

THE THICKNESS OF $K_{1,n,n}$ AND $K_{2,n,n}$

XIA GUO AND YAN YANG

ABSTRACT. The thickness of a graph G is the minimum number of planar subgraphs whose union is G . In this paper, we obtain the thickness of complete 3-partite graph $K_{1,n,n}$, $K_{2,n,n}$ and complete 4-partite graph $K_{1,1,n,n}$.

1. INTRODUCTION

The *thickness* $\theta(G)$ of a graph G is the minimum number of planar subgraphs whose union is G . It was first defined by W.T.Tutte [7] in 1963, then a few authors obtained the thickness of hypercubes [5], complete graphs [1, 2, 8] and complete bipartite graphs [3]. Naturally, people wonder about the thickness of the complete multipartite graphs.

A *complete k -partite graph* is a graph whose vertex set can be partitioned into k parts, such that every edge has its ends in different parts and every two vertices in different parts are adjacent. Let K_{p_1, p_2, \dots, p_k} denote a complete k -partite graph in which the i th part contains p_i ($1 \leq i \leq k$) vertices. For the complete 3-partite graph, Poranen proved $\theta(K_{n,n,n}) \leq \lceil \frac{n}{2} \rceil$ in [6], then Yang [10] gave a new upper bound for $\theta(K_{n,n,n})$, i.e., $\theta(K_{n,n,n}) \leq \lceil \frac{n+1}{3} \rceil + 1$ and obtained $\theta(K_{n,n,n}) = \lceil \frac{n+1}{3} \rceil$, when $n \equiv 3 \pmod{6}$. And also Yang [9] gave the thickness number of $K_{l,m,n}$ ($l \leq m \leq n$) when $l + m \leq 5$ and showed that $\theta(K_{l,m,n}) = \lceil \frac{l+m}{2} \rceil$ when $l + m$ is even and $n > \frac{1}{2}(l + m - 2)^2$; or $l + m$ is odd and $n > (l + m - 2)(l + m - 1)$.

In this paper, we obtain the thickness of complete 3-partite graph $K_{1,n,n}$ and $K_{2,n,n}$, and we also deduce the thickness of complete 4-partite graph $K_{1,1,n,n}$ from that of $K_{2,n,n}$.

2. THE THICKNESS OF $K_{1,n,n}$

In [3], Beineke, Harary and Moon gave the thickness of complete bipartite graphs $K_{m,n}$ for most value of m and n , and their theorem implies the following result immediately.

Lemma 2.1. [3] *The thickness of the complete bipartite graph $K_{n,n}$ is*

$$\theta(K_{n,n}) = \lceil \frac{n+2}{4} \rceil.$$

2010 *Mathematics Subject Classification.* 05C10.

Key words and phrases. thickness; complete 3-partite graph; complete 4-partite graph.

This work was supported by NNSF of China under Grant No. 11401430.

In [4], Chen and Yin gave a planar decomposition of the complete bipartite graph $K_{4p,4p}$ with $p+1$ planar subgraphs. Figure 1 shows their planar decomposition of $K_{4p,4p}$, in which $\{u_1, \dots, u_{4p}\} = U$ and $\{v_1, \dots, v_{4p}\} = V$ are the 2-partite vertex sets of it. Based on their decomposition, we give a planar decomposition of $K_{2,n,n}$ with $p+1$ subgraphs when $n \equiv 0$ or $3 \pmod{4}$ and prove the following lemma.

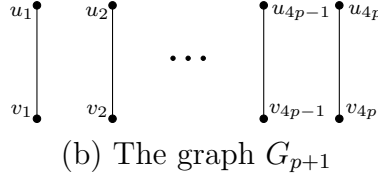
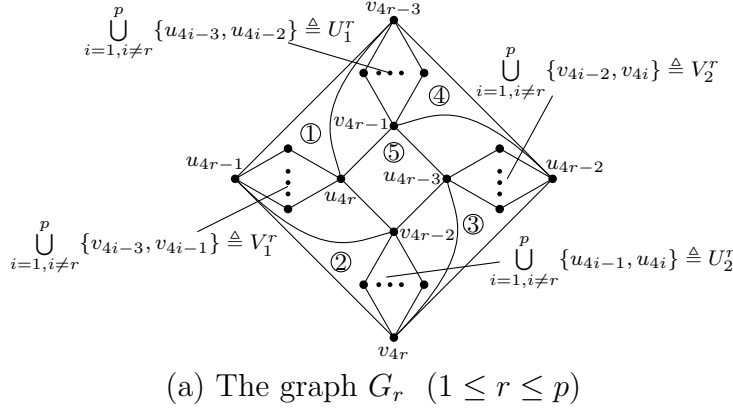


FIGURE 1. A planar decomposition of $K_{4p,4p}$

Lemma 2.2. *The thickness of the complete 3-partite graph $K_{1,n,n}$ and $K_{2,n,n}$ is*

$$\theta(K_{1,n,n}) = \theta(K_{2,n,n}) = \lceil \frac{n+2}{4} \rceil,$$

when $n \equiv 0$ or $3 \pmod{4}$.

Proof. Let the vertex partition of $K_{2,n,n}$ be (X, U, V) , where $X = \{x_1, x_2\}$, $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$.

When $n \equiv 0 \pmod{4}$, let $n = 4p$ ($p \geq 1$). Let $\{G_1, \dots, G_{p+1}\}$ be the planar decomposition of $K_{n,n}$ constructed by Chen and Yin in [4]. As shown in Figure 1, the graph G_{p+1} consists of n paths of length one. We put all the n paths in a row, place vertex x_1 on one side of the row and the vertex x_2 on the other side of the row, join both x_1 and x_2 to all vertices in G_{p+1} . Then we get a planar graph, denote it by \widehat{G}_{p+1} . It is easy to see that $\{G_1, \dots, G_p, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,n,n}$. Therefore, we have $\theta(K_{2,n,n}) \leq p+1$. Since $K_{n,n} \subset K_{1,n,n} \subset K_{2,n,n}$, combining it with Lemma 2.1, we have

$$p+1 = \theta(K_{n,n}) \leq \theta(K_{1,n,n}) \leq \theta(K_{2,n,n}) \leq p+1,$$

that is, $\theta(K_{1,n,n}) = \theta(K_{2,n,n}) = p+1$ when $n \equiv 0 \pmod{4}$.

When $n \equiv 3 \pmod{4}$, then $n = 4p + 3$ ($p \geq 0$). When $p = 0$, from [9], we have $\theta(K_{1,3,3}) = \theta(K_{2,3,3}) = 2$. When $p \geq 1$, since $K_{n,n} \subset K_{1,n,n} \subset K_{2,n,n} \subset K_{2,n+1,n+1}$, according to Lemma 2.1 and $\theta(K_{2,4p,4p}) = p + 1$, we have

$$p + 2 = \theta(K_{n,n}) \leq \theta(K_{1,n,n}) \leq \theta(K_{2,n,n}) \leq \theta(K_{2,n+1,n+1}) = p + 2.$$

Then, we get $\theta(K_{1,n,n}) = \theta(K_{2,n,n}) = p + 2$ when $n \equiv 3 \pmod{4}$.

Summarizing the above, the lemma is obtained. \square

Lemma 2.3. *There exists a planar decomposition of the complete 3-partite graph $K_{1,4p+2,4p+2}$ ($p \geq 0$) with $p + 1$ subgraphs.*

Proof. Suppose the vertex partition of the complete 3-partite graph $K_{1,n,n}$ is (X, U, V) , where $X = \{x\}$, $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$. When $n = 4p + 2$, we will construct a planar decomposition of $K_{1,4p+2,4p+2}$ with $p + 1$ planar subgraphs to complete the proof. Our construction is based on the planar decomposition $\{G_1, G_2, \dots, G_{p+1}\}$ of $K_{4p,4p}$ given in [4], as shown in Figure 1 and the reader is referred to [4] for more details about this decomposition.

For convenience, we denote the vertex set $\bigcup_{i=1, i \neq r}^p \{u_{4i-3}, u_{4i-2}\}$, $\bigcup_{i=1, i \neq r}^p \{u_{4i-1}, u_{4i}\}$,

$\bigcup_{i=1, i \neq r}^p \{v_{4i-3}, v_{4i-1}\}$ and $\bigcup_{i=1, i \neq r}^p \{v_{4i-2}, v_{4i}\}$ by U_1^r , U_2^r , V_1^r and V_2^r respectively. We also label some faces of G_r ($1 \leq r \leq p$), as indicated in Figure 1, for example, the face 1 is bounded by $v_{4r-3}u_{4r}v_ju_{4r-1}$ in which v_j is some vertex from V_1^r .

In the following, for $1 \leq r \leq p+1$, by adding vertices $x, u_{4p+1}, u_{4p+2}, v_{4p+1}, v_{4p+2}$ and some edges to G_r , and deleting some edges from G_r such edges will be added to the graph G_{p+1} , we will get a new planar graph \widehat{G}_r such that $\{\widehat{G}_1, \dots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{1,4p+2,4p+2}$. Because v_{4r-3} and v_{4r-1} in G_r ($1 \leq r \leq p$) is joined by $2p - 2$ edge-disjoint paths of length two that we call parallel paths, we can change the order of these parallel paths without changing the planarity of G_r . For the same reason, we can do changes like this for parallel paths between u_{4r-1} and u_{4r} , v_{4r-2} and v_{4r} , u_{4r-3} and u_{4r-2} . We call this change by parallel paths modification for simplicity. All the subscripts of vertices are taken modulo $4p$, except that of $v_{4p+1}, v_{4p+2}, u_{4p+1}$ and u_{4p+2} (the vertices we added to G_r).

Case 1. When p is even and $p > 2$.

(a) The construction for \widehat{G}_r , $1 \leq r \leq p$, and r is odd.

Step 1: Place the vertex x in the face 1 of G_r , delete edges $v_{4r-3}u_{4r}$ and $u_{4r}v_{4r-1}$ from G_r . Do parallel paths modification, such that $u_{4r+6} \in U_1^r$, $v_{4r+1} \in V_1^r$ and $u_{4r-3}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ are incident with a common face which the vertex x is in. Join x to $u_{4r-3}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ and u_{4r+6}, v_{4r+1} .

Step 2: Do parallel paths modification, such that $u_{4r+11}, u_{4r+12} \in U_2^r$ are incident with a common face. Place the vertex v_{4p+1} in the face, and join it to both u_{4r+11} and u_{4r+12} .

Step 3: Do parallel paths modification, such that $u_{4r+7}, u_{4r+8} \in U_2^r$ are incident with a common face. Place the vertex v_{4p+2} in the face, and join it to both u_{4r+7} and u_{4r+8} .

Step 4: Do parallel paths modification, such that $v_{4r+10}, v_{4r+12} \in V_2^r$ are incident with a common face. Place the vertex u_{4p+1} in the face, and join it to both v_{4r+10} and v_{4r+12} .

Step 5: Do parallel paths modification, such that $v_{4r+6}, v_{4r+8} \in V_2^r$ are incident with a common face. Place the vertex u_{4p+2} in the face, and join it to both v_{4r+6} and v_{4r+8} .

(b) The construction for \widehat{G}_r , $1 \leq r \leq p$, and r is even.

Step 1: Place the vertex x in the face 3 of G_r , delete edges $v_{4r}u_{4r-3}$ and $u_{4r-3}v_{4r-2}$ from G_r . Do parallel paths modification, such that $u_{4r+7} \in U_2^r$, $v_{4r+4} \in V_2^r$ and $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ are incident with a common face which the vertex x is in. Join x to $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ and u_{4r+7}, v_{4r+4} .

Step 2: Do parallel paths modifications, such that $u_{4r+5}, u_{4r+6} \in U_1^r$, $u_{4r+1}, u_{4r+2} \in U_1^r$, $v_{4r+5}, v_{4r+7} \in V_1^r$, $v_{4r+1}, v_{4r+3} \in V_1^r$ are incident with a common face, respectively. Join v_{4p+1} to both u_{4r+5} and u_{4r+6} , join v_{4p+2} to both u_{4r+1} and u_{4r+2} , join u_{4p+1} to both v_{4r+5} and v_{4r+7} , join u_{4p+2} to both v_{4r+1} and v_{4r+3} .

Table 1 shows how we add edges to $G_r(1 \leq r \leq p)$ in Case 1. The first column lists the edges we added, the second and third column lists the subscript of vertices, and we also indicate the vertex set which they belong to in brackets.

TABLE 1. The edges we add to $G_r(1 \leq r \leq p)$ in Case 1

subscript \ edge	case	r is odd		r is even	
xu_j		$4r-3, 4r-1, 4r$	$4r+6 (U_1^r)$	$4r-3, 4r-2, 4r$	$4r+7 (U_2^r)$
xv_j		$4r-3, 4r-2, 4r-1$	$4r+1 (V_1^r)$	$4r-2, 4r-1, 4r$	$4r+4 (V_2^r)$
$v_{4p+1}u_j$		$4r+11, 4r+12 (U_2^r)$		$4r+5, 4r+6 (U_1^r)$	
$v_{4p+2}u_j$		$4r+7, 4r+8 (U_2^r)$		$4r+1, 4r+2 (U_1^r)$	
$u_{4p+1}v_j$		$4r+10, 4r+12 (V_2^r)$		$4r+5, 4r+7 (V_1^r)$	
$u_{4p+2}v_j$		$4r+6, 4r+8 (V_2^r)$		$4r+1, 4r+3 (V_1^r)$	

(c) The construction for \widehat{G}_{p+1} .

From the construction in (a) and (b), the subscript set of u_j that xu_j is an edge in \widehat{G}_r for some $r \in \{1, \dots, p\}$ is

$$\begin{aligned} & \{4r-3, 4r-1, 4r, 4r+6(\bmod 4p) \mid 1 \leq r \leq p, \text{ and } r \text{ is odd}\} \\ & \cup \{4r-3, 4r-2, 4r, 4r+7(\bmod 4p) \mid 1 \leq r \leq p, \text{ and } r \text{ is even}\} \\ & = \{1, \dots, p\}. \end{aligned}$$

The subscript set of u_j that $v_{4p+1}u_j$ is an edge in \widehat{G}_r for some $r \in \{1, \dots, p\}$ is

$$\begin{aligned} & \{4r+11, 4r+12(\bmod 4p) \mid 1 \leq r \leq p, \text{ and } r \text{ is odd}\} \\ & \cup \{4r+5, 4r+6(\bmod 4p) \mid 1 \leq r \leq p, \text{ and } r \text{ is even}\} \\ & = \{4r-3, 4r-2, 4r-1, 4r \mid 1 \leq r \leq p, \text{ and } r \text{ is even}\}. \end{aligned}$$

Using the same procedure, we can list all the edges incident with $x, v_{4p+1}, v_{4p+2}, u_{4p+1}$ and u_{4p+2} in $\widehat{G}_r(1 \leq r \leq p)$, so we can also list the edges that are incident

with $x, v_{4p+1}, v_{4p+2}, u_{4p+1}$ in $K_{1,4p+2,4p+2}$ but not in any \widehat{G}_r ($1 \leq r \leq p$). Table 2 shows the edges that belong to $K_{1,4p+2,4p+2}$ but not to any \widehat{G}_r , $1 \leq r \leq p$, in which the fourth and fifth rows list the edges deleted from G_r ($1 \leq r \leq p$) in step one of (a) and (b), and the sixth row lists the edges of G_{p+1} . The \widehat{G}_{p+1} is the graph consists of the edges in Table 2, Figure 2 shows \widehat{G}_{p+1} is a planar graph.

TABLE 2. The edges of \widehat{G}_{p+1} in Case 1

edges	subscript
$xv_{4p+1}, xu_{4p+1}, v_{4p+1}u_j, u_{4p+1}v_j$	$j = 4r - 3, 4r - 2, 4r - 1, 4r, 4p + 2. (r = 1, 3, \dots, p - 1.)$
$xv_{4p+2}, xu_{4p+2}, v_{4p+2}u_j, u_{4p+2}v_j$	$j = 4r - 3, 4r - 2, 4r - 1, 4r, 4p + 1. (r = 2, 4, \dots, p.)$
$v_{4r-3}u_{4r}, u_{4r}v_{4r-1}$	$r = 1, 3, \dots, p - 1.$
$v_{4r}u_{4r-3}, u_{4r-3}v_{4r-2}$	$r = 2, 4, \dots, p.$
u_jv_j	$j = 1, \dots, 4p + 2.$

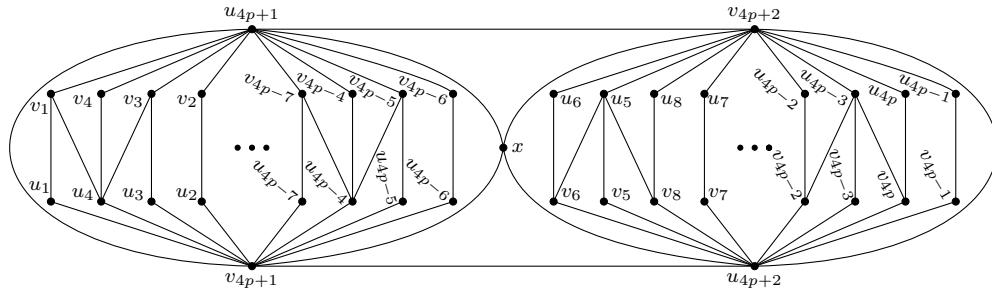
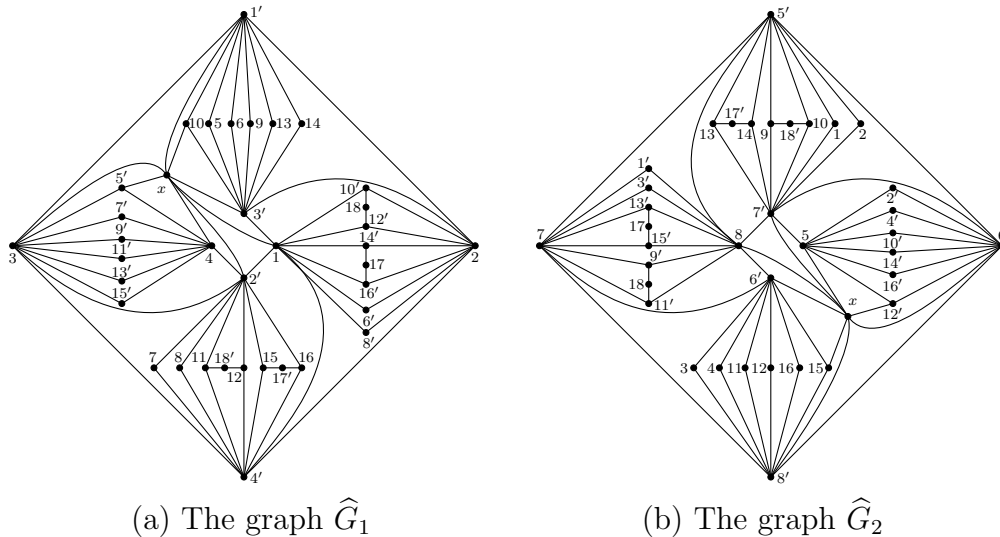


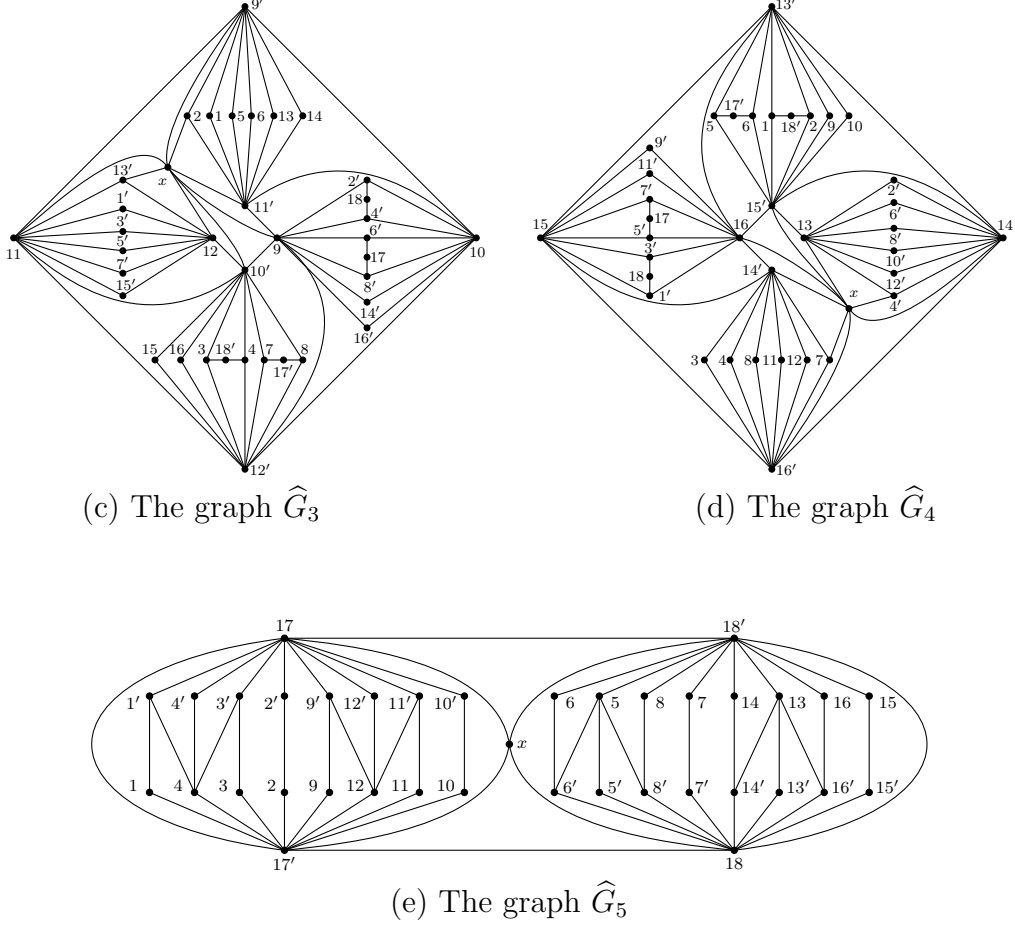
FIGURE 2. The graph \widehat{G}_{p+1} in Case 1

A planar decomposition $\{\widehat{G}_1, \dots, \widehat{G}_{p+1}\}$ of $K_{1,4p+2,4p+2}$ is obtained as above in this case. In Figure 3, we draw the planar decomposition of $K_{1,18,18}$, it is the smallest example for the Case 1. We denote vertex u_i and v_i by i and i' respectively in this figure.



(a) The graph \widehat{G}_1

(b) The graph \widehat{G}_2

FIGURE 3. A planar decomposition of $K_{1,18,18}$

Case 2. When p is odd and $p > 3$. The process is similar to that in Case 1.

(a) The construction for \widehat{G}_r , $1 \leq r \leq p$, and r is odd.

Step 1: Place the vertex x in the face 1 of G_r , delete edges $v_{4r-3}u_{4r}$ and $u_{4r}v_{4r-1}$ from G_r , for $1 \leq r \leq p$, and delete v_2u_1 from G_1 additionally.

For $1 < r < p$, do parallel paths modification to G_r , such that $u_{4r+6} \in U_1^r$, $v_{4r+1} \in V_1^r$ and $u_{4r-3}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ are incident with a common face which the vertex x is in. Join x to $u_{4r-3}, u_{4r-1}, u_{4r}, v_{4r-3}, v_{4r-2}, v_{4r-1}$ and u_{4r+6}, v_{4r+1} .

Similarly, in G_1 , join x to $u_1, u_3, u_4, v_1, v_2, v_3, v_4$ and $u_{10} \in U_1^1, v_5 \in V_1^1$. In G_p , join x to $u_{4p-3}, u_{4p-1}, u_{4p}, v_{4p-3}, v_{4p-2}, v_{4p-1}$ and $u_2 \in U_1^p$.

Step 2: For $1 \leq r < p$, do parallel paths modification to G_r , such that $u_{4r+11}, u_{4r+12} \in U_2^r, u_{4r+7}, u_{4r+8} \in U_2^r, v_{4r+10}, v_{4r+12} \in V_2^r$ and $v_{4r+6}, v_{4r+8} \in V_2^r$ are incident with a common face, respectively. Join v_{4p+1} to both u_{4r+11} and u_{4r+12} , join v_{4p+2} to both u_{4r+7} and u_{4r+8} , join u_{4p+1} to both v_{4r+10} and v_{4r+12} , join u_{4p+2} to both v_{4r+6} and v_{4r+8} .

Similarly, in G_p , join v_{4p+1} to $u_5, u_6 \in U_1^p$, join v_{4p+2} to $u_7, u_8 \in U_2^p$, join u_{4p+1} to $v_6, v_8 \in V_2^p$, join u_{4p+2} to $v_5, v_7 \in V_1^p$.

(b) The construction for \widehat{G}_r , $1 \leq r \leq p$, and r is even.

Step 1: Place the vertex x in the face 3 of G_r , delete edges $v_{4r}u_{4r-3}$ and $u_{4r-3}v_{4r-2}$ from G_r , $1 \leq r \leq p-1$.

Do parallel paths modification to G_r , $1 \leq r < p-1$, such that $u_{4r+7} \in U_2^r$, $v_{4r+4} \in V_2^r$ and $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ are incident with a common face which the vertex x is in. Join x to $u_{4r-3}, u_{4r-2}, u_{4r}, v_{4r-2}, v_{4r-1}, v_{4r}$ and u_{4r+7}, v_{4r+4} . Similarly, in G_{p-1} , join x to $u_{4p-7}, u_{4p-6}, u_{4p-4}, v_{4p-6}, v_{4p-5}, v_{4p-4}$ and $u_7 \in U_2^{p-1}$, $v_{4p} \in V_2^{p-1}$.

Step 2: Do parallel paths modifications, such that $u_{4r+5}, u_{4r+6} \in U_1^r$, $u_{4r+1}, u_{4r+2} \in U_1^r$, $v_{4r+5}, v_{4r+7} \in V_1^r$, $v_{4r+1}, v_{4r+3} \in V_1^r$ are incident with a common face, respectively. Join v_{4p+1} to both u_{4r+5} and u_{4r+6} , join v_{4p+2} to both u_{4r+1} and u_{4r+2} , join u_{4p+1} to both v_{4r+5} and v_{4r+7} , join u_{4p+2} to both v_{4r+1} and v_{4r+3} .

Table 3 shows how we add edges to G_r ($1 \leq r \leq p$) in Case 2.

TABLE 3. The edges we add to G_r ($1 \leq r \leq p$) in Case 2

subscript \ case	r is odd		r is even	
edge				
xu_j	$4r-3, 4r-1, 4r$	$4r+6, r \neq p (U_1^r)$ $2, r = p (U_1^r)$	$4r-3, 4r-2, 4r$	$4r+7, r \neq p-1 (U_2^r)$ $7, r = p-1 (U_2^r)$
xv_j	$4r-3, 4r-2, 4r-1$	$4, 5, r = 1$ $4r+1, r \neq 1, p (V_1^r)$	$4r-2, 4r-1, 4r$	$4r+4 (V_2^r)$
$v_{4p+1}u_j$	$4r+11, 4r+12, r \neq p (U_2^r)$ $5, 6, r = p (U_1^r)$		$4r+5, 4r+6 (U_1^r)$	
$v_{4p+2}u_j$	$4r+7, 4r+8 (U_2^r)$		$4r+1, 4r+2 (U_1^r)$	
$u_{4p+1}v_j$	$4r+10, 4r+12, r \neq p (V_2^r)$ $6, 8, r = p (V_2^r)$		$4r+5, 4r+7 (V_1^r)$	
$u_{4p+2}v_j$	$4r+6, 4r+8, r \neq p (V_2^r)$ $5, 7, r = p (V_1^r)$		$4r+1, 4r+3 (V_1^r)$	

(c) The construction for \widehat{G}_{p+1} .

With a similar argument to that in Case 1, we can list the edges that belong to $K_{1,4p+2,4p+2}$ but not to any \widehat{G}_r , $1 \leq r \leq p$, in this case, as shown in Table 4. Then \widehat{G}_{p+1} is the graph that consists of the edges in Table 4, Figure 4 shows \widehat{G}_{p+1} is a planar graph.

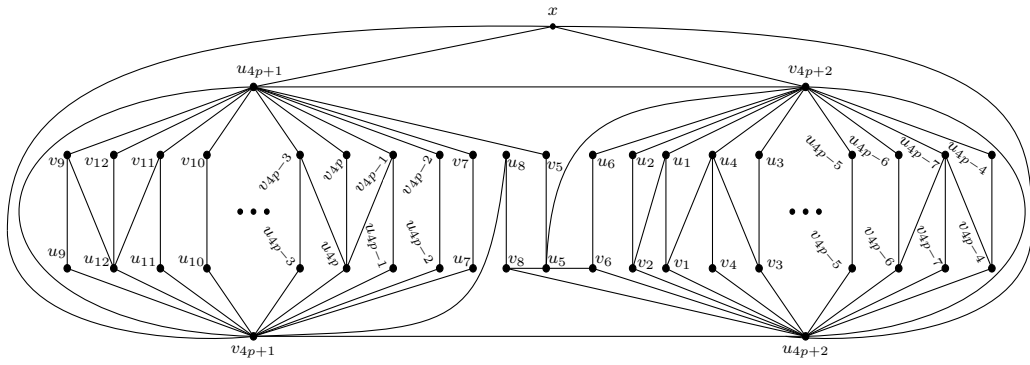
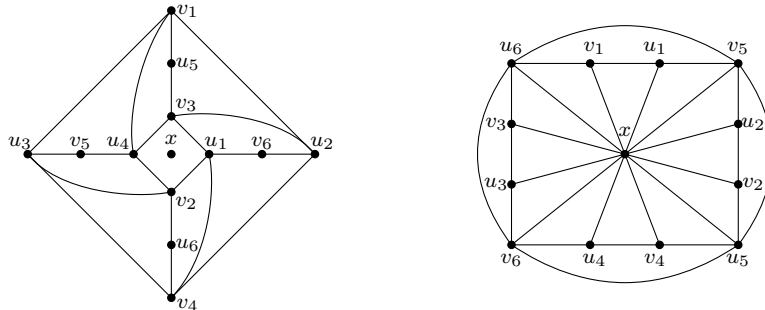
Therefore, $\{\widehat{G}_1, \dots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{1,4p+2,4p+2}$ in this case.

Case 3. When $p \leq 3$.

When $p = 0$, $K_{1,2,2}$ is a planar graph. When $p = 1, 2, 3$, we give a planar decomposition for $K_{1,6,6}$, $K_{1,10,10}$ and $K_{1,14,14}$ with 2, 3 and 4 subgraphs respectively, as shown in Figure 5, Figure 6 and Figure 7.

TABLE 4. The edges of \widehat{G}_{p+1} in Case 2

edges	subscript
$xv_{4p+1}, v_{4p+1}u_j$	$j = 4r - 3, 4r - 2, 4r - 1, 4r, 7, 8, 4p + 2. (r = 3, 5, 7, \dots, p.)$
$xu_{4p+1}, u_{4p+1}v_j$	$j = 4r - 3, 4r - 2, 4r - 1, 4r, 5, 7, 4p + 2. (r = 3, 5, 7, \dots, p.)$
$xv_{4p+2}, v_{4p+2}u_j$	$j = 4r - 3, 4r - 2, 4r - 1, 4r, 5, 6, 4p + 1. (r = 1, 4, 6, 8, \dots, p - 1.)$
$xu_{4p+2}, u_{4p+2}v_j$	$j = 4r - 3, 4r - 2, 4r - 1, 4r, 6, 8, 4p + 1. (r = 1, 4, 6, 8, \dots, p - 1.)$
$u_1v_2, v_{4r-3}u_{4r}, u_{4r}v_{4r-1}$	$r = 1, 3, \dots, p.$
$v_{4r}u_{4r-3}, u_{4r-3}v_{4r-2}$	$r = 2, 4, \dots, p - 1.$
u_jv_j	$j = 1, \dots, 4p + 2.$

FIGURE 4. The graph \widehat{G}_{p+1} in Case 2FIGURE 5. A planar decomposition of $K_{1,6,6}$

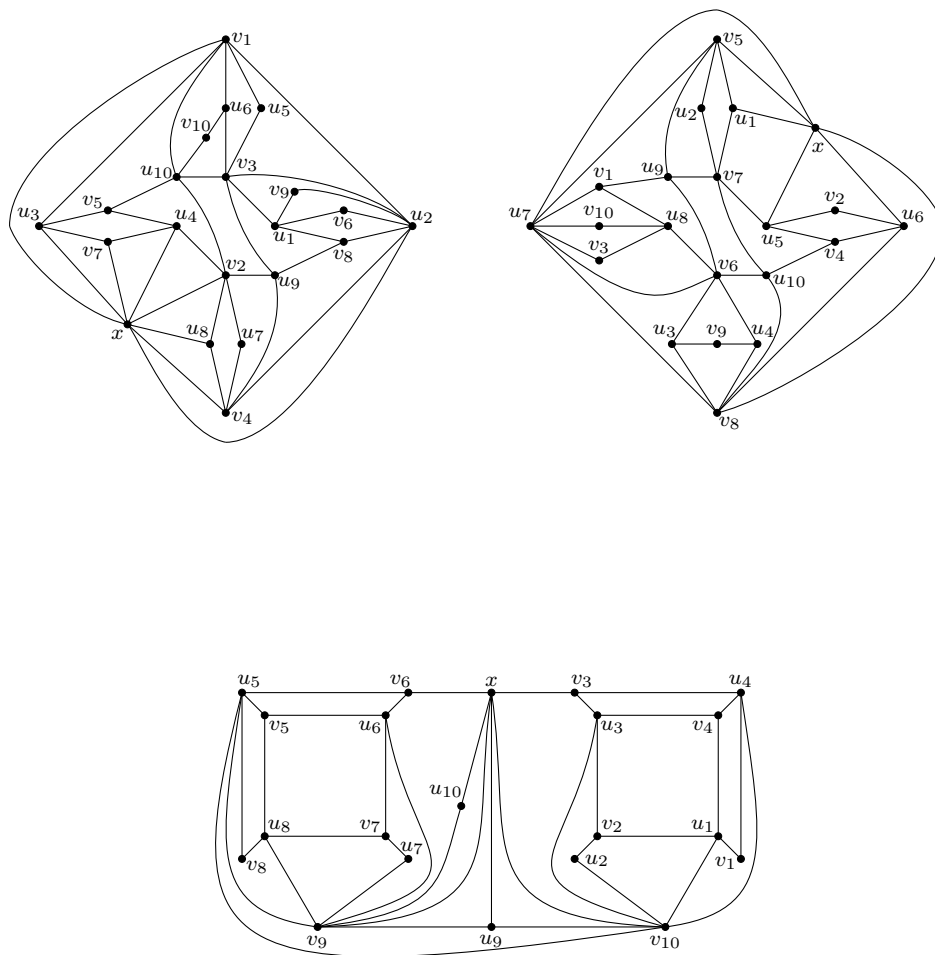
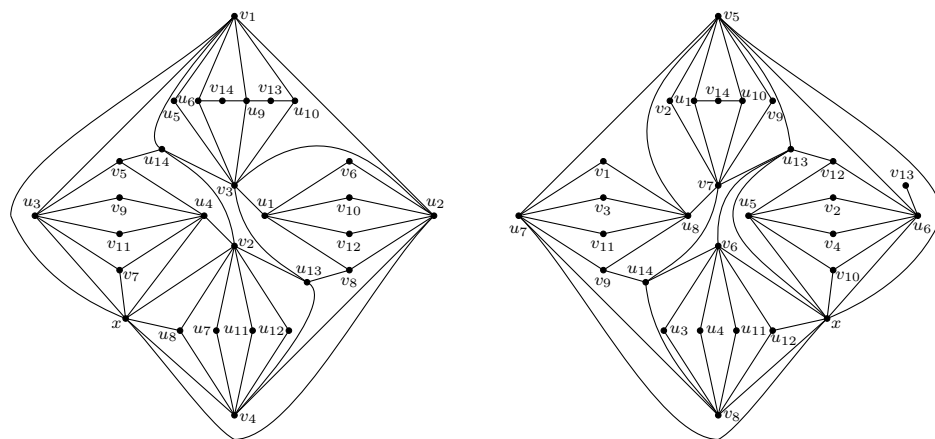
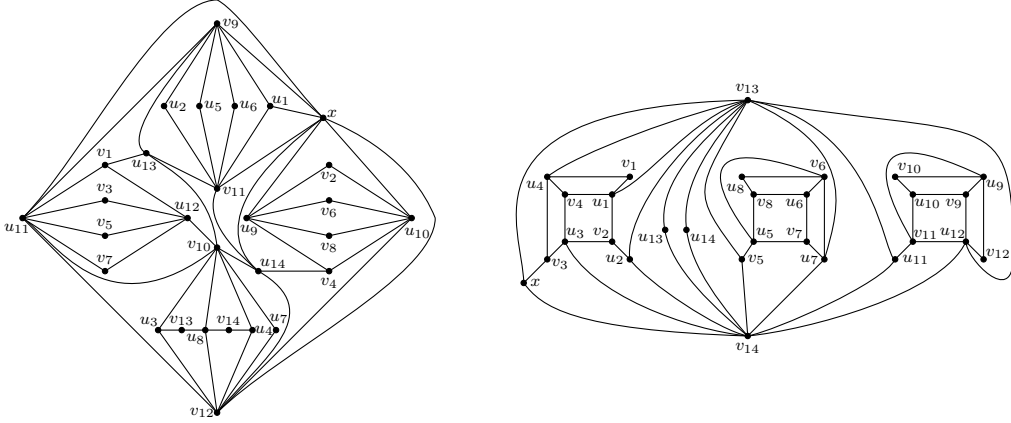


FIGURE 6. A planar decomposition of $K_{1,10,10}$



FIGURE 7. A planar decomposition of $K_{1,14,14}$

Lemma follows from Cases 1, 2 and 3. \square

Theorem 2.4. *The thickness of the complete 3-partite graph $K_{1,n,n}$ is*

$$\theta(K_{1,n,n}) = \lceil \frac{n+2}{4} \rceil.$$

Proof. When $n = 4p, 4p + 3$, the theorem follows from Lemma 2.2.

When $n = 4p + 1, n = 4p + 2$, from Lemma 2.3, we have $\theta(K_{1,4p+2,4p+2}) \leq p + 1$. Since $\theta(K_{4p,4p}) = p + 1$ and $K_{4p,4p} \subset K_{1,4p+1,4p+1} \subset K_{1,4p+2,4p+2}$, we obtain

$$p + 1 \leq \theta(K_{1,4p+1,4p+1}) \leq \theta(K_{1,4p+2,4p+2}) \leq p + 1.$$

Therefore, $\theta(K_{1,4p+1,4p+1}) = \theta(K_{1,4p+2,4p+2}) = p + 1$.

Summarizing the above, the theorem is obtained. \square

3. THE THICKNESS OF $K_{2,n,n}$

Lemma 3.1. *There exists a planar decomposition of the complete 3-partite graph $K_{2,4p+1,4p+1}$ ($p \geq 0$) with $p + 1$ subgraphs.*

Proof. Let (X, U, V) be the vertex partition of the complete 3-partite graph $K_{2,n,n}$, in which $X = \{x_1, x_2\}$, $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$. When $n = 4p + 1$, we will construct a planar decomposition of $K_{2,4p+1,4p+1}$ with $p + 1$ planar subgraphs.

The construction is analogous to that in Lemma 2.3. Let $\{G_1, G_2, \dots, G_{p+1}\}$ be a planar decomposition of $K_{4p,4p}$ given in [4]. In the following, for $1 \leq r \leq p + 1$, by adding vertices $x_1, x_2, u_{4p+1}, v_{4p+1}$ to G_r , deleting some edges from G_r and adding some edges to G_r , we will get a new planar graph \widehat{G}_r such that $\{\widehat{G}_1, \dots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,4p+1,4p+1}$. All the subscripts of vertices are taken modulo $4p$, except that of u_{4p+1} and v_{4p+1} (the vertices we added to G_r).

Case 1. When p is even and $p > 2$.

(a) The construction for \widehat{G}_r , $1 \leq r \leq p$.

Step 1: When r is odd, place the vertex x_1, x_2 and u_{4p+1} in the face 1,2 and 5 of G_r respectively. Delete edges $v_{4r-3}u_{4r}$ and $u_{4r-1}v_{4r-2}$ from G_r .

When r is even, place the vertex x_1, x_2 and u_{4p+1} in the face 3,4 and 5 of G_r , respectively. Delete edge $v_{4r}u_{4r-3}$ and $u_{4r-2}v_{4r-1}$ from G_r .

Step 2: Do parallel paths modifications, then join x_1, x_2, u_{4p+1} and v_{4p+1} to some u_j and v_j , as shown in Table 5.

TABLE 5. The edges we add to $G_r (1 \leq r \leq p)$ in Case 1

subscript edge	case	r is odd		r is even	
x_1u_j		$4r-1, 4r$	$4r+5 (U_1^r)$	$4r-3, 4r-2$	$4r+8 (U_2^r)$
x_1v_j		$4r-3, 4r-1$	$4r+1 (V_1^r)$	$4r-2, 4r$	$4r+4 (V_2^r)$
x_2u_j		$4r-1, 4r$	$4r+3 (U_2^r)$	$4r-3, 4r-2$	$4r+2 (U_1^r)$
x_2v_j		$4r-2, 4r$	$4r+7 (V_1^r)$	$4r-3, 4r-1$	$4r+6 (V_2^r)$
$u_{4p+1}v_j$		$4r-2, 4r-1$			
$v_{4p+1}u_j$		$4r+4, 4r+8 (U_2^r)$		$4r-11, 4r-7 (U_1^r)$	

(b) The construction for \widehat{G}_{p+1} .

We list the edges that belong to $K_{2,4p+1,4p+1}$ but not to any $\widehat{G}_r, 1 \leq r \leq p$, as shown in Table 6. Then \widehat{G}_{p+1} is the graph that consists of the edges in Table 6, Figure 8 shows \widehat{G}_{p+1} is a planar graph.

TABLE 6. The edges of \widehat{G}_{p+1} in Case 1

edges	subscript
x_1u_j	$j = 4r-2, 4r+3, 4p+1. (r = 1, 3, \dots, p-1.)$
x_1v_j	
x_2u_j	$j = 4r-7, 4r, 4p+1. (r = 2, 4, \dots, p.)$
x_2v_j	
$u_{4p+1}v_j$	$j = 4r-3, 4r. (r = 1, 2, \dots, p.)$
$v_{4p+1}u_j$	$j = 4r-2, 4r-1. (r = 1, 2, \dots, p.)$
$v_{4r-3}u_{4r}, v_{4r-2}u_{4r-1}$	$r = 1, 3, \dots, p-1.$
$u_{4r-3}v_{4r}, u_{4r-2}v_{4r-1}$	$r = 2, 4, \dots, p.$
u_jv_j	$j = 1, \dots, 4p+1.$

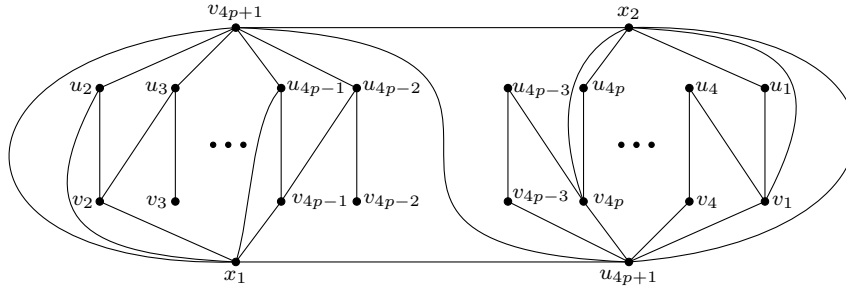
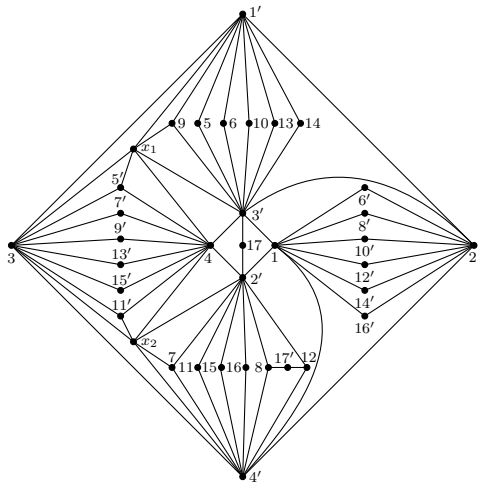
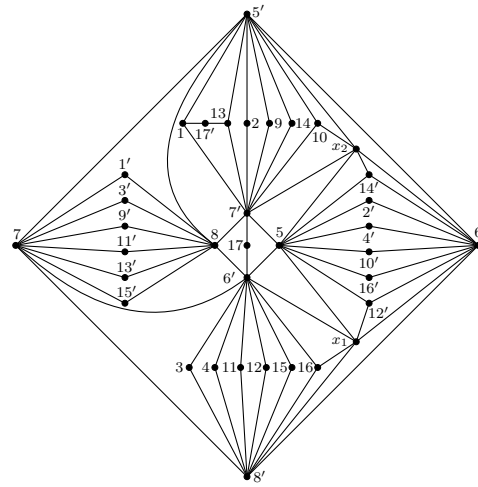


FIGURE 8. The graph \widehat{G}_{p+1} in Case 1

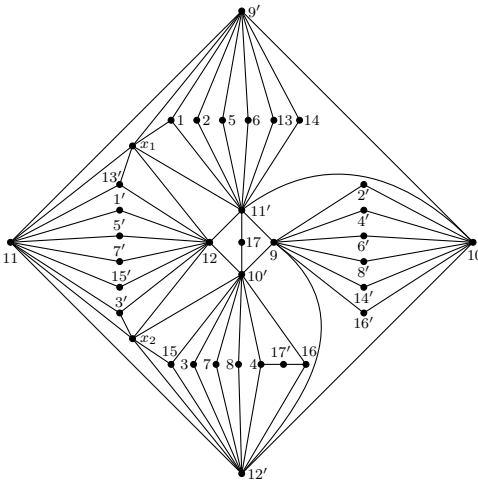
Therefore, $\{\widehat{G}_1, \dots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,4p+1,4p+1}$ in this case. In Figure 9, we draw the planar decomposition of $K_{2,17,17}$ it is the smallest example for the Case 1. We denote vertex u_i and v_i by i and i' respectively in this figure.



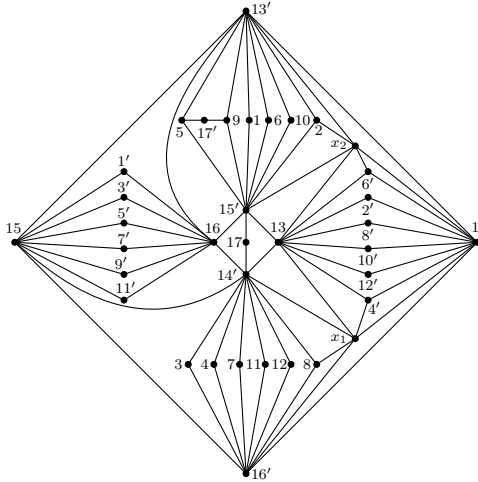
(a) The graph \widehat{G}_1



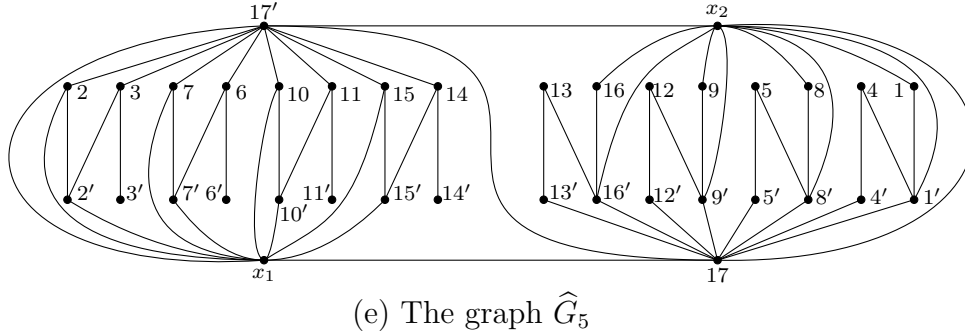
(b) The graph \widehat{G}_2



(c) The graph \widehat{G}_3



(d) The graph \widehat{G}_4


 FIGURE 9. A planar decomposition of $K_{2,17,17}$

Case 2. When p is odd and $p > 3$.

(a) The construction for \widehat{G}_r , $1 \leq r \leq p$.

Step 1: When r is odd, place the vertex x_1, x_2 and u_{4p+1} in the face 1,2 and 5 of G_r respectively. Delete edges $v_{4r-3}u_{4r}$ and $u_{4r-1}v_{4r-2}$ from G_r .

When r is even, place the vertex x_1, x_2 and u_{4p+1} in the face 3,4 and 5 of G_r , respectively. Delete edge $v_{4r}u_{4r-3}$ and $u_{4r-2}v_{4r-1}$ from G_r .

Step 2: Do parallel paths modifications, then join x_1, x_2, u_{4p+1} and v_{4p+1} to some u_j and v_j , as shown in Table 7.

 TABLE 7. The edges we add to $G_r (1 \leq r \leq p)$ in Case 2

subscript \ case	r is odd		r is even	
edge				
$x_1 u_j$	$4r - 1, 4r$	$4r + 5, r \neq p (U_1^r)$ $1, r = p (U_1^r)$	$4r - 3, 4r - 2$	$4r + 8, r \neq p - 1 (U_2^r)$ $8, r = p - 1 (U_2^r)$
$x_1 v_j$	$4r - 3, 4r - 1$	$4r + 1, r \neq p (V_1^r)$	$4r - 2, 4r$	$4r + 4 (V_2^r)$
$x_2 u_j$	$4r - 1, 4r$	$4r + 3, r \neq p (U_2^r)$ $8, r = p (U_2^r)$	$4r - 3, 4r - 2$	$4r + 2 (U_1^r)$
$x_2 v_j$	$4r - 2, 4r$	$4r + 7, r \neq p (V_1^r)$ $3, r = p (V_1^r)$	$4r - 3, 4r - 1$	$4r + 6, r \neq p - 1 (V_2^r)$ $6, r = p - 1 (V_2^r)$
$u_{4p+1} v_j$	$4r - 2, 4r - 1$			
$v_{4p+1} u_j$	$4r + 4, 4r + 8, r \neq p (U_2^r)$ $4, r = p (U_2^r)$		$4r - 11, 4r - 7 (U_1^r)$	

(b) The construction for \widehat{G}_{p+1} .

We list the edges that belong to $K_{2,4p+1,4p+1}$ but not to any \widehat{G}_r , $1 \leq r \leq p$, as shown in Table 8. Then \widehat{G}_{p+1} is the graph that consists of the edges in Table 8, Figure 10 shows \widehat{G}_{p+1} is a planar graph.

TABLE 8. The edges of \widehat{G}_{p+1} in Case 2

edges	subscript
$x_1 u_j$	$j = 2, 4r + 3, 4r + 6, 4p + 1. (r = 1, 3, \dots, p - 2.)$
$x_1 v_j$	$j = 2, 4, 4r + 3, 4r + 6, 4p + 1. (r = 1, 3, \dots, p - 2.)$
$x_2 u_j$	$j = 1, 2, 9, 4r, 4r + 1, 4p + 1. (r = 4, \dots, p - 1.)$
$x_2 v_j$	$j = 1, 8, 9, 4r, 4r + 1, 4p + 1. (r = 4, \dots, p - 1.)$
$u_{4p+1} v_j$	$j = 4r - 3, 4r. (r = 1, 2, \dots, p.)$
$v_{4p+1} u_j$	$j = 4r - 2, 4r - 1, 4p - 7. (r = 1, 2, \dots, p.)$
$v_{4r-3} u_{4r}, v_{4r-2} u_{4r-1}$	$r = 1, 3, \dots, p.$
$u_{4r-3} v_{4r}, u_{4r-2} v_{4r-1}$	$r = 2, 4, \dots, p - 1.$
$u_j v_j$	$j = 1, \dots, 4p + 1.$

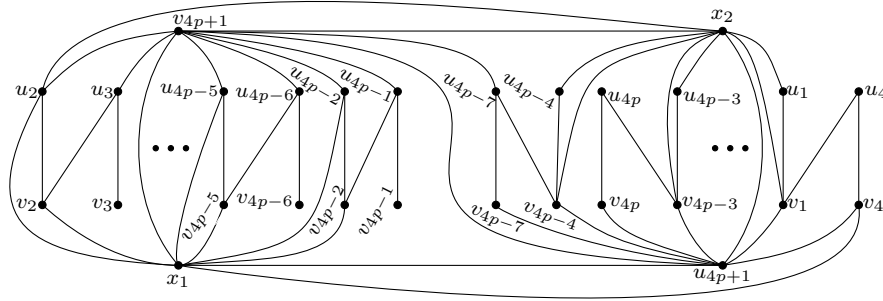
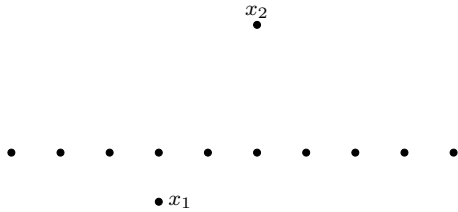


FIGURE 10. The graph \widehat{G}_{p+1} in Case 2

Therefore, $\{\widehat{G}_1, \dots, \widehat{G}_{p+1}\}$ is a planar decomposition of $K_{2,4p+1,4p+1}$ in this case.

Case 3. When $p \leq 3$.

When $p = 0$, $K_{2,1,1}$ is a planar graph. When $p = 1, 2, 3$, we give a planar decomposition for $K_{2,5,5}$, $K_{2,9,9}$ and $K_{2,13,13}$ with 2, 3 and 4 subgraphs respectively, as shown in Figure 11, Figure 12 and Figure 13.



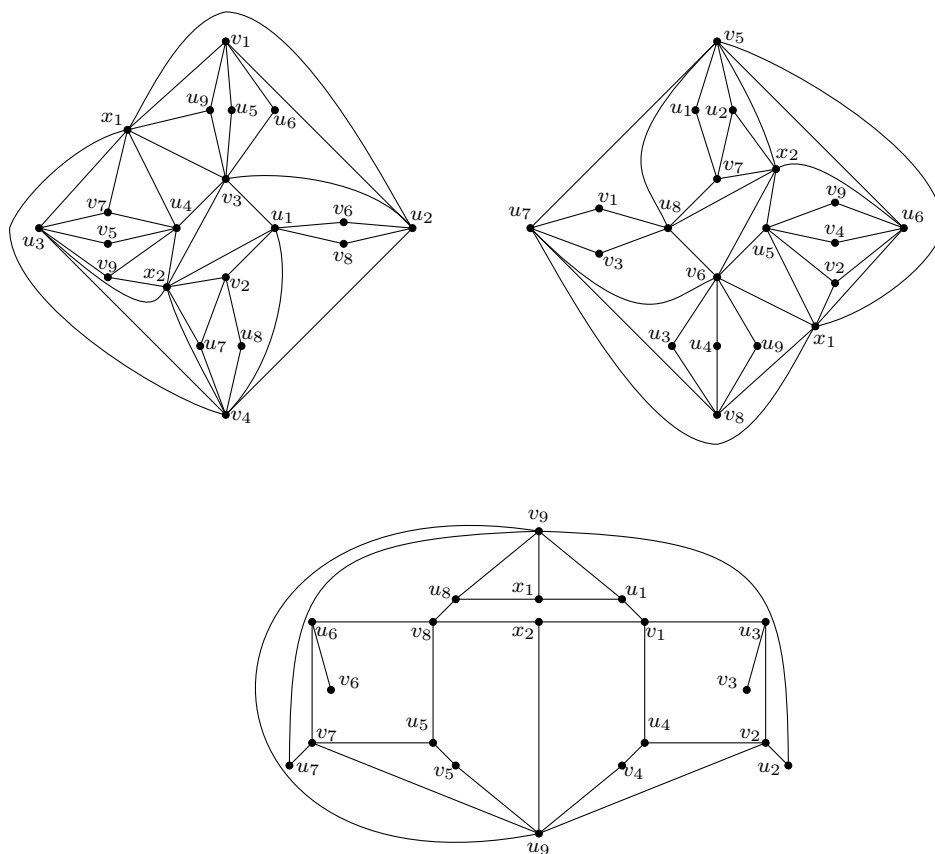
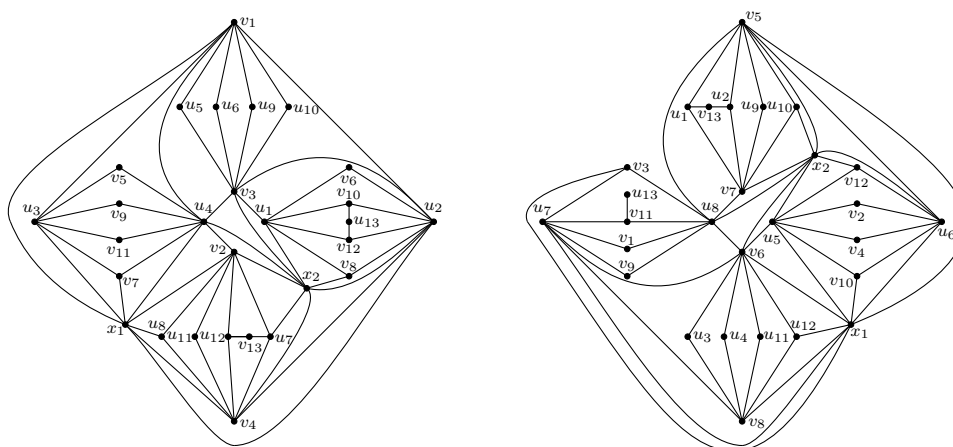
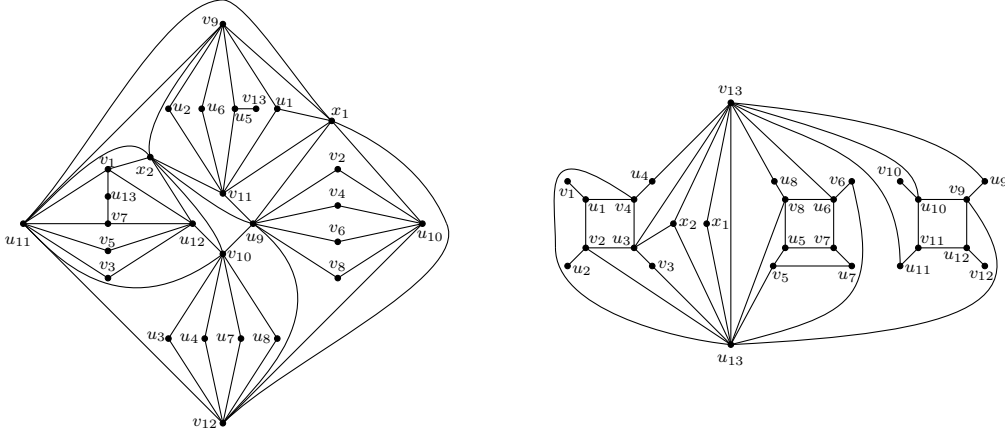


FIGURE 12. A planar decomposition $K_{2,9,9}$



FIGURE 13. A planar decomposition of $K_{2,13,13}$

Summarizing Cases 1,2 and 3, the lemma follows. \square

Theorem 3.2. *The thickness of the complete 3-partite graph $K_{2,n,n}$ is*

$$\theta(K_{2,n,n}) = \lceil \frac{n+3}{4} \rceil.$$

Proof. When $n = 4p, 4p+3$, from Lemma 2.2, the theorem holds.

When $n = 4p+1$, from Lemma 3.1, we have $\theta(K_{2,4p+1,4p+1}) \leq p+1$. Since $\theta(K_{4p,4p}) = p+1$ and $K_{4p,4p} \subset K_{2,4p+1,4p+1}$, we have

$$p+1 = \theta(K_{4p,4p}) \leq \theta(K_{2,4p+1,4p+1}) \leq p+1.$$

Therefore, $\theta(K_{2,4p+1,4p+1}) = p+1$.

When $n = 4p+2$, since $K_{4p+3,4p+3} \subset K_{2,4p+2,4p+2}$, from Lemma 2.1, we have $p+2 = \theta(K_{4p+3,4p+3}) \leq \theta(K_{2,4p+2,4p+2})$. On the other hand, it is easy to see $\theta(K_{2,4p+2,4p+2}) \leq \theta(K_{2,4p+1,4p+1}) + 1 = p+2$, so we have $\theta(K_{2,4p+2,4p+2}) = p+2$.

Summarizing the above, the theorem is obtained. \square

4. THE THICKNESS OF $K_{1,1,n,n}$

Theorem 4.1. *The thickness of the complete 4-partite graph $K_{1,1,n,n}$ is*

$$\theta(K_{1,1,n,n}) = \lceil \frac{n+3}{4} \rceil.$$

Proof. When $n = 4p+1$, we can get a planar decomposition for $K_{1,1,4p+1,4p+1}$ from that of $K_{2,4p+1,4p+1}$ as follows.

(1) When $p = 0$, $K_{1,1,1,1}$ is a planar graph, $\theta(K_{1,1,1,1}) = 1$. When $p = 1, 2$ and 3, we join the vertex x_1 to x_2 in the last planar subgraph in the planar decomposition for $K_{2,5,5}$, $K_{2,9,9}$ and $K_{2,13,13}$ which was shown in Figure 11, 12 and 13. Then we get the planar decomposition for $K_{1,1,5,5}$, $K_{1,1,9,9}$ and $K_{1,1,13,13}$ with 2, 3 and 4 planar subgraphs respectively.

(2) When $p \geq 4$, we join the vertex x_1 to x_2 in \widehat{G}_{p+1} in the planar decomposition for $K_{2,4p+1,4p+1}$ which was constructed in Lemma 3.1. The \widehat{G}_{p+1} is shown

in Figure 8 or 10 according to p is even or odd. Because x_1 and x_2 lie on the boundary of the same face, we will get a planar graph by adding edge x_1x_2 to \widehat{G}_{p+1} . Then a planar decomposition for $K_{1,1,4p+1,4p+1}$ with $p+1$ planar subgraphs can be obtained.

Summarizing (1) and (2), we have $K_{1,1,4p+1,4p+1} \leq p+1$.

On the other hand, from Lemma 2.1, we have $\theta(K_{4p+1,4p+1}) = p+1$. Due to $K_{4p+1,4p+1} \subset K_{1,1,4p,4p} \subset K_{1,1,4p+1,4p+1}$, we get $p+1 \leq \theta(K_{1,1,4p,4p}) \leq \theta(K_{1,1,4p+1,4p+1})$. So we have

$$\theta(K_{1,1,4p,4p}) = \theta(K_{1,1,4p+1,4p+1}) = p+1.$$

When $n = 4p+3$, from Theorem 3.2, we have $\theta(K_{2,4p+2,4p+2}) = p+2$. Since $K_{2,4p+2,4p+2} \subset K_{1,1,4p+2,4p+2} \subset K_{1,1,4p+3,4p+3} \subset K_{1,1,4(p+1),4(p+1)}$, and the ideas from the previous case establish, we have $p+2 \leq \theta(K_{1,1,4p+2,4p+2}) \leq \theta(K_{1,1,4p+3,4p+3}) \leq \theta(K_{1,1,4(p+1),4(p+1)}) = p+2$, which shows

$$\theta(K_{1,1,4p+2,4p+2}) = \theta(K_{1,1,4p+3,4p+3}) = p+2.$$

Summarizing the above, the theorem follows. \square

REFERENCES

- [1] V.B. Alekseev and V.S. Gončakov, The thickness of an arbitrary complete graph, *Math. Sbornik.*, **30(2)** (1976), 187–202.
- [2] L.W. Beineke and F. Harary, The thickness of the complete graph, *Canad. J. Math.*, **17** (1965), 850–859.
- [3] L.W. Beineke, F. Harary and J.W. Moon, On the thickness of the complete bipartite graph, *Proc. Cambridge Philos. Soc.*, **60** (1964), 1–5.
- [4] Y.C. Chen and X.L. Yin, The thickness of the Cartesian product of two graphs, *Canad. Math. Bull.*, **59(4)** (2016), 705–720.
- [5] M. Kleinert, Die Dicke des n -dimensionalen Würfel-Graphen, *J. Combin. Theory*, **3** (1967), 10–15.
- [6] T. Poranen, A simulated annealing algorithm for determining the thickness of a graph, *Inform. Sci.*, **172** (2005), 155–172.
- [7] W.T. Tutte, The thickness of a graph, *Indag. Math.*, **25** (1963), 567–577.
- [8] J.M. Vasak, The thickness of the complete graph, *Notices Amer. Math. Soc.*, **23** (1976), A-479.
- [9] Y. Yang, A note on the thickness of $K_{l,m,n}$, *Ars Combin.*, **117** (2014), 349–351.
- [10] Y. Yang, Remarks on the thickness of $K_{n,n,n}$, *Ars Math. Contemp.*, **12(1)** (2017), 135–144.

DEPARTMENT OF MATHEMATICS, TIANJIN UNIVERSITY, 300072, TIANJIN, CHINA
E-mail address: guoxi a@tj u. edu. cn

DEPARTMENT OF MATHEMATICS, TIANJIN UNIVERSITY, 300072, TIANJIN, CHINA
E-mail address: yanyang@tj u. edu. cn (Corresponding author: Yan YANG)