can derive (3.3) and (3.4).

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